Mathematics and Its Applications

Efstathios Vassiliou

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Springer

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Volume 578

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by

Efstathios Vassiliou

University of Athens,
Greece

Springer

A C.I.P. Catalogue record for this book is available from the Library of Congress.

ISBN-10 1-4020-3415-6 (HB) Springer Dordrecht, Berlin, Heidelberg, New York
ISBN-10 1-4020-3416-4 (e-book) Springer Dordrecht, Berlin, Heidelberg, New York
ISBN-13 978-1-4020-3415-2 (HB) Springer Dordrecht, Berlin, Heidelberg, New York
ISBN-13 978-1-4020-3416-9 (e-book) Springer Dordrecht, Berlin, Heidelberg, New York

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Published by Springer,
P.O. Box 17, 3300 AA Dordrecht, The Netherlands.

Printed on acid-free paper

Printed in the Netherlands.

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Preface

L' injénuité même d' un regard neuf (celui de la science l'est toujours) peut parfois éclairer d' un jour nouveau d' anciens problèmes.

J. MONOD [77, p. 13]

THIS book is intended as a comprehensive introduction to the theory of *principal sheaves* and their *connections* in the setting of Abstract Differential Geometry (ADG), the latter being initiated by A. Mallios's *Geometry of Vector Sheaves* [62]. Based on sheaf-theoretic methods and sheaf cohomology, the present *Geometry of Principal Sheaves* embodies the classical theory of connections on principal and vector bundles, and connections on vector sheaves, thus paving the way towards a unified (abstract) gauge theory and other potential applications to theoretical physics. We elaborate on the aforementioned brief description in the sequel.

Abstract (ADG) vs. Classical Differential Geometry (CDG). Modern differential geometry is built upon the fundamental notions of *differential (smooth) manifolds* and *fiber bundles*, based, in their turn, on ordinary differential calculus.

However, the theory of smooth manifolds is inadequate to cope, for instance, with spaces like orbifolds, spaces with corners, or other spaces with more complicated singularities. This is a rather unfortunate situation, since one cannot apply the powerful methods of differential geometry to them or to any spaces that do not admit an ordinary method of differentiation. The same inadequacy manifests in physics, where many geometrical models of physical phenomena are non-smooth.

These deficiencies gave rise, long ago, to the study of a variety of structures extending that of a differential manifold. In this regard we may cite, e.g., the *differential spaces* according to R. Sikorski [113] (see also [112]), and the *generalized spaces* in the sense of M. A. Mostow [79], J. W. Smith [114] and I. Satake [107], to name but a few. The central idea behind all these approaches is essentially to single out a family of functions, enlarging the set of usual smooth functions so that the singularities (with respect to the latter) are overcome. In customary terminology, one is thus led to a new *structure sheaf*, characterizing the (enlarged) "smooth" functions of the space under consideration.

Circa 1989, A. Mallios, in an entirely different direction and independent of the other approaches, motivated by S. Selesnick's paper [109] (holding no relevance to differential spaces and the like) arrived at a quite general theory pointing out the essential, algebraic in effect, differential-geometric mechanism of the classical theory in that context. In this respect, the structure sheaf of a "smooth" space –in the general sense– was freed from its functional character, prevalent in the case of the aforementioned generalizations. As a matter of fact, he defined *algebraized spaces* $(X, \mathcal{A})$ to which he attached *differential triads* $(\mathcal{A}, d, \Omega)$. Here $\mathcal{A}$ is a sheaf of commutative associative and unital algebras, Ω an $\mathcal{A}$-module, and d an Ω-valued derivation of $\mathcal{A}$. Such Ω and d can be constructed in various ways from a given $\mathcal{A}$. One way is by applying the sheafification process to Kähler's algebraic theory of differentials. These notions laid the foundations for an abstract formulation of differential geometry, where *no notion of differentiability* is assumed whatsoever. Indeed, the methods applied therein are rather algebraic, being based mainly on sheaf-theoretic techniques and sheaf cohomology.

This point of view has an obvious unifying power and naturally includes smooth manifolds and various differential spaces, standard or otherwise. Nevertheless, the most important of all is, perhaps, the new insight that such a generalization gives to the meaning of a *geometric space* X, i.e., a space serving as the basis for the development of a sort of "differential" geometry reminiscent of its classical ancestor. Here the adjective differential, being deprived of its ordinary connotation (recalling differential calculus), refers rather to a wealth of effective methods and notions parallel to their classical counterparts. In such a space, the object of primary importance is the structure sheaf and not X itself, which may contain many types of singularities (relative to any classical structure or a given $\mathcal{A}$). Accordingly, the "removal" of the said singularities consists of changing the pathological $\mathcal{A}$ (expressing our old "arithmetic") to a new structure sheaf absorbing –so to speak– the singularities, while X remains unaltered.

In other words, the space X now plays the secondary rôle of the carrier of the generalized "smooth" functions, which can be thought of as the sections of the structure sheaf $\mathcal{A}$, whereas the entire differential-geometric apparatus lives in $\mathcal{A}$ and certain other sheaves over X. We refer to A. Mallios – E. E. Rosinger [71, 72] for an application of these ideas to the *highly singular* algebra of Rosinger's generalized functions and the multi-foam algebra of them, both encountered and confronted with, in problems of non-linear PDE's. An interesting discussion (with mathematical, physical, and even philosophical, repercussions) on the meaning, appearance and removal of singularities, within CDG and ADG, can be found in A. Mallios [66].

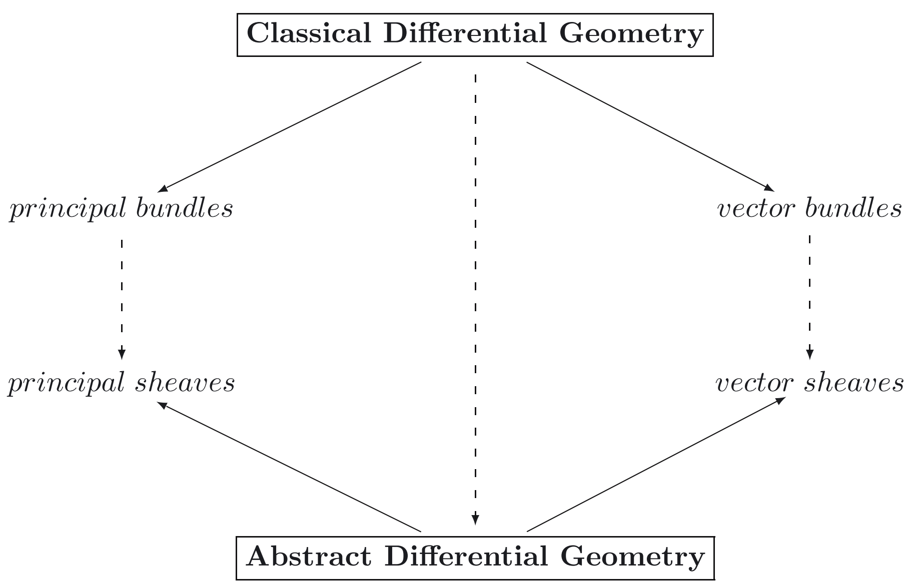
Undoubtedly, the same approach is also important to physical applications, where –as already mentioned– differentiability is quite a restrictive property (if not non-natural) and the quest for algebraic methods seems to be most desirable. In this respect we refer to A. Einstein's conclusion in [26, p. 158], the relevant comments of M. Heller [42, pp. 349–350] (in conjunction with M. Heller-W. Sasin [43]), and A. Mallios [66, 65]. Without going into details, which are beyond the scope of this book, the same geometrical mechanism can be applied to the quantum domain. Elementary particles can be treated as "geometrical" objects, without reference to any space in the usual sense, by applying the methods of ADG.

In the same spirit, the recent publication of J. Nestruev [83], though in a different setting, is another advocate of the algebraic formalism motivated by physical considerations. Indeed, Nestruev's book "...explains in detail why the differential calculus on manifolds is simply an aspect of commutative algebra" (op. cit., p. vii) and why "Perhaps, more important is the algebraic approach to the study of manifolds" (ibid., p. 6). This is in accordance with the earlier idea that "a large part of the classical differential geometry is part of linear algebra, more precisely –of the theory of modules"(R. Sikorski [113, p. 45]; see also [112]). Technically speaking, a manifold can be defined as the spectrum of a certain commutative topological algebra ([58], [83]).

The main ideas of ADG are expounded in [62] and are applied to the geometry of vector sheaves, that is, the abstract analog of vector bundles. An extensive summary of this work is also given in [63]. It is remarkable that classical topics such as connections and their curvature, various notions of flatness, Riemannian and Hermitian structures, Weil's integrality theorem, the Chern-Weil homomorphism, characteristic classes and many more are treated in this context. Two other volumes ([67]), devoted mainly to physical applications, are under preparation (see also [68], [69], [70]).

The present author had the opportunity to become acquainted, as early as 1990, with the Geometry of Vector Sheaves through a seminar organized by A. Mallios at the Mathematics Department of the University of Athens, by numerous discussions with him, and finally from various preprints and subsequent drafts of certain sections of his treatise. In this inspiring environment the author embarked on a research project concerning the geometry of principal sheaves, results of which are collected in a coherent and systematic way herein. Besides its "raison d' être" per se, which will be further justified below, this volume can also be considered as a natural companion of [62], extending and complementing various aspects of it.

The next figure depicts the relationship between the classical geometry of fiber spaces, mainly vector and principal bundles, and their abstract counterparts vector and principal sheaves. The vertical arrows of the picture represent categorical imbeddings.



The contents in brief. For the reader's convenience we continue with an outline of the book. More detailed information can be found in the table of contents and the individual introduction of each chapter.

Chapter 1 contains the very basic notions and results of the *theory of sheaves* and *sheaf-cohomology* needed throughout this exposition. It is rather sketchy, principally without formal statements and proofs, and intends to facilitate references to standard material fully treated in many excellent books. The expert may skip this chapter.

Chapter 2 is a detailed *categorical* study of differential triads, a point of view not treated in [62]. It is shown that the category of differential triads is closed under subspaces, quotients, products and various limits, unfolding evidence to its potentiality as described. Smooth manifolds, differential spaces and related structures are naturally imbedded in this category. An abstract differentiability notion, rendering smooth a very broad category of continuous mappings, is also introduced.

Chapter 3 is devoted to the study of *Lie sheaves of groups*, the abstract analog of Lie groups. They are sheaves of groups $\mathcal{G}$ admitting a representation on an appropriate sheaf of Lie algebras $\mathcal{L}$, say $\rho : \mathcal{G} \to \mathcal{A}ut(\mathcal{L})$, and are equipped with an abstract form of logarithmic differential $\partial : \mathcal{G} \to \Omega \otimes_{\mathcal{A}} \mathcal{L}$, satisfying the property

$$\partial(g \cdot h) = \rho(h^{-1}).\partial(g) + \partial(h), \qquad (g,h) \in \mathcal{G} \times_X \mathcal{G}.$$

We call such a ∂ a *Maurer-Cartan* differential. Lie sheaves of groups constitute both the structure sheaf and the structure type of our principal sheaves, and one of the building blocks of the theory of connections developed in subsequent chapters. An important example is the general linear group sheaf $\mathcal{GL}(n, \mathcal{A})$, closely related with the geometry of a vector sheaf (of rank n).

Chapter 4 aims at the theory of *principal sheaves*, originally considered by A. Grothendieck [36]. However, here we define them in a slightly different way, allowing the development of the geometry we have in mind. A principal sheaf $\mathcal{P}$, for our purpose, is a sheaf locally isomorphic to a (Lie) sheaf of groups $\mathcal{G}$, the latter also acting on $\mathcal{P}$. After studying the morphisms of principal sheaves and their relationship with 1-cocycles, we proceed to the cohomological classification of this category of sheaves. An important example is the sheaf of frames of a vector sheaf, treated in the next chapter.

Chapter 5 starts with *vector sheaves*. Each vector sheaf $\mathcal{E}$ determines a corresponding *principal sheaf of frames* $\mathcal{P}(\mathcal{E})$. By means of the latter, the geometry of vector sheaves is reduced to that of principal ones. A considerable part of the chapter is devoted to various sheaves *associated* with a given principal sheaf $\mathcal{P}$ by representations of its structure sheaf $\mathcal{G}$ on certain sheaves of groups. A vector sheaf is trivially associated with its principal sheaf of frames.

Chapter 6 introduces the fundamental notion of *connection* (or *gauge potential* of physicists) on a principal sheaf $\mathcal{P}$. A connection is defined to be a morphism of sheaves $D : \mathcal{P} \to \Omega \otimes_{\mathcal{A}} \mathcal{L}$, satisfying the fundamental property

$$D(p \cdot g) = \rho(g^{-1}).D(p) + \partial(g), \qquad (p, g) \in \mathcal{P} \times_X \mathcal{G}.$$

It turns out that D is equivalent to a family of local sections (ω_α) of $\Omega \otimes_{\mathcal{A}} \mathcal{L}$, analogous to the ordinary local connection forms. The existence of connections is ensured by the annihilation of a particular cohomology class, named the *Atiyah class* of the principal sheaf, after the classical analog of holomorphic connections. Our approach provides yet another, operator-like, definition of ordinary connections. Other results concern connections linked together by morphisms of principal sheaves, gauge transformations of connections and the moduli sheaf of connections.

Chapter 7 explains how the connections of a principal sheaf induce connections on various associated sheaves, principal or vector ones. In particular, it is shown that the theory of connections on vector sheaves, as developed in [62], can be deduced from the general theory of Chapter 6.

Chapter 8 is centered on another fundamental notion of differential geometry, namely the *curvature of a connection* (in the language of physics, the *field strength* of a gauge potential). Its existence is ensured if higher order differentials, extending the Maurer-Cartan differential ∂, exist. By the same token, we obtain the analogs of Cartan's (second) structure equation and Bianchi's identity. In the sequel we focus on flat and integrable connections, flat principal sheaves and parallelism. Unlike the classical case, these notions are not equivalent unless an appropriate Frobenius condition is assumed to be satisfied.

Chapter 9 deals exclusively with the abstraction of the Chern-Weil homomorphism using the theory of connections on principal sheaves.

Chapter 10 contains a few applications further illustrating some of the general methods and ideas of this work. They are examples somewhat more technical than those included in the previous chapters. Among them we include *infinite-dimensional connections*, *Riemannian metrics*, *the torsion of a linear connection on* Ω^*, and *non-commutative differential triads*. Finally, the concluding problems are not exercises intending to test the reader's comprehension. They are merely inviting him/her to complete our exposition by pursuing this trend of research towards certain topics not covered here.

Reverting to CDG, we would like to add that, in the course of our investigation, we show in detail how the standard theory fits into the present scheme. Apart from providing a basic example, this endeavor gives two useful by-products: the disclosure, on the one hand, of the basic tools of CDG that are susceptible of the described abstraction, and, on the other hand, the investigation of the rôle of others, less exploited in the classical context because of the abundance of means therein.

To be more precise, let us mention, as an example, the case of connections on a principal bundle. Looking carefully at them, we see they involve two fundamental tools: the adjoint representation and the logarithmic (or total) differential of the structural group of the bundle. Including them (in an appropriate, axiomatic way) in the structure of a sheaf of groups, we are able to define connections on principal sheaves with structure sheaf the said (Lie) sheaf of groups. In contrast to this, other widely used methods, based on connection mappings (defined over tangent spaces), global connection forms, horizontal subspaces and the like, are nonsensical in our framework, so they cannot serve this approach.

As a moral, we could say that a classical notion fits well here, if it is susceptible of a convenient localization.

We believe that the preceding summary of contents along with the discussion on ADG helps to clarify the framework and scope of this book, succinctly presented in the beginning of the preface.

Readership. The book is addressed to researchers and graduate students with an interest in differential geometry, wishing to become acquainted with the theory of connections on principal sheaves within ADG, and/or to look at the classical theory from a different (algebraic) point of view. Combined with [62, 67], it can be used as the platform for potential applications to theoretical physics such as gauge theories, (pre)quantization, gravitation, and quantum theory.

Although an ample summary of the theory of sheaves is included, some familiarity with its techniques would be welcome. Similarly, a working knowledge of the fundamentals of the geometry of smooth manifolds and bundles would help the reader to better bridge CDG with the present approach.

Since the book addresses a wide audience, with different backgrounds and interests, particular care is given to the details of the exposition. For the benefit of the novice, all the proofs are meticulously laid out, using as elementary as possible methods, although this may sometimes seem tedious to the expert.

Acknowledgments. It is my pleasant duty to mention here the persons who, in their way, have helped me to realize this project.

First of all, I have to express my sincere indebtedness to A. Mallios whose ideas, through his seminars at the University of Athens, along with his book and our many discussions, were a steady source of inspiration. His encouragement and enthusiasm gave me the impetus to transform a bunch of notes and scattered results into a systematic treatment.

My wife and colleague M. Papatriantafillou had a significant contribution. I owe to her almost the entire material of Chapter 2. Her critical reading of the first draft resulted in many improvements and corrections of errors (not only typing ones). My ex student and colleague G. Galanis also pointed out certain imprecisions and many typos. C. Hopkins-Panagou helped to polish the final draft. I thank all of them for their painstaking work. However, I am inevitably responsible for the remaining mistakes, seeking solace in the aphorism that "no one ever wrote five pages of mathematics without mistakes".

My sincere and heartfelt thanks also go to: Professor M. Hazewinkel, the Editor of *Mathematics and Its Applications* series, for including the present work in it; the referees who read (parts of) the preliminary draft, for their expert opinion and suggestions; the administrative and technical staff of Kluwer Academic Publishers and Springer-Verlag, in particular M. Vlot of the Mathematics and Information Unit, for their help, from the time of submission till the final stage of production.

Special thanks are reserved for the creators of TEX and LATEX as well as to the numerous TEXnicians and TEXperts who made the writing of mathematical books and papers a real joy. Among the latter, P. Taylor is mentioned for granting permission to use (for the commercial production of an academic publication) his program, with which the diagrams of the book have been drawn. Typing my manuscript turned out to be a pleasant activity, especially after seeing my notes transformed into nicely typeset pages.

Finally, I would like to acknowledge the financial support I received from the Special Research Account of the University of Athens (grant 70/4/3410) in regard to the research project that led to this book.

Athens, December 2004 E. V.

Chapter 1

Sheaves and all that

... the central message of Quantum Field Theory [is] that all information characterizing the theory is strictly local ...

R. Haag [39, p. 326]

Sheaves were introduced ... by J. Leray and have had a profound effect on several mathematical disciplines. Their major virtue is that they unify and give a mechanism for dealing with many problems concerned with passage from local information to global information.

R. O. Wells, Jr. [142, p. 37]

In this preliminary chapter we gather together the very basic notions of the theory of sheaves and their cohomology. Rather than writing a complete introduction to the subject, our intention is to fix the notations and terminology applied in the main part of the book, and to guide the reader to the sources that seem to be most appropriate for the purpose of this work.

We are primarily concerned with the relationship between sheaves and (complete) presheaves, their morphisms, and the rudiments of the Čech cohomology. The latter is almost exclusively used throughout and is developed in the general case of $\mathcal{A}$-modules, where $\mathcal{A}$ is a sheaf of algebras. Other useful topics, such as sums and products, the pull-back and the push-out of sheaves by continuous maps, and certain sheaves of functions are also treated.

For later reference, as well as for the reader's convenience, each section is divided into short subsections. Formal statements and proofs have generally been omitted to allow the reader to proceed in a leisurely manner.

For complete details we mainly refer to Bredon [16], Dowker [23], Godement [33], Grothendieck [36], Gunning [37], Hirzebruch [44], Swan [119], Tennison [121], Vaisman [123], and Mallios [62, Vol. I], covering more or less standard topics. The last book contains a detailed approach to $\mathcal{A}$-modules and their cohomology. Additional references, for more specific topics, will be given occasionally.

1.1. Sheaves

Here we deal with the basic definitions and properties of sheaves, their morphisms and sections.

1.1.1. Sheaves and morphisms

Let $X \equiv (X, \mathfrak{T}_X)$ be a topological space. A ***sheaf*** (of sets) over X is determined by a triplet

$$\mathcal{S} \equiv (\mathcal{S}, \pi, X),$$

where $\mathcal{S}$ is topological space and $\pi : \mathcal{S} \to X$ a *local homeomorphism*. The previous definition implies that π is a *continuous, open* map. However, it is *not* necessarily a surjection.

We call $\mathcal{S}$ the ***total*** or ***sheaf space***, X the ***base***, and π the ***projection*** of the sheaf. If there is no danger of confusion, a sheaf as above is simply denoted by its (total) space $\mathcal{S}$.

For any $x \in \operatorname{im} \pi$, the set

$$\mathcal{S}_x := \pi^{-1}(x) \equiv \pi^{-1}(\{x\})$$

is called the ***stalk*** of $\mathcal{S}$ at (or, over) x. It is a direct consequence of the definitions that each stalk $\mathcal{S}_x$ is a *discrete subspace* of $\mathcal{S}$, with respect to the relative topology. Some authors, influenced by the general theory of fiber spaces, use the term ***fiber*** in place of stalk.

For any open $U \subseteq X$, the ***restriction*** $\mathcal{S}|_U = \pi^{-1}(U)$ of $\mathcal{S}$ to U is a sheaf over U, with projection the restriction of π to $\mathcal{S}|_U$.

A ***subsheaf*** of $\mathcal{S}$ is an *open* subset $\mathcal{S}'$ of $\mathcal{S}$. Then the triplet $(\mathcal{S}', \pi|_{\mathcal{S}'}, X)$ is a sheaf in the usual sense, whence the terminology.

Examples of some sheaves, frequently used throughout, will be given in Section 1.3.

If $(\mathcal{S}, \pi, X)$ and $(\mathcal{T}, \pi', X)$ are two sheaves of sets (over the same base X), a ***morphism*** of $\mathcal{S}$ into $\mathcal{T}$, or a ***sheaf morphism***, is a *continuous* map $\phi : \mathcal{S} \to \mathcal{T}$ such that $\pi' \circ \phi = \pi$ (see the next diagram).

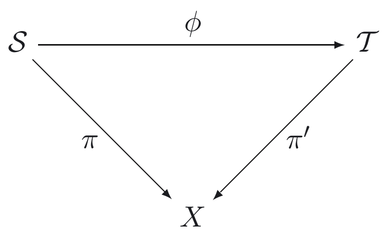


Diagram 1.1

A morphism of sheaves is necessarily a *local homeomorphism*, thus we may think of $(\mathcal{S}, \phi, \mathcal{T})$ as a sheaf. Also, ϕ is ***stalk preserving***; that is, $\phi(\mathcal{S}_x) \subseteq \mathcal{T}_x$, for every $x \in X$. Hence, one can define the restrictions of ϕ to the stalks of $\mathcal{S}$

$$\phi_x := \phi\big|_{\mathcal{S}_x} : \mathcal{S}_x \longrightarrow \mathcal{T}_x.$$

The set of morphisms between two sheaves $\mathcal{S}$ and $\mathcal{T}$ is denoted by

$$\mathrm{Hom}(\mathcal{S}, \mathcal{T}).$$

A morphism ϕ is ***injective*** (resp. ***surjective***) if all the restrictions ϕ_x share the same property. Moreover, ϕ is said to be an ***isomorphism*** if it has an inverse that is also a sheaf morphism. It is immediate that a morphism ϕ is an isomorphism if and only if ϕ_x is an isomorphism, for every $x \in X$.

It is obvious that the composition of morphisms (over the same base) is also a sheaf morphism, thus we obtain the ***category of sheaves***, over the topological space X, denoted by

$$\mathcal{S}h_X.$$

1.1.2. Sheaves with algebraic structures

Before proceeding further, we first define the fiber product of sheaves. More precisely, if $(\mathcal{S}, \pi, X)$ and $(\mathcal{T}, \pi', X)$ are two sheaves of sets, then their ***fiber***

product over X is the sheaf

$$(\mathcal{S} \times_X \mathcal{T}, \pi_X, X),$$

where the space

$$\mathcal{S} \times_X \mathcal{T} := \{(s,t) \in \mathcal{S} \times \mathcal{T} : \pi(s) = \pi'(t)\}$$

is endowed with the relative topology (as a subspace of $\mathcal{S} \times \mathcal{T}$), and

$$\pi_X(s,t) := \pi(s) = \pi'(t), \qquad (s,t) \in \mathcal{S} \times_X \mathcal{T}.$$

An alternative notation for the fiber product of two sheaves $\mathcal{S}$ and $\mathcal{T}$ is $\mathcal{S} \circ \mathcal{T}$, but it will not be applied here. The fiber product will be rediscussed in Subsection 1.3.2.

Very frequently, sheaves whose stalks have an algebraic structure will be considered. In this case, the corresponding operation(s), defined stalk-wise by some map(s) of the general form

$$\star : \mathcal{S} \times_X \mathcal{S} \ni (s_1, s_2) \longmapsto s_1 \star s_2 \in \mathcal{S},$$

is (are) *assumed to be continuous.*

For example, a ***sheaf of groups*** is a sheaf $(\mathcal{G}, \pi, X)$, whose stalks $\mathcal{G}_x$ are groups, and the ***multiplication***

$$\gamma : \mathcal{G} \times_X \mathcal{G} \ni (g,h) \longmapsto \gamma(g,h) = g \cdot h \in \mathcal{G},$$

as well as the ***inversion***

$$\alpha : \mathcal{G} \ni g \longmapsto \alpha(g) = g^{-1} \in \mathcal{G},$$

are continuous maps.

Likewise, we define ***sheaves of*** $\mathbb{K}$***-vector spaces*** ($\mathbb{K} = \mathbb{R}, \mathbb{C}$), ***sheaves of rings***, and ***sheaves of algebras***.

Another, commonly used, structure is that of an $\mathcal{A}$***-module***, where $(\mathcal{A}, \pi, X)$ is a sheaf of algebras. This is a sheaf of *abelian groups*, say $(\mathcal{E}, \pi, X)$, whose stalks $\mathcal{E}_x$ are, in addition, $\mathcal{A}_x$-modules (in the usual sense) so that the operation (: *scalar multiplication*)

$$\mathcal{A} \times_X \mathcal{E} \ni (a,u) \longmapsto a \cdot u \equiv au \in \mathcal{E}$$

is also continuous.

If $\mathcal{E}$ is an $\mathcal{A}$-module, a subsheaf $\mathcal{E}'$ such that the set $\mathcal{E}'_x := \mathcal{E}' \cap \mathcal{E}_x$ is an $\mathcal{A}_x$-submodule of $\mathcal{E}_x$, for every $x \in X$, is called an ***$\mathcal{A}$-submodule*** of $\mathcal{E}$. Clearly, $\mathcal{E}'$ is an $\mathcal{A}$-module itself.

A ***morphism of sheaves with an algebraic structure*** will be a morphism of sheaves ϕ (in the sense of Subsection 1.1.1) preserving stalk-wise the given structure; in other words, the restrictions ϕ_x to the stalks are morphisms of the given structure. For convenience, a morphism of $\mathcal{A}$-modules will be called ***$\mathcal{A}$-morphism***.

Note. In most of the previous sheaves we have denoted their projections by the same symbol π. We follow this practice whenever there is no danger of confusion, otherwise a distinctive index will be appropriately inserted.

1.1.3. The sections of a sheaf

Let $(\mathcal{S}, \pi, X)$ be a sheaf and U a subset of X. A ***section of $\mathcal{S}$ over U*** is a *continuous* map $s : U \to \mathcal{S}$ such that $\pi \circ s = id_U$. In a more pictorial way, the following diagram is commutative.

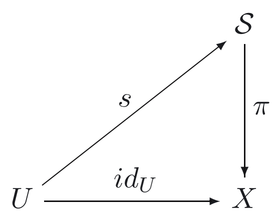


DIAGRAM 1.2

Though the definition has a meaning for any (not necessarily continuous) sections, our main concern will be that of continuous ones over an *open* domain.

It is obvious that s is an *injective* map and $s(x) \in \mathcal{S}_x$, for every $x \in U$. Moreover, if U is an open subset of X, then $s(U)$ is an open subset of $\mathcal{S}$ and $s : U \to s(U)$ is a *homeomorphism*. Thus, in particular, the sections of $\mathcal{S}$ are *open maps*, and

(**BTS**) the sets $s(U)$, for all U running the topology $\mathfrak{T}_X$ of X and all $s \in \mathcal{S}(U)$, determine a *basis for the topology* of $\mathcal{S}$.

We denote by

$$\mathcal{S}(U) \equiv \Gamma(U, \mathcal{S}) \tag{1.1.1}$$

the set of *continuous sections* of $\mathcal{S}$ over U. If $U \subsetneqq X$, then the elements of $\mathcal{S}(U)$ are also called ***local sections***, whereas the set

$$\mathcal{S}(X) \equiv \Gamma(X, \mathcal{S}) \tag{1.1.1$'$}$$

consists of the (continuous) ***global sections*** of $\mathcal{S}$.

The local structure of a sheaf $\mathcal{S}$ ensures the existence of (local) sections with given initial conditions. To be more precise, for any $p \in \mathcal{S}$ with $\pi(p) = x$, there is an open neighborhood U of x and $s \in \mathcal{S}(U)$ such that $s(x) = p$. If $t \in \mathcal{S}(U)$ is another section with $t(x) = p$, then there exists an open neighborhood $W \subseteq U$ of x such that $s|_W = t|_W$.

A morphism of sheaves $\phi : \mathcal{S} \to \mathcal{T}$ induces the morphisms of sections

$$\bar{\phi}_U : \mathcal{S}(U) \longrightarrow \mathcal{T}(U), \tag{1.1.2}$$

for every $U \in \mathfrak{T}_X$, given by

$$\bar{\phi}_U(s) := \phi \circ s. \tag{1.1.2$'$}$$

In most cases, as is the custom, the ***induced morphism*** $\bar{\phi}_U$ is simply written as ϕ. Thus,

(1.1.3) the expression $\phi(s)$ may represent either the value of the morphism ϕ at a *point* $s \in \mathcal{S}$, or the value of the induced morphism $\bar{\phi}_U$ at a *section* $s \in \mathcal{S}(U)$.

The exact meaning of such an expression will be clarified by the context or by an explicit mention of the domain of the variable s at hand.

It is evident that, when dealing with sheaves with an algebraic structure, the corresponding sets of sections inherit the same structure. For instance, if $\mathcal{G}$ is a sheaf of groups, then $\mathcal{G}(U) \equiv \Gamma(U, \mathcal{G})$ is a group under the operation defined, point-wise, by

$$(s \cdot t)(x) := s(x) \cdot t(x); \qquad x \in U,$$

for every $s, t \in \mathcal{G}(U)$. Moreover, for every $s \in \mathcal{G}(U)$, one defines the ***inverse section*** $s^{-1} \in \mathcal{G}(U)$ with

$$s^{-1}(x) := (s(x))^{-1}, \qquad x \in U. \tag{1.1.4}$$

In particular, a sheaf of groups admits a distinguished continuous global section $\mathbf{1} \in \mathcal{G}(X)$, called the ***unit*** or ***identity section***, defined by

$$\mathbf{1} : X \longrightarrow \mathcal{G} : x \mapsto \mathbf{1}(x) := e_x, \tag{1.1.5}$$

if e_x is the neutral (unit) element of $\mathcal{G}_x$.

Similarly, given a sheaf of unital algebras $(\mathcal{A}, \pi, X)$, we define the ***zero section*** $\mathbf{0} \in \mathcal{A}(X)$

$$\mathbf{0} : X \longrightarrow \mathcal{A} : x \mapsto \mathbf{0}(x) := 0_x, \tag{1.1.6}$$

as well as the ***unit section*** $1 \in \mathcal{A}(X)$

$$1 : X \longrightarrow \mathcal{A} : x \mapsto 1(x) := 1_x, \tag{1.1.7}$$

where 0_x and 1_x are, respectively, the zero and unit of $\mathcal{A}_x$.

Any $\mathcal{A}$-module $\mathcal{E}$ has also a zero section $\mathbf{0} \in \mathcal{E}(X)$.

1.2. Presheaves

We give the basic definitions and properties of presheaves, along with a detailed description of their relationship with sheaves.

1.2.1. Presheaves and morphisms

Let $X \equiv (X, \mathfrak{T}_X)$ be a topological space. A ***presheaf of sets*** consists of two kinds of data:

– a family of sets $S(U)$, for all $U \in \mathfrak{T}_X$, and

– a family of maps

$$\rho_V^U : S(U) \to S(V), \tag{1.2.1}$$

associated to every pair (U, V) of open subset of X with $U \supseteq V$, and such that the conditions

$$\rho_U^U = id_{S(U)}, \tag{1.2.1a}$$

$$\rho_W^V \circ \rho_V^U = \rho_W^U, \tag{1.2.1b}$$

hold for every $U, V, W \in \mathfrak{T}_X$ with $U \supseteq V \supseteq W$.

A presheaf, as above, is denoted by

$$S \equiv \left(S(U), \rho_V^U\right).$$

Sometimes, the symbol ρ_{VU} is used instead of ρ_V^U, in order to give (1.2.1b) a more symmetrical form. The maps ρ_V^U are called ***restriction maps***, although, in the real sense, there is no restriction unless the sets $S(U)$ themselves consist of maps. This is the case, for instance, of the sections of a sheaf (see below). In the same spirit, the elements of the set $S(U)$ are often called ***sections*** (over U) of the presheaf.

Condition (1.2.1b) is illustrated in the commutative diagram:

$$\begin{array}{ccc} S(U) & \xrightarrow{\quad \rho^U_V \quad} & S(V) \\ & \rho^U_W \searrow \qquad \swarrow \rho^V_W & \\ & S(W) & \end{array}$$

DIAGRAM 1.3

Schematically, we may think of S as a double correspondence

$$U \longmapsto S(U),$$
$$\big(i^V_U : V \hookrightarrow U\big) \longmapsto \big(\rho^U_V : S(U) \to S(V)\big),$$

where i^V_U is the natural inclusion. Hence, categorically speaking, S is a contravariant functor from the category of open subsets of X and inclusions to the category of sets. We customarily refer to S by the first correspondence.

By its very definition, a presheaf S determines an *inductive* or *direct* system (of sets, groups, modules etc.), with index set the topology $\mathfrak{T}_X$ directed by the relation

$$U \preceq V \quad \Longleftrightarrow \quad V \subseteq U,$$

for every $U, V \in \mathfrak{T}_X$.

Examples of certain presheaves, required later on, are given in Section 1.3 below.

A *weaker definition* of a presheaf allows U to run in a *basis* $\mathcal{B}$ for the topology of X (see Gunning [37, p. 16], Eisenbud-Harris [27, p. 16]). Such a presheaf extends to an ordinary one in the following way: Assume that $\big(S(V), \rho^V_W\big)$ is a presheaf over a basis $\mathcal{B}$. Then, for any open $U \subseteq X$, we set

$$S(U) := \varprojlim_{\mathcal{B} \ni V \subseteq U} S(V).$$

(For the relevant definitions and properties of projective (or inverse) limits see, e.g., Bourbaki [12], Eilenberg-Steenrod [25].) The projection of the limit to each $S(V)$ is taken, by definition, as the restriction $\rho^U_V : S(U) \to S(V)$.

The restriction $\rho^U_{U'}$, for arbitrary open U, U' with $U' \subseteq U$, is given by the projective limit of the morphism of projective systems

$$\left\{\rho^V_{U' \cap V} : S(V) \to S(U' \cap V) \,\middle|\, V \in \mathcal{B}\right\},$$

after the observation that

$$S(U') := \varprojlim_{\mathcal{B} \ni V \subseteq U'} S(V) = \varprojlim_{\mathcal{B} \ni V \subseteq U} S(U' \cap V).$$

Such an extension is unique, up to isomorphism, as a result of the universal property of the projective limit.

If the sets $S(U)$ have an algebraic structure, then the restriction maps ρ^U_V are *assumed to be morphisms* with respect to this structure. Thus, e.g., a presheaf of groups is a contravariant functor from the category of open subsets of X and inclusions to the category of groups. In particular, if S is a sheaf of abelian groups, we agree that $S(\emptyset) = 0 \equiv \{0\}$ (: the trivial group).

Assume now that we are given two presheaves (of sets)

$$S \equiv \left(S(U), \rho^U_V\right) \qquad \text{and} \qquad T \equiv \left(T(U), \tau^U_V\right),$$

over the same topological space X. A ***morphism*** of S into T, or a ***presheaf morphism***, $\phi : S \to T$ is a family of maps $\phi \equiv (\phi_U)$, with U running in $\mathfrak{T}_X$, where the maps

$$\phi_U : S(U) \longrightarrow T(U)$$

satisfy equality

$$\tau^U_V \circ \phi_U = \phi_V \circ \rho^U_V, \tag{1.2.2}$$

for every open U and V in X, with $V \subseteq U$. In other words, the following diagram is commutative.

$$\begin{array}{ccc} S(U) & \xrightarrow{\phi_U} & T(U) \\ {\scriptstyle \rho^U_V}\big\downarrow & & \big\downarrow{\scriptstyle \tau^U_V} \\ S(V) & \xrightarrow{\phi_V} & T(V) \end{array}$$

Diagram 1.4

The set of morphisms between the presheaves S and T is denoted by

$$\mathrm{Hom}(S, T).$$

A morphism of presheaves is an ***injection***, ***surjection*** or ***isomorphism***, if the maps $\phi_U : S(U) \to T(U)$ have the corresponding property, for all $U \in \mathfrak{T}_X$. Clearly, the composition of two presheaf morphisms $\phi : S \to T$ and $\psi : T \to R$ is defined to be

$$\phi \circ \psi \equiv (\phi_U \circ \psi_U), \qquad U \in \mathfrak{T}_X. \tag{1.2.3}$$

Thus we obtain the ***category of presheaves*** (over X), denoted by

$$\mathcal{PS}h_X.$$

Given a sheaf $\mathcal{S}$, its sections $\mathcal{S}(U) \equiv \Gamma(U, \mathcal{S})$, for all open $U \subseteq X$ (see Subsection 1.1.3) , together with the natural restrictions

$$\rho_V^U : \mathcal{S}(U) \longrightarrow \mathcal{S}(V) : s \mapsto s|_V; \qquad V \subseteq U,$$

determine the presheaf

$$\Gamma(\mathcal{S}) := \big(\Gamma(U, \mathcal{S}) \equiv \mathcal{S}(U), \rho_V^U\big), \tag{1.2.4}$$

called the ***presheaf of sections*** of the sheaf $\mathcal{S}$.

The previous process determines the so-called ***section functor***

$$\Gamma : \mathcal{S}h_X \longrightarrow \mathcal{PS}h_X,$$

between the categories of sheaves and presheaves over X. In fact, to any object $\mathcal{S} \in \mathcal{S}h_X$, Γ assigns the presheaf $\Gamma(\mathcal{S})$, whereas to any sheaf morphism $\phi : \mathcal{S} \to \mathcal{T}$, Γ assigns the morphism of presheaves

$$\Gamma(\phi) \equiv \bar{\phi} : \Gamma(\mathcal{S}) \longrightarrow \Gamma(\mathcal{T}), \tag{1.2.4a}$$

determined by the family of induced morphisms (see (1.1.2) and (1.1.2′))

$$\bar{\phi}_U : \mathcal{S}(U) \longrightarrow \mathcal{T}(U), \tag{1.2.4b}$$

for all $U \in \mathfrak{T}_X$.

In particular, as we shall see in Subsection 1.2.3, the image of Γ is contained in the subcategory of *complete* presheaves over X.

1.2.2. Sheaves generated by presheaves

We shall show that a presheaf generates, in a canonical way, a sheaf. In point of fact, many sheaves arise from presheaves in this way.

Let $S \equiv \big(S(U), \rho^U_V\big)$ be a presheaf (of sets) over the topological space X and let $x \in X$ be any point. On the *disjoint union*

$$\bigsqcup_{U \in \mathcal{N}(x)} S(U),$$

where $\mathcal{N}(x)$ is the *filter of open neighborhoods* of x, we define the following equivalence relation: if $s \in S(U)$ and $t \in S(V)$, then

$$s \sim t \iff \exists \quad W \in \mathcal{N}(x) : \ \rho^U_W(s) = \rho^V_W(t). \tag{1.2.5}$$

We denote by $\mathcal{S}_x$ the resulting quotient space, which is precisely the *inductive* or *direct limit* of all $S(U)$, with U running in $\mathcal{N}(x)$; that is,

$$\mathcal{S}_x := \bigsqcup_{x \in U} S(U) \Big/ \sim \ \equiv \ \varinjlim_{U \in \mathcal{N}(x)} S(U).$$

The equivalence class of an element $s \in S(U)$ is denoted by $[s]_x \in \mathcal{S}_x$. We call $[s]_x$ the ***germ of** s **at** x*.

For each open $U \in \mathfrak{T}_X$ and any $x \in U$, we have the corresponding ***canonical map*** (into germs)

$$\rho_{U,x} : S(U) \longrightarrow \mathcal{S}_x : s \mapsto [s]_x. \tag{1.2.6}$$

The relationship between the restriction maps and the canonical maps into germs is pictured in the following commutative diagram:

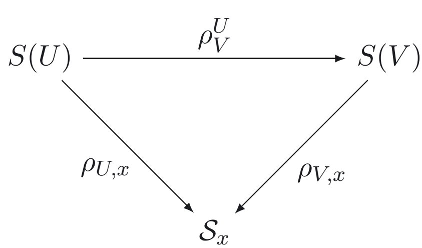


Diagram 1.5

Varying x in the entire X, we define the set

$$\mathcal{S} := \bigcup_{x \in X} \mathcal{S}_x$$

(actually a disjoint union too) and the obvious projection $\pi : \mathcal{S} \to X$ with $\pi(\mathcal{S}_x) := x$, for every $x \in X$. We topologize $\mathcal{S}$ by taking as a *basis* for the topology all the sets of the form

$$\{\rho_{U,x}(s) \mid x \in U\} \subseteq \mathcal{S},$$

for all $s \in S(U)$ and all $U \in \mathfrak{T}_X$. We show that the triplet $\mathcal{S} \equiv (\mathcal{S}, \pi, X)$ is a sheaf, called the ***sheaf generated by*** (or ***associated with***) ***the presheaf*** S. It is also denoted by

$$\mathcal{S} = \mathbf{S}(S) = \mathbf{S}\big(U \longmapsto S(U)\big). \tag{1.2.7}$$

Let $\mathcal{S} = \mathbf{S}(S)$. Then, for every open $U \subseteq X$, there is a ***canonical map (morphism)*** of sections

$$\rho_U : S(U) \longrightarrow \mathcal{S}(U), \tag{1.2.8}$$

defined as follows: For any $s \in S(U)$, the section $\rho_U(s)$ is given by

$$(\rho_U(s))\,(x) := [s]_x \in \mathcal{S}_x, \qquad x \in U. \tag{1.2.8$'$}$$

Another useful notation is

$$\tilde{s} := \rho_U(s). \tag{1.2.9}$$

Thus, taking into account (1.2.6), (1.2.8′) and (1.2.9), it is seen that

$$\tilde{s}(x) = (\rho_U(s))\,(x) = [s]_x = \rho_{U,x}(s), \tag{1.2.10}$$

for every $s \in S(U)$ and $x \in U$. Therefore,

(**BTGS**) the basis for the topology of the sheaf $\mathcal{S}$, generated by the presheaf S, consists of the subsets $\tilde{s}(U)$ of $\mathcal{S}$, obtained from all $s \in S(U)$ and all open $U \subseteq X$.

The symbol $\mathbf{S}$ in (1.2.7) indicates a process known as the ***sheafification***, by which, to every presheaf, we associate a sheaf. In fact, $\mathbf{S}$ determines the ***sheafification functor***

$$\mathbf{S} : \mathcal{PS}h_X \longrightarrow \mathcal{S}h_X$$

between the categories of presheaves (of sets) and sheaves (of sets) over the topological space X. To prove the previous claim one has to explain how the sheafification process generates a morphism of sheaves

$$\widetilde{\phi} \equiv \mathbf{S}(\phi) : \mathcal{S} \equiv \mathbf{S}(S) \longrightarrow \mathcal{T} \equiv \mathbf{S}(T) \tag{1.2.11}$$

from a given morphism of presheaves

$$\phi \equiv (\phi_U) : S \equiv \big(S(U), \rho_V^U\big) \longrightarrow T \equiv \big(T(U), \tau_V^U\big).$$

To this end, one sets

$$\widetilde{\phi}|_{\mathcal{S}_x} \equiv \mathbf{S}(\phi)\big|_{\mathcal{S}_x} := \phi_x; \qquad x \in X, \tag{1.2.12}$$

where the map

$$\phi_x := \varinjlim_{U \in \mathcal{N}(x)} (\phi_U) : \varinjlim_{U \in \mathcal{N}(x)} S(U) \equiv \mathcal{S}_x \longrightarrow \mathcal{T}_x \equiv \varinjlim_{U \in \mathcal{N}(x)} T(U)$$

is defined, in turn, by

$$\phi_x([s]_x) := [\phi_U(s)]_x = \widetilde{\phi_U(s)}(x), \tag{1.2.13}$$

for every $s \in S(U)$.

Equalities (1.2.10) and (1.2.13) imply the commutativity of the diagram

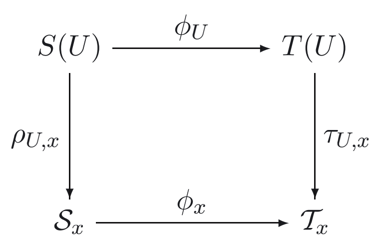


Diagram 1.6

which, along with Diagram 1.5, proves that (1.2.13) is well defined. Applying the topology described in (**BTGS**), it is verified that $\mathbf{S}(\phi)$ is indeed a continuous map (commuting with the projections).

Combining (1.2.13) with (1.2.10), it follows that

$$\widetilde{\phi}(\tilde{s}(x)) = \phi_x([s]_x) = \widetilde{\phi_U(s)}(x), \tag{1.2.13$'$}$$

for every $s \in S(U)$ and $x \in U$.

On the other hand, if

$$\overline{(\widetilde{\phi})}_U \equiv \overline{\mathbf{S}(\phi)}_U \equiv \overline{\mathbf{S}((\phi_U))}, \qquad U \in \mathfrak{T}_X$$

are the morphisms of sections induced by $\widetilde{\phi} \equiv \mathbf{S}(\phi)$ (see (1.1.2)), we also obtain the next commutative diagram.

$$\begin{array}{ccc} S(U) & \xrightarrow{\ \phi_U\ } & T(U) \\ {\scriptstyle \rho_U}\downarrow & & \downarrow{\scriptstyle \tau_U} \\ \mathcal{S}(U) & \xrightarrow{\ \overline{(\widetilde{\phi})}_U\ } & \mathcal{T}(U) \end{array}$$

DIAGRAM 1.7

If we start with a presheaf S, endowed with an algebraic structure determined by an operation, say $\star$, then the same structure passes to (the stalks of) the sheaf $\mathcal{S} = \mathbf{S}(S)$. Indeed, for any $a, b \in \mathcal{S}_x$, we let

$$a \star b := \rho_{W,x}\left(\rho_W^U(s) \star \rho_W^V(t)\right),$$

if $s \in S(U)$, $t \in T(V)$ are presheaf sections representing a and b, respectively, in $\mathcal{S}_x$; that is,

$$a = [s]_x \equiv \tilde{s}(x) \quad \text{and} \quad b = [t]_x \equiv \tilde{t}(x),$$

where U, V, $W \in \mathcal{N}(x)$, with $W \subseteq U \cap V$. It is not difficult to show that $a \star b$ is well defined and has the properties of the original operation on the presheaf. In this case, the canonical maps ρ_U and $\rho_{U,x}$ are morphisms with respect to $\star$.

Note. Let $S \equiv (S(U), \rho_V^U)$ be a presheaf, with U running in the topology $\mathfrak{T}_X$, and let $\mathcal{S}$ be the sheafification of S. If U is restricted to run in a basis $\mathcal{B}$ for $\mathfrak{T}_X$, then we can repeat verbatim the sheafification process and obtain a sheaf, say $\widetilde{\mathcal{S}}$. The sheaves $\mathcal{S}$ and $\widetilde{\mathcal{S}}$ are identical set-theoretically and topologically. Indeed, it is an easy exercise to verify that

$$\mathcal{S}_x = \varinjlim_{U \in \mathcal{N}(x)} S(U) = \varinjlim_{x \in U \subseteq \mathcal{B}} S(U) = \widetilde{\mathcal{S}}_x,$$

thus $\mathcal{S} = \widetilde{\mathcal{S}}$, as sets. Moreover, their topologies coincide since the corresponding bases (see (BTGS), p. 12) are equivalent.

Similarly, assume that $T \equiv (T(V), \rho^V_W)$ is a presheaf with $V \in \mathcal{B}$ and let $\overline{T}$ be its extension to a presheaf over $\mathfrak{T}_X$ (see page 8). Since T is the restriction of $\overline{T}$ to $\mathcal{B}$, the previous arguments imply that the sheaves generated by T and $\overline{T}$ coincide, i.e., $\mathbf{S}(T) = \mathbf{S}(\overline{T})$.

1.2.3. Complete presheaves

In the last two subsections we defined the functors $\Gamma : \mathcal{S}h_X \to \mathcal{PS}h_X$ and $\mathbf{S} : \mathcal{PS}h_X \to \mathcal{S}h_X$. If we start with a sheaf $\mathcal{S}$ and apply successively Γ and $\mathbf{S}$, we obtain a canonical sheaf isomorphism

$$\mathbf{S}\,(\Gamma(\mathcal{S})) \xrightarrow{\ \simeq\ } \mathcal{S} \tag{1.2.14}$$

in the following manner: For any $u \in \mathbf{S}\,(\Gamma(\mathcal{S}))_x$, there is a section $s \in \mathcal{S}(U)$, for some $U \in \mathcal{N}(x)$, such that $u = [s]_x$. The desired isomorphism is realized by the assignment

$$\mathbf{S}\,(\Gamma(\mathcal{S}))_x \ni u = [s]_x \longmapsto s(x) \in \mathcal{S}_x.$$

It is a well defined map, independent of the choice of s. Since the elements of $\mathbf{S}\,(\Gamma(\mathcal{S}))$ are the germs of sections of $\mathcal{S}$ (see Subsection 1.2.2), the former sheaf is called the ***sheaf of germs of the sections of*** $\mathcal{S}$. Therefore,

(1.2.15) a sheaf coincides, *up to isomorphism*, with the sheaf of germs of its sections; hence, for any open $U \subseteq X$ and any section $s \in \mathcal{S}(U)$, $[s]_x = s(x)$ within an isomorphism, for every $x \in U$. Accordingly, a morphism $\phi : \mathcal{S} \to \mathcal{T}$ identifies with the sheaf morphism generated by $\{\bar{\phi}_U : \mathcal{S}(U) \to \mathcal{T}(U) \,|\, U \in \mathfrak{T}_X\}$; that is, $\phi \equiv \mathbf{S}\big((\bar{\phi}_U)\big)$.

In particular, if $u \in \mathcal{S}_x$, there is an $s \in \mathcal{S}(U)$, for some $U \in \mathcal{N}(x)$, such that $u = s(x)$. Thus, in virtue of (1.1.2′) and (1.1.3),

$$\phi(u) = \phi(s(x)) = \bar{\phi}_U(s)(x) \equiv \phi(s)(x).$$

In other words, we are led to the following useful conclusion:

(1.2.15′) A morphism of sheaves is completely known once it is known section-wise.

However, if we start with a presheaf S and apply successively $\mathbf{S}$ and Γ, the resulting presheaf $\Gamma\,(\mathbf{S}(S))$ is not necessarily isomorphic with S (for counter-examples see, e.g., Gunning [37, p. 19], Warner [140, p. 168]). We do obtain isomorphic presheaves if we consider the ***category of complete presheaves***, denoted by

$$\mathcal{C}o\mathcal{P}\mathcal{S}h_X.$$

More precisely, a presheaf $S \equiv \big(S(U), \rho_V^U\big)$ is said to be ***complete*** if, for every open $U \subseteq X$ and every open covering $(U_\alpha)_{\alpha\in I}$ of U, the following two conditions are fulfilled:

(**CP. 1**) If $s, t \in S(U)$ are any sections such that

$$\rho_{U_\alpha}^U(s) = \rho_{U_\alpha}^U(t),$$

for all $\alpha \in I$, then $s = t$.

(**CP. 2**) If $s_\alpha \in S(U_\alpha)$, $\alpha \in I$, is a family of sections such that

$$\rho_{U_{\alpha\beta}}^{U_\alpha}(s_\alpha) = \rho_{U_{\alpha\beta}}^{U_\beta}(s_\beta); \qquad U_{\alpha\beta} := U_\alpha \cap U_\beta,$$

for all $\alpha, \beta \in I$, then there exists a section $s \in S(U)$ such that

$$\rho_{U_\alpha}^U(s) = s_\alpha, \qquad \alpha \in I.$$

The element $s \in S(U)$, ensured by (CP. 2), is unique, in virtue of (CP. 1).

Now, based on (1.2.9) and (1.2.10), we can prove that, for all open $U \subseteq X$, the maps

$$\rho_U : S(U) \longrightarrow \Gamma\,(U,\ \mathbf{S}(S)) = \mathbf{S}(S)(U) : s \mapsto \tilde{s}$$

are bijections and determine the desired isomorphism

$$S \xrightarrow{\ \simeq\ } \Gamma\,(\mathbf{S}(S)). \tag{1.2.16}$$

It is worthy to note that the isomorphism (1.2.16) is the *necessary and sufficient* condition for the completeness of the presheaf S. Therefore,

> a presheaf over a topological space X identifies with the presheaf of sections of some sheaf (over X) if and only if the initial presheaf is complete.

Furthermore, if $\{\phi_U : S(U) \to T(U) \,|\, U \in \mathfrak{T}_X\}$ is a morphism between the *complete* presheaves S and T, and $\widetilde{\phi} = \mathbf{S}((\phi_U))$ is the morphism induced by the sheafification of (ϕ_U) (cf. (1.2.11) and the notations before Diagram 1.7), then, *within an isomorphism*,

$$\overline{(\widetilde{\phi})}_U := \overline{\mathbf{S}\,((\phi_U))}_U = \phi_U, \qquad U \in \mathfrak{T}_X. \tag{1.2.17}$$

A typical example of a complete presheaf is provided by the sections of a sheaf. Other examples are obtained from various spaces of functions. For instance, if X is a topological space, then, for any open $U \subseteq X$, we denote by $C^0(U, \mathbb{K})$ the algebra of $\mathbb{K}$-valued *continuous* functions on U. The assignment

$$U \longmapsto C^0(U, \mathbb{K}), \qquad U \in \mathfrak{T}_X$$

is a complete presheaf of algebras generating the ***sheaf of germs of continuous functions on*** X, denoted by $\mathcal{C}_X$. It is a sheaf of algebras such that, in virtue of (1.1.1) and (1.2.16),

$$\mathcal{C}_X(U) \cong C^0(U, \mathbb{K}), \qquad U \in \mathfrak{T}_X.$$

Likewise, if M is a *smooth manifold* and $C^\infty(U, \mathbb{K})$ is the algebra of $\mathbb{K}$-valued *smooth* functions on U, then the complete presheaf of algebras

$$U \longmapsto C^\infty(U, \mathbb{K}), \qquad U \in \mathfrak{T}_M$$

generates the ***sheaf of germs of smooth functions on*** X, denoted by $\mathcal{C}^\infty_M$. It is again a sheaf of algebras such that

$$\mathcal{C}^\infty_M(U) \cong C^\infty(U, \mathbb{K}), \qquad U \in \mathfrak{T}_M.$$

Sheaf theory jargon. Some authors use the terms *sheaf* and ***étalé space*** where we use *complete presheaf* and *sheaf*, respectively.

1.3. Some useful sheaves and presheaves

We collect below certain sheaves and presheaves frequently encountered in this work.

1.3.1. The constant sheaf

Let F be any set equipped with the *discrete* topology. Then the ***constant sheaf*** (over the topological space X) with ***stalks of type*** F is the triplet

$$F_X \equiv (X \times F, p_1, X),$$

where $p_1 : X \times F \to X$ is the projection to the first factor. If there is no danger of confusion, often F_X is simply denoted by F.

The above characterization of the stalks is due to the fact that

$$p_1^{-1}(x) = \{x\} \times F \cong F, \qquad x \in X.$$

We note that the assumption "F is a discrete space" is essential in order to prove that p_1 is a local homeomorphism. In fact, these conditions are *equivalent*.

It is shown that the continuous local sections of F_X are precisely the *locally constant functions* with values in F.

We obtain F_X also as the sheafification of the ***constant presheaf***

$$\boldsymbol{F} = \big(F(U), \rho_V^U\big),$$

where $F(U) := F$, for all open $U \subseteq X$, and $\rho_V^U := id_F$. Since

$$\varinjlim_{U \in \mathcal{N}(x)} F(U) \cong \{x\} \times F,$$

we see that

$$\mathbf{S}(\boldsymbol{F}) = \bigcup_{x \in X} (\{x\} \times F) = X \times F = F_X.$$

We note that $\boldsymbol{F}$ is *not* a complete presheaf; hence, if $f \in F(U) = F$ is a constant (viewed as a constant function ("section")), then the correspondence (see (1.2.9))

$$F(U) \ni f \longmapsto \tilde{f} \in F_X(U)$$

is not a bijection. It is only an *injection*; therefore, in a suitable terminology, $\boldsymbol{F}$ is known as a ***monopresheaf***.

1.3.2. Products and sums of sheaves

In Subsection 1.1.2 we defined the fiber product $\mathcal{S} \times_X \mathcal{T}$ of two sheaves $(\mathcal{S}, \pi, X)$ and $(\mathcal{T}, \pi', X)$. Its stalks are

$$(\mathcal{S} \times_X \mathcal{T})_x = \mathcal{S}_x \times \mathcal{T}_x.$$

It is obvious that, for any open $U \subseteq X$, a section $f \in (\mathcal{S} \times_X \mathcal{T})(U)$ has the form $f = (s, t) \in \mathcal{S}(U) \times \mathcal{T}(U)$; hence,

$$(\mathcal{S} \times_X \mathcal{T})(U) \cong \mathcal{S}(U) \times \mathcal{T}(U). \tag{1.3.1}$$

On the other hand, given two presheaves

$$S \equiv \big(S(U), \rho_V^U\big) \qquad \text{and} \qquad T \equiv \big(T(U), \tau_V^U\big),$$

their ***direct product*** is the presheaf

$$S \times T := \big(S(U) \times T(U), \rho_V^U \times \tau_V^U\big).$$

Thus, if we consider the direct product $\Gamma(\mathcal{S}) \times \Gamma(\mathcal{T})$ of the presheaves of sections of two sheaves $\mathcal{S}$ and $\mathcal{T}$ (see (1.2.4)), we immediately check that

$$\mathcal{S} \times_X \mathcal{T} \cong \mathbf{S}\big(\Gamma(\mathcal{S}) \times \Gamma(\mathcal{T})\big);$$

in other words,

$\mathcal{S} \times_X \mathcal{T}$ is generated by the presheaf $U \longmapsto \mathcal{S}(U) \times \mathcal{T}(U)$.

Similarly, if $\mathcal{S} = \mathbf{S}\big(U \longmapsto S(U)\big)$ and $\mathcal{T} = \mathbf{S}\big(U \longmapsto T(U)\big)$,

$$\mathcal{S} \times_X \mathcal{T} \cong \mathbf{S}\big(U \longmapsto S(U) \times T(U)\big).$$

In particular, if $\mathcal{S}$ and $\mathcal{T}$ are $\mathcal{A}$-*modules* (see the last part of Subsection 1.1.2), then their fiber product is again an $\mathcal{A}$-module. In this case we also use the symbol

$$\mathcal{S} \oplus \mathcal{T} = \mathcal{S} \times_X \mathcal{T},$$

and we call it the ***direct*** or ***Whitney sum of the*** $\mathcal{A}$***-modules*** $\mathcal{S}$ ***and*** $\mathcal{T}$.

We can define the ***fiber product of any family of sheaves*** $(\mathcal{S}_i)_{i\in I}$. This is denoted by

$$\prod_{i\in I} \mathcal{S}_i.$$

It coincides (within an isomorphism) with the sheaf generated by the (complete) presheaf

$$U \longmapsto \prod_{i\in I} \mathcal{S}_i(U); \qquad U \in \mathfrak{T}_X,$$

that is,

$$\prod_{i\in I} \mathcal{S}_i \;\cong\; \mathbf{S}\Big(U \longmapsto \prod_{i\in I} \mathcal{S}_i(U)\Big).$$

However, the equality of stalks

$$\Big(\prod_{i\in I} \mathcal{S}_i\Big)_x = \prod_{i\in I} \mathcal{S}_{i,x}$$

is valid only if I is *finite*, otherwise the stalk of the product is, in general, *injectively* mapped into the product of the stalks.

The ***direct*** or ***Whitney sum of any family*** $(\mathcal{S}_i)$ ***of*** $\mathcal{A}$***-modules***

$$\bigoplus_{i\in I}\mathcal{S}_i$$

is defined analogously. As before, we have that

$$\bigoplus_{i\in I}\mathcal{S}_i \;\cong\; \mathbf{S}\Big(U \longmapsto \bigoplus_{i\in I}\mathcal{S}_i(U)\Big).$$

If $(\mathcal{S}_i)_{i=1,\dots,n}$ is a *finite family* of $\mathcal{A}$-modules, then

$$\prod_{i=1}^{n}\mathcal{S}_i \;\cong\; \bigoplus_{i=1}^{n}\mathcal{S}_i,$$

as $\mathcal{A}$-modules. In particular, if $\mathcal{S}_i = \mathcal{S}$, for every $i = 1,\dots,n$, we also set

$$\mathcal{S}^n = \underbrace{\mathcal{S}\times_X\cdots\times_X\mathcal{S}}_{\text{n-factors}} \cong \underbrace{\mathcal{S}\oplus\cdots\oplus\mathcal{S}}_{\text{n-summands}} = \bigoplus^{n}\mathcal{S}$$

> Occasionally, in order to distinguish the fiber product/direct sum $\mathcal{S}^n$ from the exterior power $\mathcal{S}\bigwedge\cdots\bigwedge\mathcal{S}$ (n times), also denoted by $\mathcal{S}^n$ (see Subsection 1.3.4), the former product/sum is denoted by $\mathcal{S}^{(n)}$.

This will be applied, in particular, in Chapters 9 and 10.

1.3.3. The tensor product of $\mathcal{A}$-modules

This is another very important notion needed in the sequel. More precisely, we assume that $\mathcal{S}$ and $\mathcal{T}$ are two $\mathcal{A}$-modules over the topological space X, where

$\mathcal{A}$ is a *unital commutative associative* $\mathbb{K}$-algebra sheaf.

This kind of algebra sheaf will systematically be used throughout.

We further consider the presheaf determined by the correspondence

$$U \longmapsto \mathcal{S}(U)\otimes_{\mathcal{A}(U)}\mathcal{T}(U); \qquad U\in\mathfrak{T}_X,$$

and the restriction maps

$$\rho^U_V\otimes\tau^U_V : \mathcal{S}(U)\otimes_{\mathcal{A}(U)}\mathcal{T}(U)\longrightarrow\mathcal{S}(V)\otimes_{\mathcal{A}(V)}\mathcal{T}(V),$$

if $V \subseteq U$. Here ρ_V^U, τ_V^U are the natural restrictions of sections of the respective sheaves. The tensor product figuring in both cases is a generalization of the usual tensor product, having now coefficients in the algebra $\mathcal{A}(U)$. Its construction is analogous to the general construction of the tensor product of modules over a commutative ring (see Bourbaki [14]). The previous presheaf, which is *not* necessarily complete, is denoted by

$$\Gamma(\mathcal{S}) \otimes_{\Gamma(\mathcal{A})} \Gamma(\mathcal{T}), \tag{1.3.2}$$

and is called the ***tensor product of the presheaves (of sections)*** $\Gamma(\mathcal{S})$ and $\Gamma(\mathcal{T})$. It is a $\Gamma(\mathcal{A})$-module in the sense that each individual product $\mathcal{S}(U) \otimes_{\mathcal{A}(U)} \mathcal{T}(U)$ is an $\mathcal{A}(U)$-module.

Now, the ***tensor product (over $\mathcal{A}$) of the $\mathcal{A}$-modules $\mathcal{S}$ and $\mathcal{T}$*** is defined to be the sheaf $\mathcal{S} \otimes_{\mathcal{A}} \mathcal{T}$ generated by the presheaf (1.3.2); that is,

$$\mathcal{S} \otimes_{\mathcal{A}} \mathcal{T} := \mathbf{S}\big(\Gamma(\mathcal{S}) \otimes_{\Gamma(\mathcal{A})} \Gamma(\mathcal{T})\big). \tag{1.3.3}$$

This is again an $\mathcal{A}$-module, whose stalks satisfy (see Mallios [62, p. 130])

$$(\mathcal{S} \otimes_{\mathcal{A}} \mathcal{T})_x \cong \mathcal{S}_x \otimes_{\mathcal{A}_x} \mathcal{T}_x. \tag{1.3.4}$$

Moreover, as in the ordinary case of the tensor product over a commutative ring,

$$\mathcal{S} \otimes_{\mathcal{A}} \mathcal{A} \cong \mathcal{A} \otimes_{\mathcal{A}} \mathcal{S} \cong \mathcal{S}. \tag{1.3.5}$$

The usual properties of the *tensor algebra* hold true in the present framework.

Finally, if $\phi : \mathcal{S} \to \mathcal{S}'$ and $\psi : \mathcal{T} \to \mathcal{T}'$ are two morphisms of $\mathcal{A}$-modules, we consider the induced morphisms

$$\Gamma(\phi) : \Gamma(\mathcal{S}) \longrightarrow \Gamma(\mathcal{S}') \quad \text{and} \quad \Gamma(\psi) : \Gamma(\mathcal{T}) \longrightarrow \Gamma(\mathcal{T}').$$

We recall from Subsection 1.2.1 that

$$\Gamma(\phi) \equiv \big\{\bar{\phi}_U : \mathcal{S}(U) \longrightarrow \mathcal{S}'(U)\big\}_{U \in \mathfrak{T}_X},$$

and similarly for $\Gamma(\psi)$. Then the ***tensor product of the $\mathcal{A}$-morphisms*** ϕ and ψ is the $\mathcal{A}$-morphism

$$\phi \otimes \psi : \mathcal{S} \otimes_{\mathcal{A}} \mathcal{T} \longrightarrow \mathcal{S}' \otimes_{\mathcal{A}} \mathcal{T}',$$

defined by

$$\phi \otimes \psi := \mathbf{S}\,(\Gamma(\phi) \otimes \Gamma(\psi)). \tag{1.3.6}$$

More explicitly, $\phi \otimes \psi$ is the morphism generated by the presheaf morphism

$$\{\bar{\phi}_U \otimes \bar{\psi}_U : \mathcal{S}(U) \otimes_{\mathcal{A}(U)} \mathcal{T}(U) \longrightarrow \mathcal{S}'(U) \otimes_{\mathcal{A}(U)} \mathcal{T}'(U)\}_{U \in \mathfrak{T}_X}.$$

Regarding the behavior of $\phi \otimes \psi$ on the stalks, we add that

$$\phi \otimes \psi\big|_{(\mathcal{S} \otimes_{\mathcal{A}} \mathcal{T})_x} \cong \big(\phi\big|_{\mathcal{S}_x}\big) \otimes \big(\psi\big|_{\mathcal{T}_x}\big).$$

Note. To clarify that the tensor product of $\mathcal{A}$-morphisms (1.3.6) is defined with respect to the algebra sheaf $\mathcal{A}$, we should have written $\phi \otimes_{\mathcal{A}} \psi$. However, for the sake of simplicity, this practice is generally not applied.

1.3.4. The exterior power of $\mathcal{A}$-modules

As in the previous section, we consider a unital commutative associative $\mathbb{K}$-algebra sheaf $\mathcal{A}$ and an $\mathcal{A}$-module $\mathcal{S}$. The ***p-th exterior power*** of $\mathcal{S}$ is the $\mathcal{A}$-module

$$\textstyle\bigwedge^p \mathcal{S} := \underbrace{\mathcal{S} \bigwedge_{\mathcal{A}} \cdots \bigwedge_{\mathcal{A}} \mathcal{S}}_{p\text{-factors}}. \tag{1.3.7}$$

obtained by sheafification of the (not necessarily complete) $\Gamma(\mathcal{A})$-presheaf

$$U \longrightarrow \textstyle\bigwedge^p(\mathcal{S}(U)) := \underbrace{\mathcal{S}(U) \bigwedge_{\mathcal{A}(U)} \cdots \bigwedge_{\mathcal{A}(U)} \mathcal{S}(U)}_{p\text{-factors}}. \tag{1.3.8}$$

The exterior product $\mathcal{S}(U) \bigwedge_{\mathcal{A}(U)} \cdots \bigwedge_{\mathcal{A}(U)} \mathcal{S}(U)$ is defined as in the ordinary case of modules over a commutative ring (see Bourbaki [14]).

It is proved that

$$\big(\textstyle\bigwedge^p \mathcal{S}\big)_x \cong \textstyle\bigwedge^p(\mathcal{S}_x); \qquad x \in X, \tag{1.3.9}$$

and

$$\mathcal{S} \textstyle\bigwedge_{\mathcal{A}} \mathcal{A} \cong \mathcal{A} \bigwedge_{\mathcal{A}} \mathcal{S} \cong \mathcal{S}. \tag{1.3.10}$$

The exterior power $\bigwedge^p \mathcal{S}$ is defined for all $p = 0, 1, \ldots$, by agreeing that $\bigwedge^0 \mathcal{S} = \mathcal{A}$ and $\bigwedge^1 \mathcal{S} = \mathcal{S}$. Hence, the usual properties of the exterior algebra hold true also in the present framework.

We shall return to the exterior algebra of a particular $\mathcal{A}$-module in Section 2.5, where a simplified notation of the exterior power will be introduced (see also the final remark of Subsection 1.3.2).

The ***exterior product of two $\mathcal{A}$-morphisms***

$$\phi : \mathcal{S} \to \mathcal{T} \quad \text{and} \quad \psi : \mathcal{S} \to \mathcal{T}$$

is the $\mathcal{A}$-morphism

$$\phi \wedge \psi : \mathcal{S} \textstyle\bigwedge_{\mathcal{A}} \mathcal{S} \longrightarrow \mathcal{T} \textstyle\bigwedge_{\mathcal{A}} \mathcal{T}$$

defined by

$$\phi \wedge \psi := \mathbf{S}\big(\Gamma(\phi) \wedge \Gamma(\psi)\big). \tag{1.3.11}$$

The exterior product of p morphisms is defined analogously. As in the case of the tensor product of two $\mathcal{A}$-morphisms (see the note at the end of Subsection 1.3.3), we write $\phi \wedge \psi$ instead of $\phi \wedge_{\mathcal{A}} \psi$.

1.3.5. Sheaves of morphisms

In Subsection 1.1.1 we defined the set $\mathrm{Hom}(\mathcal{S}, \mathcal{T})$ consisting of the morphisms between two sheaves of sets $\mathcal{S}$ and $\mathcal{T}$. Analogously, we define the set of morphisms with respect to a given structure. So, if $\mathcal{S}$ and $\mathcal{T}$ are $\mathcal{A}$-modules, the set of $\mathcal{A}$-morphisms between them is an $\mathcal{A}(X)$-module, denoted by

$$\mathrm{Hom}_{\mathcal{A}}(\mathcal{S}, \mathcal{T}).$$

Let us clarify the scalar multiplication of the previous $\mathcal{A}(X)$-module structure. For any $\alpha \in \mathcal{A}(X)$ and $f \in \mathrm{Hom}_{\mathcal{A}}(\mathcal{S}, \mathcal{T})$, the morphism αf is defined by

$$(\alpha f)(u) := \alpha(\pi(u)) f(U); \qquad u \in \mathcal{S},$$

if π is the projection of $\mathcal{S}$ (cf. also the notations of page 4).

Since, for every open $U \in X$, the restricted sheaves $\mathcal{S}|_U$ and $\mathcal{T}|_U$ are $\mathcal{A}|_U$-modules, we may define the $\mathcal{A}(U)$-modules

$$\mathrm{Hom}_{\mathcal{A}|_U}(\mathcal{S}|_U, \mathcal{T}|_U), \qquad U \in \mathfrak{T}_X.$$

Thus the correspondence

$$U \longmapsto \mathrm{Hom}_{\mathcal{A}|_U}(\mathcal{S}|_U, \mathcal{T}|_U); \qquad U \in \mathfrak{T}_X,$$

determines a complete presheaf, with restriction maps the usual restrictions of morphisms to subsheaves. The sheaf generated by the previous presheaf is called the ***sheaf of germs of $\mathcal{A}$-morphisms of $\mathcal{S}$ in $\mathcal{T}$***, denoted by

$$\mathcal{H}om_{\mathcal{A}}(\mathcal{S}, \mathcal{T}).$$

It is clearly an $\mathcal{A}$-module (over X) such that, by the completeness of the generating presheaf,

$$\mathcal{H}om_{\mathcal{A}}(\mathcal{S}, \mathcal{T})(U) \cong \mathrm{Hom}_{\mathcal{A}|_U}(\mathcal{S}|_U, \mathcal{T}|_U),$$

for every $U \in \mathfrak{T}_X$. In particular,

$$\mathcal{H}om_{\mathcal{A}}(\mathcal{S}, \mathcal{T})(X) \cong \mathrm{Hom}_{\mathcal{A}}(\mathcal{S}, \mathcal{T}).$$

Given an $\mathcal{A}$-module $\mathcal{S}$, we define its ***dual*** to be the $\mathcal{A}$-module

$$\mathcal{S}^* := \mathcal{H}om_{\mathcal{A}}(\mathcal{S}, \mathcal{A}),$$

while the ***sheaf of germs of $\mathcal{A}$-endomorphisms*** of $\mathcal{S}$ is

$$\mathcal{E}nd_{\mathcal{A}}(\mathcal{S}) := \mathcal{H}om_{\mathcal{A}}(\mathcal{S}, \mathcal{S}).$$

We have the identifications

$$\begin{aligned} \mathcal{S}^*(U) &\cong \mathrm{Hom}_{\mathcal{A}|_U}(\mathcal{S}|_U, \mathcal{A}|_U), \\ \mathcal{E}nd_{\mathcal{A}}(\mathcal{S})(U) &\cong \mathrm{Hom}_{\mathcal{A}|_U}(\mathcal{S}|_U, \mathcal{S}|_U), \end{aligned} \tag{1.3.12}$$

for every $U \in \mathfrak{T}_X$. We note that $\mathcal{E}nd_{\mathcal{A}}(\mathcal{S})$ is an $\mathcal{A}$-algebra sheaf, since each $\mathrm{Hom}_{\mathcal{A}|_U}(\mathcal{S}|_U, \mathcal{S}|_U)$ has a ring multiplication provided by the composition of endomorphisms.

Finally, if $\mathcal{S}$ is a sheaf endowed with *any algebraic structure* (not only that of an $\mathcal{A}$-module, as before) we denote by

$$\mathcal{A}ut(\mathcal{S})$$

the ***sheaf of germs of automorphisms of $\mathcal{S}$***. Sometimes, an appropriate index indicates, if necessary, the particular structure involved. By definition,

$$\mathcal{A}ut(\mathcal{S}) = \mathbf{S}\big(U \longmapsto \mathrm{Aut}(\mathcal{S}|_U)\big),$$

where $\mathrm{Aut}(\mathcal{S}|_U) = \mathrm{Iso}(\mathcal{S}|_U, \mathcal{S}|_U)$ is the set of *automorphisms of $\mathcal{S}|_U$, with respect to the given structure*. It is immediate that $\mathcal{A}ut(\mathcal{S})$ is a *sheaf of groups*. In virtue of (1.3.12), we obtain

$$\mathcal{A}ut(\mathcal{S})(U) \cong \mathrm{Aut}(\mathcal{S}|_U). \tag{1.3.13}$$

1.3.6. Multiple operations

In certain cases we shall encounter sheaves derived from the successive application of various products such as the tensor, exterior, and fiber product. Since it is often convenient to work with presheaves, one may wonder what the right presheaf may be in such a case.

Instead of stating a general, technical result, let us describe a few concrete examples which will clarify our point.

Let $\mathcal{E}, \mathcal{F}, \mathcal{R}$ be $\mathcal{A}$-modules over the topological space $X \equiv (X, \mathfrak{T}_X)$ and consider the $\mathcal{A}$-module $\mathcal{S} = (\mathcal{E} \otimes_{\mathcal{A}} \mathcal{F}) \times_X \mathcal{R}$. Typically, the fiber product is generated by the presheaf $P : U \longmapsto (\mathcal{E} \otimes_{\mathcal{A}} \mathcal{F})(U) \times \mathcal{R}(U)$, $U \in \mathfrak{T}_X$. However, the tensor product figuring in the previous expression is not easy to handle. Though $(\mathcal{E} \otimes_{\mathcal{A}} \mathcal{F})(U) \neq \mathcal{E}(U) \otimes_{\mathcal{A}(U)} \mathcal{F}(U)$, we can consider the presheaf $\tilde{P} : U \longmapsto \big(\mathcal{E}(U) \otimes_{\mathcal{A}(U)} \mathcal{F}(U)\big) \times \mathcal{R}(U)$, generating an $\mathcal{A}$-module $\tilde{\mathcal{S}}$. It is not hard to show that $\mathcal{S} = \tilde{\mathcal{S}}$, within an isomorphism. Therefore, $\mathcal{S}$ may be thought of as generated by the second presheaf, i.e.,

$$\big(\mathcal{E} \otimes_{\mathcal{A}} \mathcal{F}\big) \times_X \mathcal{R} \equiv \mathbf{S}\big(U \longmapsto ((\mathcal{E}(U) \otimes_{\mathcal{A}(U)} \mathcal{F}(U)) \times \mathcal{R}(U)\big).$$

Similarly we show that

$$\big(\mathcal{E} {\textstyle\bigwedge}_{\mathcal{A}} \mathcal{E}\big) \times_X \mathcal{R} \equiv \mathbf{S}\big(U \longmapsto ((\mathcal{E}(U) {\textstyle\bigwedge}_{\mathcal{A}(U)} \mathcal{E}(U)) \times \mathcal{R}(U)\big).$$

The same principle applies, e.g., to $\big(\mathcal{E} \bigwedge_{\mathcal{A}} \mathcal{E}\big) \otimes_{\mathcal{A}} \mathcal{F}$, thus

$$\big(\mathcal{E} {\textstyle\bigwedge}_{\mathcal{A}} \mathcal{E}\big) \otimes_{\mathcal{A}} \mathcal{F} \equiv \mathbf{S}\big(U \longmapsto ((\mathcal{E}(U) {\textstyle\bigwedge}_{\mathcal{A}(U)} \mathcal{E}(U)) \otimes_{\mathcal{A}(U)} \mathcal{F}(U)\big).$$

Other combinations of products are dealt with analogously.

1.4. Change of the base space

We give two important constructions allowing the shifting of a sheaf over a given base to a sheaf over a new base, in a way preserving the algebraic structures.

1.4.1. The pull-back of a sheaf

In this case the change of the base space is accomplished by moving, so to speak, the base space backwards by means of a continuous map.

Let $f : Y \to X$ be a *continuous* map between the topological spaces Y and X. If $\mathcal{S} \equiv (\mathcal{S}, \pi, X)$, then the ***pull-back*** or ***inverse image*** of $\mathcal{S}$ by f is the sheaf

$$f^*(\mathcal{S}) \equiv (f^*(\mathcal{S}), \pi^*, Y),$$

whose sheaf space

$$f^*(\mathcal{S}) \equiv Y \times_X \mathcal{S} := \{(y, z) \in Y \times \mathcal{S} \mid f(y) = \pi(z)\} \tag{1.4.1}$$

is equipped with the relative topology as subspace of $Y \times \mathcal{S}$, and its projection is $\pi^* := \mathrm{pr}_1|_{Y \times_X \mathcal{S}}$. The space (1.4.1) is known as the *fiber product, over X, of the topological spaces Y and $\mathcal{S}$* (compare with the fiber product of sheaves in Subsections 1.1.2 and 1.3.2).

For every open $U \subseteq X$, there is a ***canonical,*** or ***adjunction, map of sections***

$$f^*_U : \mathcal{S}(U) \longrightarrow f^*(\mathcal{S})(f^{-1}(U)), \tag{1.4.2}$$

defined as follows: If $s \in \mathcal{S}(U)$, then the section $f^*_U(s)$ is determined by

$$f^*_U(s)(y) := \big(y, s(f(y))\big), \qquad y \in f^{-1}(U). \tag{1.4.3}$$

The totality of the sets of the form

$$f^*_U(s)(f^{-1}(U)) = \big\{\big(y, s(f(y))\big) \,\big|\, y \in f^{-1}(U)\big\} = f^{-1}(U) \times_U \mathcal{S}|_U, \tag{$\mathcal{B}$}$$

obtained by taking all the sections $s \in \mathcal{S}(U)$, for all U varying in the topology $\mathfrak{T}_X$ of X, provides a basis for the topology of $f^*(\mathcal{S})$.

On the other hand, for each $y \in Y$, there is a *canonical bijection*

$$f^*_y : \mathcal{S}_{f(y)} \longrightarrow f^*(\mathcal{S})_y : z \mapsto (y, z). \tag{1.4.4}$$

Therefore, for a given open $U \subseteq X$, (1.4.3) yields

$$f^*_y\big(s(f(y))\big) = \big(y, s(f(y))\big) = f^*_U(s)(y), \tag{1.4.4$'$}$$

for every $s \in \mathcal{S}(U)$ and $y \in f^{-1}(U)$.

Now let $\mathcal{S}$ and $\mathcal{T}$ be two sheaves over the topological space X and let $f : Y \to X$ be a continuous map. A sheaf morphism $\phi : \mathcal{S} \to \mathcal{T}$ induces a morphism between the corresponding pull-backs (by f)

$$f^*(\phi) : f^*(\mathcal{S}) \equiv Y \times_X \mathcal{S} \longrightarrow f^*(\mathcal{T}) \equiv Y \times_X \mathcal{T},$$

with $f^*(\phi) := (id_Y \times \phi)\big|_{f^*(\mathcal{S})}$, i.e.,

$$f^*(\phi)(y,z) := (y, \phi(z)), \qquad (y,z) \in f^*(\mathcal{S}). \tag{1.4.5}$$

As a result, for a fixed continuous map $f : Y \to X$, the correspondence (see the final notation in Subsection 1.1.1)

$$f^* : \mathcal{S}h_X \longrightarrow \mathcal{S}h_Y,$$

associating $f^*(\mathcal{S})$ to every $\mathcal{S}$, and $f^*(\phi)$ to every morphism ϕ between sheaves in $\mathcal{S}h_X$, is a *covariant functor* between the aforementioned categories of sheaves. This is the ***pull-back functor***.

For any continuous maps $g : Z \to Y$ and $f : Y \to X$, and for every $\mathcal{S} \in \mathcal{S}h_X$, the sheaves $(f \circ g)^*(\mathcal{S})$ and $g^*(f^*(\mathcal{S}))$ are isomorphic, as a consequence of the universal property of the pull-back. Thus we obtain the equality

$$(f \circ g)^* = g^* \circ f^*, \tag{1.4.6}$$

within an isomorphism.

To complete our brief exposition on the pull-back of a sheaf, we describe how this arises from a presheaf of appropriate sections. In fact, for an open $V \subseteq Y$, we define the following set of continuous ***sections (over*** V***) of*** $\mathcal{S}$ ***along*** f

$$\mathcal{S}_f(V) := \big\{ s : V \to \mathcal{S} \; : \; \text{continuous with } \pi \circ s = f|_V \big\}.$$

The correspondence $\mathcal{S}_f : V \longmapsto \mathcal{S}_f(V)$, $V \in \mathfrak{T}_Y$, together with the usual restrictions of sections, determines a *complete* presheaf. It can be shown that

$$f^*(\mathcal{S}) \cong \mathbf{S}(\mathcal{S}_f).$$

Therefore, for every open $U \subseteq X$,

$$f^*(\mathcal{S})\big(f^{-1}(U)\big) \cong \mathcal{S}_f\big(f^{-1}(U)\big).$$

If $\mathcal{S}$ now has an algebraic structure determined by an operation, say $\star : \mathcal{S} \times_X \mathcal{S} \to \mathcal{S}$, then this structure is inherited by $f^*(\mathcal{S})$. Indeed, using the same operation symbol for both $\mathcal{S}$ and $f^*(\mathcal{S})$, we may set

$$(x,z) \star (x,z') := (x, z \star z'), \tag{1.4.7}$$

for every $(x,z), (x,z') \in f^*(\mathcal{S})$.

As a useful application of the last considerations, let us mention that, if $\mathcal{A}$ is a sheaf of unital algebras (over X) and $\mathbf{1} \in \mathcal{A}(X)$ denotes the unit section, then $f^*(\mathcal{A})$ is a sheaf of unital algebras (over Y) with unit section $\mathbf{1}^*$ given by $\mathbf{1}^* = f_X^*(\mathbf{1})$, if $f_X^* : \mathcal{A}(X) \to f^*(\mathcal{A})(Y)$ is the (global) adjunction map.

Similar conclusions hold for the zero section of $f^*(\mathcal{A})$ and the zero section of $f^*(\mathcal{E})$, if $\mathcal{E}$ is an $\mathcal{A}$-module.

1.4.2. The push-out of a sheaf

In this case, the base space of the sheaf is “moved” forward by means of a continuous map.

More precisely, we assume that $\mathcal{S} \equiv (\mathcal{S}, \pi, X)$ is a sheaf and $f : X \to Y$ a *continuous* map between the topological spaces X and Y. Then, for every open $V \subseteq Y$, we consider the continuous sections $\mathcal{S}(f^{-1}(V))$ of $\mathcal{S}$ over the open set $f^{-1}(V) \subseteq X$. We check that the correspondence

$$V \longmapsto \mathcal{S}(f^{-1}(V)); \qquad V \in \mathfrak{T}_Y, \tag{1.4.8}$$

along with the natural restrictions of sections, is a *complete* presheaf. The ***push-out*** or ***direct image*** of $\mathcal{S}$ by f is the sheafification of (1.4.8), denoted by $f_*(\mathcal{S}) \equiv (f_*(\mathcal{S}), Y, \pi_*)$; that is,

$$f_*(\mathcal{S}) := \mathbf{S}\left(V \longmapsto \mathcal{S}(f^{-1}(V))\right);$$

hence, for any open $V \subseteq Y$,

$$f_*(\mathcal{S})(V) \cong \mathcal{S}(f^{-1}(V)). \tag{1.4.9}$$

By the same token, if $\phi : \mathcal{S} \to \mathcal{T}$ is a morphism of sheaves (over X), there is an induced morphism

$$f_*(\phi) : f_*(\mathcal{S}) \to f_*(\mathcal{T})$$

generated by the presheaf morphism

$$\left\{\bar{\phi}_{f^{-1}(V)} : \mathcal{S}(f^{-1}(V)) \longrightarrow \mathcal{T}(f^{-1}(V))\right\}_{V \in \mathfrak{T}_Y}. \tag{1.4.10}$$

To transfer an algebraic structure of $\mathcal{S}$ to $f_*(\mathcal{S})$, we first consider the same structure on every $\mathcal{S}(f^{-1}(V))$, and then we pass to $f_*(\mathcal{S})$, by applying the procedure described at the end of Subsection 1.2.2.

It is useful to describe the behavior of $f_*(\phi) : f_*(\mathcal{S}) \to f_*(\mathcal{T})$ with regard to the local sections. For every open $V \subseteq Y$, there is an induced morphism of sections (see (1.1.3))

$$f_*(\phi) \equiv \overline{f_*(\phi)}_V : f_*(\mathcal{S})(V) \longrightarrow f_*(\mathcal{T})(V).$$

Now, for a $\sigma \in f_*(\mathcal{S})(V)$, there is an $s \in \mathcal{S}(f^{-1}(V))$ such that $\sigma = \tilde{s}$, where $\tilde{s}$ is the image of s under the isomorphism $\mathcal{S}(f^{-1}(V)) \xrightarrow{\simeq} f_*(\mathcal{S})(V)$ (see (1.2.9)). Hence, from the following analog of Diagram 1.7, namely

$$\begin{array}{ccc}
\mathcal{S}(f^{-1}(V)) & \xrightarrow{\ \bar{\phi}_{f^{-1}(V)}\ } & \mathcal{T}(f^{-1}(V)) \\
\Big\downarrow \cong & & \Big\downarrow \cong \\
f_*(\mathcal{S})(V) & \xrightarrow{\ \overline{f_*(\phi)}_V\ } & f_*(\mathcal{T})(V)
\end{array}$$

DIAGRAM 1.8

we obtain (after $\bar{\phi}_{f^{-1}(V)} \equiv \phi$)

$$f_*(\phi)(\sigma) = f_*(\phi)(\tilde{s}) = \widetilde{\phi(s)},$$

from which one infers that

$$f_*(\phi)(\tilde{s}) = \widetilde{\phi(s)}, \qquad s \in \mathcal{S}(f^{-1}(V)).$$

As in the case of the pull-back, we obtain the (*covariant*) ***push-out functor***

$$f_* : \mathcal{S}h_X \longrightarrow \mathcal{S}h_Y.$$

Since $(g \circ f)_*(\mathcal{S}) = g_*(f_*(\mathcal{S}))$ within an isomorphism, we have the identification

$$(g \circ f)_* \equiv g_* \circ f_*, \tag{1.4.11}$$

for any continuous maps $f : X \to Y$ and $g : Y \to Z$.

To illustrate further the construction of the push-out, let us consider the case of a sheaf of unital algebras $\mathcal{A}$ (over X) with unit section $\mathbf{1} \in \mathcal{A}(X)$. The push-out $f_*(\mathcal{A})$ is sheaf of unital algebras (over Y). By (1.4.9) and (1.2.8), the corresponding unit section $\mathbf{1}_* \in f_*(\mathcal{A})(Y)$ is given by $\mathbf{1}_* = \rho_Y(\mathbf{1})$, where

$\rho_Y : \mathcal{A}(X) = \mathcal{A}(f^{-1}(Y)) \xrightarrow{\simeq} f_*(\mathcal{A})(Y)$ is the canonical (unital algebra) isomorphism of sections. Hence, applying (1.2.9), we have that $\mathbf{1}_* = \widetilde{\mathbf{1}}$.

Similar results hold for the zero sections of $f_*(\mathcal{A})$ and $f_*(\mathcal{E})$, if $\mathcal{E}$ is an $\mathcal{A}$-module.

1.5. Exact sequences

Given an $\mathcal{A}$-morphism $\phi : \mathcal{S} \to \mathcal{T}$ between two $\mathcal{A}$-modules $\mathcal{S} \equiv (\mathcal{S}, \pi, X)$ and $\mathcal{T} \equiv (\mathcal{T}, \pi', X)$, the ***kernel*** of ϕ is defined by

$$\ker\phi := \{z \in \mathcal{S} : \phi(z) = 0_{\pi(z)}\}.$$

If $\mathbf{0}$ is the zero section of $\mathcal{S}$, clearly $\ker\phi = \phi^{-1}(\mathbf{0}(X))$. Since, by the properties of sections, $\mathbf{0}(X)$ is open in $\mathcal{S}$, it follows that

$$\ker\phi \equiv (\ker\phi, \pi|_{\ker\phi}, X)$$

is a subsheaf of $\mathcal{S}$. It is also an $\mathcal{A}$-module, because $(\ker\phi)_x = \ker\phi_x$ is an $\mathcal{A}_x$-submodule of $\mathcal{S}_x$, for every $x \in X$. Thus, $\ker\phi$ is an $\mathcal{A}$-submodule of $\mathcal{S}$.

In a similar way, the ***image*** of ϕ

$$\operatorname{im}\phi := \phi(\mathcal{S}),$$

being an open subset of $\mathcal{T}$, determines the $\mathcal{A}$-submodule of $\mathcal{T}$

$$\operatorname{im}\phi \equiv (\operatorname{im}\phi, \pi'|_{\operatorname{im}\phi}, X).$$

Its stalks $(\operatorname{im}\phi)_x = \operatorname{im}\phi_x$ are $\mathcal{A}_x$-submodules of $\mathcal{T}_x$, for every $x \in X$.

Note. The kernel and image can also be defined for morphisms of sheaves bearing other appropriate algebraic structures, such as groups, rings etc. They also inherit the respective structure.

A sequence of $\mathcal{A}$-modules and $\mathcal{A}$-morphisms

$$\cdots \longrightarrow \mathcal{S}_{i-1} \xrightarrow{\phi_{i-1}} \mathcal{S}_i \xrightarrow{\phi_i} \mathcal{S}_{i+1} \longrightarrow \cdots \tag{1.5.1}$$

is said to be ***exact*** if $\ker\phi_i = \operatorname{im}\phi_{i-1}$, for every index i. If this happens only at certain term(s), e.g., $\mathcal{S}_i$, then we say that the sequence is ***exact at*** $\mathcal{S}_i$. The sequence (1.5.1) is exact if and only if the induced sequence of stalks

$$\cdots \longrightarrow \mathcal{S}_{i-1,x} \xrightarrow{\phi_{i-1,x}} \mathcal{S}_{i,x} \xrightarrow{\phi_{i,x}} \mathcal{S}_{i+1,x} \longrightarrow \cdots$$

is exact, for every $x \in X$.

In particular, a ***short exact sequence*** of $\mathcal{A}$-modules is a sequence of the form

$$0 \longrightarrow \mathcal{R} \xrightarrow{\psi} \mathcal{S} \xrightarrow{\phi} \mathcal{T} \longrightarrow 0. \tag{1.5.2}$$

Here 0 denotes the constant $\mathcal{A}$-module 0_X (see Subsection 1.3.1).

Exact sequences of presheaves of modules are obtained analogously. To this end, we first consider a presheaf of (unital commutative associative) $\mathbb{K}$-algebras $A \equiv \big(A(U),\ \sigma_V^U\big)$. An ***A-module*** or ***A-presheaf*** is a presheaf $S \equiv \big(S(U),\ \rho_V^U\big)$ such that:

$S(U)$ is an $A(U)$-module, for every open $U \in \mathfrak{T}_X$, and

$$\rho_V^U(a \cdot s) = \sigma_V^U(a) \cdot \rho_V^U(s); \qquad (a, s)A(U) \times S(U),$$

for every U, $V \in \mathfrak{T}_X$ with $V \subseteq U$.

Accordingly, an ***A*-morphism** $\phi \equiv (\phi_U)$ of $S \equiv \big(S(U),\ \rho_V^U\big)$ into $T \equiv \big(T(U), \tau_V^U\big)$ is a morphism of presheaves such that every $\phi_U : S(U) \to T(U)$ is a morphism of $A(U)$-modules.

Given an A-morphism ϕ, the A-modules

$$\ker \phi \equiv \ker\left((\phi_U)\right) := \big(\ker \phi_U,\ \rho_V^U|_{\ker \phi_U}\big),$$

$$\operatorname{im} \phi \equiv \operatorname{im}\left((\phi_U)\right) := \big(\operatorname{im} \phi_U, \tau_V^U|_{\operatorname{im} \phi_U}\big),$$

(for all $U \in \mathfrak{T}_X$) are called, respectively, the ***kernel*** and the ***image of the A-morphism*** ϕ.

A sequence of A-modules and A-morphisms

$$\cdots \longrightarrow S_{i-1} \xrightarrow{\phi_{i-1}} S_i \xrightarrow{\phi_i} S_{i+1} \longrightarrow \cdots \tag{1.5.3}$$

is said to be ***exact*** if $\ker \phi_i = \operatorname{im} \phi_{i-1}$, for every i. This means that, for every open $U \subseteq X$, the corresponding sequence

$$\cdots \longrightarrow S_{i-1}(U) \xrightarrow{\phi_{i-1,U}} S_i(U) \xrightarrow{\phi_{i,U}} S_{i+1}(U) \longrightarrow \cdots$$

is exact.

Starting with an exact sequence of the form (1.5.1), the sequence of $\Gamma(\mathcal{A})$-modules

$$\cdots \longrightarrow \Gamma(\mathcal{S}_{i-1}) \xrightarrow{\Gamma(\phi_{i-1})} \Gamma(\mathcal{S}_i) \xrightarrow{\Gamma(\phi_i)} \Gamma(\mathcal{S}_{i+1}) \longrightarrow \cdots$$

is not necessarily exact. On the contrary, the sheafification of an exact sequence of A-modules of the form (1.5.3) leads to the exact sequence of $\mathbf{S}(A)$-modules

$$\cdots \longrightarrow \mathbf{S}(S_{i-1}) \xrightarrow{\mathbf{S}(\phi_{i-1})} \mathbf{S}(S_i) \xrightarrow{\mathbf{S}(\phi_i)} \mathbf{S}(S_{i+1}) \longrightarrow \cdots$$

since the inductive limits preserve the exactness.

By the same token, in particular, a short exact sequence of A-modules

$$0 \longrightarrow R \xrightarrow{\psi} S \xrightarrow{\phi} T \longrightarrow 0, \tag{1.5.4}$$

leads to the corresponding short exact sequence of $\mathbf{S}(A)$-modules

$$0 \longrightarrow \mathbf{S}(R) \xrightarrow{\mathbf{S}(\psi)} \mathbf{S}(S) \xrightarrow{\mathbf{S}(\phi)} \mathbf{S}(T) \longrightarrow 0.$$

But, as already said, the converse is *not* always true; that is, starting with a short exact sequence of the form (1.5.2), we obtain the exact sequence

$$0 \longrightarrow \Gamma(\mathcal{R}) \xrightarrow{\Gamma(\psi)} \Gamma(\mathcal{S}) \xrightarrow{\Gamma(\phi)} \Gamma(\mathcal{T}), \tag{1.5.5}$$

where the morphism $\Gamma(\phi)$ is *not* necessarily surjective (for relevant counter-examples we refer, e.g., to Warner [140, Section 5.11], Wells [142, p. 52]).

The cohomology theory, discussed in the next section, measures the deviation of (1.5.5) from being exact at $\Gamma(\mathcal{T})$.

1.6. Sheaf cohomology

Sheaf cohomology can be approached from various points of view. Its full treatment is adequately covered in most of the references mentioned in the introduction of this chapter.

Guided by the needs of the present work, we first give a brief account of the Čech cohomology (see Subsections 1.6.1 and 1.6.2), some aspects of which will be encountered in subsequent chapters. A very short account of cohomology via resolutions, aiming mainly at the abstract de Rham theorem, is given in Subsection 1.6.3. The aforementioned theorem will only be needed in Chapter 9 (Chern-Weil theory), whence the brevity of our discussion.

The cohomology with coefficients in a sheaf of non abelian groups restricts only to the 1st cohomology set. This case will be treated in Subsection 1.6.4. We close with Subsection 1.6.5, where we sketch the construction of the 1st hypercohomology group, in order to make the results of Section 6.7 readable.

1.6.1. Čech cohomology with coefficients in a sheaf

This is a popular approach to cohomology, allowing direct computations without use of resolutions (cf. Subsection 1.6.3). Some of its drawbacks are remedied by assuming that we work over paracompact topological spaces. The sheaf of coefficients are mainly $\mathcal{A}$-modules, a fact generalizing the ordinary cohomology theory with coefficients in a sheaf of K-modules, where K is a ring.

We begin with an *arbitrary* topological space X and a fixed open covering $\mathcal{U} = (U_\alpha)_{\alpha \in I}$ of it. If $\mathcal{S}$ is an $\mathcal{A}$-module over X, then, for any integer $q \geq 0$, we define the set of (Čech) ***q-cochains of $\mathcal{U}$ with coefficients in $\mathcal{S}$***

$$C^q(\mathcal{U}, \mathcal{S}) := \prod_{(\alpha_0,\dots,\alpha_q)} \mathcal{S}(U_{\alpha_0\dots\alpha_q}),$$

where, for the sake of convenience, we have set

$$U_{\alpha_0\dots\alpha_q} := U_{\alpha_0} \cap \dots \cap U_{\alpha_q}. \tag{1.6.1}$$

From Subsection 1.2.1 we recall that $\mathcal{S}(\emptyset) = 0$.

Thus, by definition, a q-cochain is a map f which, to every $q+1$ indices from I, assigns a section $f_{\alpha_0\dots\alpha_q} \equiv f(\alpha_0 \dots \alpha_q) \in \mathcal{S}(U_{\alpha_0\dots\alpha_q})$. Accordingly, setting

$$(s \cdot f)_{\alpha_0\dots\alpha_q} := s\big|_{U_{\alpha_0\dots\alpha_q}} \cdot f_{\alpha_0\dots\alpha_q},$$

for every $s \in \mathcal{A}(X)$ and $f \in C^q(\mathcal{U}, \mathcal{S})$, we see that $C^q(\mathcal{U}, \mathcal{S})$ is an $\mathcal{A}(X)$-module.

The ***q-th coboundary operator*** or ***homomorphism***

$$\delta^q : C^q(\mathcal{U}, \mathcal{S}) \longrightarrow C^{q+1}(\mathcal{U}, \mathcal{S})$$

is defined by

$$\begin{aligned} \delta^q(f)_{\alpha_0\dots\alpha_{q+1}} &:= \sum_{i=0}^{q+1} (-1)^i \rho_{U_{\alpha_0\dots\alpha_{q+1}}}^{U_{\alpha_0\dots\widehat{\alpha}_i\dots\alpha_{q+1}}} \left(f_{\alpha_0\dots\widehat{\alpha}_i\dots\alpha_{q+1}} \right) \\ &= \sum_{i=0}^{q+1} (-1)^i f_{\alpha_0\dots\widehat{\alpha}_i\dots\alpha_{q+1}}\big|_{U_{\alpha_0\dots\alpha_{q+1}}}, \end{aligned} \tag{1.6.2}$$

for every $f \in C^q(\mathcal{U}, \mathcal{S})$. Clearly, ρ_V^U are now the ordinary restrictions of sections and the "hat" ($\widehat{\ }$) indicates omission of the corresponding entry. A useful convention is the following:

$$C^q(\mathcal{U}, \mathcal{S}) = 0, \quad \delta^q = 0, \qquad q < 0.$$

(1.6.3) If there is no danger of confusion, we simply write δ in place of δ^q, for all $q \geq 0$.

In a different terminology, any collection $\sigma = (U_{\alpha_0}, \ldots, U_{\alpha_q})$ of open sets from the covering $\mathcal{U}$, with $U_{\alpha_0 \ldots \alpha_q} \neq \emptyset$, is called a ***q-simplex*** with ***support*** $|\sigma| := U_{\alpha_0 \ldots \alpha_q}$. Using the previous notations and terminology, a q-cochain f can be interpreted as a map assigning, to each q-simplex σ, a section $f(\sigma) \in \mathcal{S}(|\sigma|)$.

If we define the ***i-th face*** of a simplex σ to be the $(q-1)$-simplex

$$\sigma^i := (U_{\alpha_0}, \ldots, \widehat{U}_{\alpha_i}, \ldots, U_{\alpha_q}),$$

then (by interpretation a cochain as a map of simplexes) the coboundary operator can be written in the following condensed form:

$$\delta^q(f) = \sum_{i=0}^{q+1} (-1)^i \rho_{|\sigma|}^{|\sigma^i|} f(\sigma^i).$$

To illustrate (1.6.2), we calculate two frequently occurring cases. For a 0-cochain $f \equiv (f_\alpha) \in C^0(\mathcal{U}, \mathcal{S})$, we have that

$$\delta(f) \equiv \delta^0((f_\alpha)) = (f_\beta - f_\alpha)\big|_{U_{\alpha\beta}}, \tag{1.6.4}$$

while, for an 1-cochain $f \equiv (f_{\alpha\beta}) \in C^1(\mathcal{U}, \mathcal{S})$,

$$\delta(f) \equiv \delta^1((f_{\alpha\beta})) = (f_{\alpha\beta} + f_{\beta\gamma} - f_{\alpha\gamma})\big|_{U_{\alpha\beta\gamma}}. \tag{1.6.5}$$

A fundamental property of δ is that

$$\delta^{q+1} \circ \delta^q = 0; \qquad q \in \mathbb{Z}_0^+, \tag{1.6.6}$$

where, for convenience, we let

$$\mathbb{Z}_0^+ := \mathbb{N} \cup \{0\}. \tag{1.6.6$'$}$$

Thus, the collection of cochains, together with the coboundary operators, determines the ***(Čech) cochain complex*** of $\mathcal{A}(X)$-modules

$$\cdots \longrightarrow C^{q-1}(\mathcal{U}, \mathcal{S}) \xrightarrow{\delta^{q-1}} C^q(\mathcal{U}, \mathcal{S}) \xrightarrow{\delta^q} C^{q+1}(\mathcal{U}, \mathcal{S}) \longrightarrow \cdots,$$

briefly denoted by

$$\check{C}^\bullet(\mathcal{U}, \mathcal{S}) := \big(C^q(\mathcal{U}, \mathcal{S}), \delta^q\big)_{q \in \mathbb{Z}_0^+}. \tag{1.6.7}$$

The ***q-th*** (or ***q-dimensional***) ***Čech cohomology*** $\mathcal{A}(X)$***-module of*** $\mathcal{U}$ ***with coefficients in*** $\mathcal{S}$ is defined to be the quotient

$$\check{H}^q(\mathcal{U},\mathcal{S}) := \ker\delta^q \big/ \mathrm{im}\,\delta^{q-1}.$$

Still, it is customary to set

$$\check{Z}^q(\mathcal{U},\mathcal{S}) := \ker\delta^q, \tag{1.6.8a}$$

$$\check{B}^q(\mathcal{U},\mathcal{S}) := \mathrm{im}\,\delta^{q-1}, \tag{1.6.8b}$$

by which we denote, respectively, the $\mathcal{A}(X)$-modules of q-th (Čech) ***cocycles*** and ***coboundaries*** of $\mathcal{U}$ with coefficients in $\mathcal{S}$. Therefore,

$$\check{H}^q(\mathcal{U},\mathcal{S}) = \check{Z}^q(\mathcal{U},\mathcal{S}) \big/ \check{B}^q(\mathcal{U},\mathcal{S}). \tag{1.6.9}$$

For any cocycle $f \in \check{Z}^q(\mathcal{U},\mathcal{S})$,

$$[f]_{\mathcal{U}} \in \check{H}^q(\mathcal{U},\mathcal{S}) \tag{1.6.10}$$

stands for its cohomology class, with respect to $\mathcal{U}$.

Note. Influenced by the case of cohomology with coefficients in a sheaf of *abelian groups*, the cohomology module (1.6.9) is also called the q-th (Čech) ***cohomology group***.

From (1.6.4) it follows that *a* 0*-cochain* f *is a* 0*-cocycle if and only if* f *determines a global section of* $\mathcal{S}$. Hence, (1.6.9) implies that

$$\check{H}^0(\mathcal{U},\mathcal{S}) = \Gamma(X,\mathcal{S}) \equiv \mathcal{S}(X). \tag{1.6.11}$$

To examine the behavior of the quotients (1.6.9) with respect to the open coverings of X, we first consider an open refinement $\mathcal{V} = (V_j)_{j\in J}$ of $\mathcal{U}$. In this case we can find a ***refining map*** $\tau : J \to I$; that is, a map such that $V_j \subseteq U_{\tau(j)}$. For every $q \in \mathbb{Z}_0^+$, τ induces a corresponding cochain map (actually an $\mathcal{A}(X)$-morphism)

$$\tau_q : C^q(\mathcal{U},\mathcal{S}) \longrightarrow C^q(\mathcal{V},\mathcal{S}),$$

determined by

$$\tau_q(f)_{j_0\dots j_q} := f_{\tau(j_0)\dots\tau(j_q)}\big|_{V_{j_0\cdots j_q}}, \tag{1.6.12}$$

for every $f \in C^q(\mathcal{U}, \mathcal{S})$ and $j_0, \ldots, j_q \in J$. Since the morphisms τ_q commute with the coboundary operators as in the diagram

$$\begin{array}{ccc} C^q(\mathcal{U}, \mathcal{S}) & \xrightarrow{\tau_q} & C^q(\mathcal{V}, \mathcal{S}) \\ \downarrow{\scriptstyle \delta^q} & & \downarrow{\scriptstyle \delta^q} \\ C^{q+1}(\mathcal{U}, \mathcal{S}) & \xrightarrow{\tau_{q+1}} & C^{q+1}(\mathcal{V}, \mathcal{S}) \end{array}$$

Diagram 1.9

we obtain the $\mathcal{A}(X)$-morphisms

(1.6.13) $$\tau_q^* : \check{H}^q(\mathcal{U}, \mathcal{S}) \longrightarrow \check{H}^q(\mathcal{V}, \mathcal{S}) : [f]_{\mathcal{U}} \longmapsto [\tau_q(f)]_{\mathcal{V}}, \qquad q \in \mathbb{Z}_0^+.$$

If $\bar{\tau} : J \to I$ is another refining map (thus, $V_j \subseteq U_{\tau(j)} \cap U_{\bar{\tau}(j)}$, for every $j \in J$), then we show that $\bar{\tau}_q^* = \tau_q^*$, for every $q \geq 0$. The proof is based on a *homotopy* argument.

More precisely, we define the ***homotopy operators***

$$h_q : C^q(\mathcal{U}, \mathcal{S}) \longrightarrow C^{q-1}(\mathcal{V}, \mathcal{S}); \qquad q \geq 1,$$

by setting

(1.6.14) $$\begin{aligned} h_q(f)_{j_0 \ldots j_{q-1}} &:= \sum_{i=0}^{q-1} (-1)^i \rho_{\overline{V}}^{\overline{U}} f_{\tau(j_0) \ldots \tau(j_i) \bar{\tau}(j_i) \ldots \bar{\tau}(j_{q-i})} \\ &= \sum_{i=0}^{q-1} (-1)^i f_{\tau(j_0) \ldots \tau(j_i) \bar{\tau}(j_i) \ldots \bar{\tau}(j_{q-i})} \big|_{V_{j_0 \cdots j_{q-1}}}, \end{aligned}$$

for every $f \in C^q(\mathcal{U}, \mathcal{S})$, where

$$\overline{U} = U_{\tau(j_0) \ldots \tau(j_i) \bar{\tau}(j_i) \ldots \bar{\tau}(j_{q-i})} \quad \text{and} \quad \overline{V} = V_{j_0 \ldots j_{q-1}}.$$

We check that

$$\bar{\tau}_q - \tau_q = \delta^{q-1} \circ h_q + h_{q+1} \circ \delta^q,$$

if $q \geq 1$, whereas

$$\bar{\tau}_0 - \tau_0 = h_1 \circ \delta^0.$$

The following diagram may be useful. To avoid any confusion, we note that its sub-diagrams are *not* commutative.

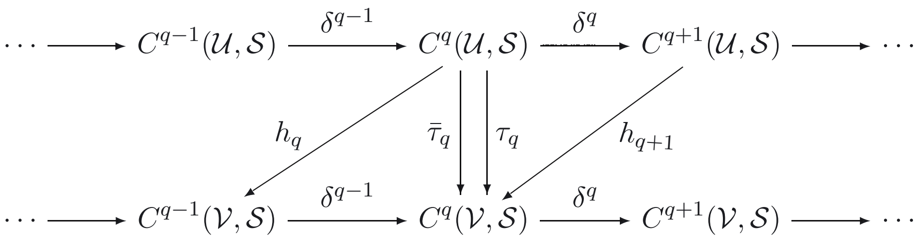


DIAGRAM 1.10

As a consequence of the preceding,

$$\bar{\tau}_q(f) - \tau_q(f) = \delta^{q-1}(h_q(f)) \in \check{B}^q(\mathcal{U}, \mathcal{S}),$$

for every $f \in \check{Z}^q(\mathcal{U}, \mathcal{S})$. Hence, (1.6.13) leads to

$$\tau_q^*([f]_{\mathcal{U}}) = [\tau_q(f)]_{\mathcal{V}} = [\bar{\tau}_q(f)]_{\mathcal{V}} = \bar{\tau}_q^*([f]_{\mathcal{U}}); \qquad f \in \check{Z}^q(\mathcal{U}, \mathcal{S}), \tag{1.6.15}$$

which proves the assertion stated after (1.6.13).

Since $[f]_{\mathcal{U}} \mapsto \tau_q^*([f]_{\mathcal{U}})$ is independent of the choice of the refining map, there exists a unique $\mathcal{A}(X)$-morphism

$$t^{\mathcal{U}}_{\mathcal{V}} : \check{H}^q(\mathcal{U}, \mathcal{S}) \longrightarrow \check{H}^q(\mathcal{V}, \mathcal{S}) \tag{1.6.16}$$

with $t^{\mathcal{U}}_{\mathcal{V}} = \tau_q^*$, for an arbitrary refining map $\tau : J \to I$. It is immediate that

$$t^{\mathcal{U}}_{\mathcal{U}} = id \qquad \text{and} \qquad t^{\mathcal{U}}_{\mathcal{W}} = t^{\mathcal{V}}_{\mathcal{W}} \circ t^{\mathcal{U}}_{\mathcal{V}},$$

for any open coverings $\mathcal{U}$, $\mathcal{V}$ and $\mathcal{W}$ of X, with $\mathcal{W} \subseteq \mathcal{V} \subseteq \mathcal{U}$. Thus, considering *the set of all proper open coverings of* X directed by the relation $\preceq$, defined by

$$\mathcal{U} \preceq \mathcal{V} \iff \mathcal{V} \subseteq \mathcal{U},$$

we obtain an *inductive system* of $\mathcal{A}(X)$-modules $\big(\check{H}^q(\mathcal{U}, \mathcal{S}),\, t^{\mathcal{U}}_{\mathcal{V}}\big)$. Then the *inductive* or *direct* limit

$$\check{H}^q(X, \mathcal{S}) := \varinjlim_{\mathcal{U}} \check{H}^q(\mathcal{U}, \mathcal{S}) \tag{1.6.17}$$

(with respect to the directed set of all proper open coverings of X) is the ***q-th (dimensional) Čech cohomology*** $\mathcal{A}(X)$***-module of*** X ***with coefficients in the sheaf (***$\mathcal{A}$***-module)*** $\mathcal{S}$. Traditionally, the previous cohomology module is also called the ***q-th cohomology group of*** X ***with coefficients in*** $\mathcal{S}$.

Note. The necessity of considering proper open coverings stems from the logical difficulties concerning the set of all open coverings of X (see Dowker [23, pp. 60–61], Hirzebruch [44, p. 17]).

For each open covering $\mathcal{U}$, there exists a ***canonical map***

$$t_{\mathcal{U}} : \check{H}^q(\mathcal{U}, \mathcal{S}) \longrightarrow \check{H}^q(X, \mathcal{S}). \tag{1.6.18}$$

Then, for a cocycle $f \in \check{Z}^q(\mathcal{U}, \mathcal{S})$, the class

$$[f] := t_{\mathcal{U}}([f]_{\mathcal{U}}) \tag{1.6.19}$$

denotes the cohomology class of f in the module (1.6.17).

For any refinement $\mathcal{V}$ of $\mathcal{U}$, we obtain the next commutative diagram, as a result of the general theory of inductive systems and their limits (see, e.g., Bourbaki [12, p. 89]).

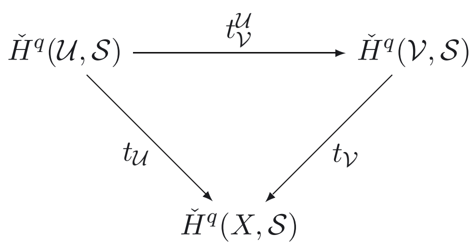


Diagram 1.11

However, in order to get a feeling of the mechanism expounded so far, we give a direct proof of the commutativity of Diagram 1.11. To this end, we take the refinement $\mathcal{V}$ of $\mathcal{U}$ and a refining map $\tau : J \to I$. We consider a common refinement $\mathcal{W} = (W_k)_{k \in K}$ of both $\mathcal{U}$ and $\mathcal{V}$, as well as any refining maps $\sigma : K \to I$ and $\rho : K \to J$ with $W_k \subseteq U_{\sigma(k)}$ and $W_k \subseteq V_{\rho(k)}$, thus $W_k \subseteq U_{\sigma(k)} \cap V_{\rho(k)}$. Clearly, $\tau \circ \rho : K \to I$ is also a refining map such that $(\tau \circ \rho)_q = \rho_q \circ \tau_q$, in virtue of (1.6.12). Therefore, for an arbitrary class $[f]_{\mathcal{U}} \in \check{H}^q(\mathcal{U}, \mathcal{S})$, equalities (1.6.15) and (1.6.16) imply that

$$\begin{aligned} t^{\mathcal{U}}_{\mathcal{W}}([f]_{\mathcal{U}}) &= [\sigma_q(f)]_{\mathcal{W}} = [(\tau \circ \rho)_q(f)]_{\mathcal{W}} \\ &= [(\rho_q(\tau_q(f))]_{\mathcal{W}} = t^{\mathcal{V}}_{\mathcal{W}}(t^{\mathcal{U}}_{\mathcal{V}}([f]_{\mathcal{U}})). \end{aligned}$$

This proves that the classes $[f]_{\mathcal{U}} \in \check{H}^q(\mathcal{U}, \mathcal{S})$ and $t^{\mathcal{U}}_{\mathcal{V}}([f]_{\mathcal{U}}) \in \check{H}^q(\mathcal{V}, \mathcal{S})$ are equivalent (in the inductive limit); hence,

$$t_{\mathcal{U}}([f]_{\mathcal{U}}) = t_{\mathcal{V}}\left(t^{\mathcal{U}}_{\mathcal{V}}([f]_{\mathcal{U}})\right),$$

as required.

The particular case of $q = 1$ is worthy to be noted because the morphism (see (1.6.13) and (1.6.16))

$$\tau_1^* \equiv t^{\mathcal{U}}_{\mathcal{V}} : \check{H}^1(\mathcal{U}, \mathcal{S}) \longrightarrow \check{H}^1(\mathcal{V}, \mathcal{S}) \tag{1.6.20}$$

is *injective*, for every $\mathcal{V} \subseteq \mathcal{U}$. Therefore,

$$\check{H}^1(X, \mathcal{S}) := \varinjlim_{\mathcal{U}} \check{H}^1(\mathcal{U}, \mathcal{S}) \cong \bigsqcup_{\mathcal{U}} \check{H}^1(\mathcal{U}, \mathcal{S})$$

(disjoint union). For details we refer to Mallios [62, Vol. I, p. 183].

The collection of the $\mathcal{A}(X)$-modules $\check{H}^q(X, \mathcal{S})$, for all $q \in \mathbb{Z}_0^+$, is the ***Čech cohomology of*** X ***with coefficients in the sheaf (***$\mathcal{A}$***-module)*** $\mathcal{S}$, denoted by $\check{H}^*(X, \mathcal{S})$. It can be proved that:

(1.6.21) *If* X *is a (Hausdorff) paracompact space, then* $\check{H}^*(X, \mathcal{S})$ *satisfies the axioms of a* cohomology theory.

The assumption that X is a (Hausdorff) paracompact space is a sufficient condition ensuring the existence of the long cohomology sequence, derived from a short exact sequence of sheaves. Details are given in the next subsection. For a neat exposition of the axioms of a cohomology theory, one may consult, e.g., Warner [140, pp. 176–177].

1.6.2. Čech cohomology with coefficients in a presheaf

Let $S \equiv \left(S(U), \rho^U_V\right)$ be an A-module over *any* topological space X (see the discussion following (1.5.2)). The construction of the ***q-th Čech cohomology module of*** X ***with coefficients in the presheaf*** S, namely

$$\check{H}^q(X, S),$$

is analogous to the one with coefficients in a sheaf ($\mathcal{A}$-module). The only difference is that, instead of the $\mathcal{A}(U)$-modules of sections $\mathcal{S}(U) \equiv \Gamma(U, \mathcal{S})$, we consider the $A(U)$-modules $S(U)$. Hence, in all the expressions involving restrictions of sections, we use the restriction morphisms of the form ρ^U_V.

For instance, the coboundary and homotopy operators are now given by the first equalities of (1.6.2) and (1.6.14), respectively.

Accordingly, the ***Čech cohomology of X with coefficients in the presheaf (A-module)*** S is the collection of $A(X)$-modules

$$\check{H}^*(X, S) = \{\check{H}^q(X, S)\}_{q \in \mathbb{Z}_0^+}.$$

It should be noted that (compare with (1.6.21)):

> $\check{H}^*(X, S)$ *satisfies all the axioms of a cohomology theory, without any restriction on the topology of* X.

In the context of the cohomology theory with coefficients in a presheaf, we obtain long exact cohomology sequences, whereas this is problematic for the cohomology with coefficients in a sheaf, unless X is a (Hausdorff) paracompact space (see the end of Subsection 1.6.1).

To be more specific, regarding the previous comment, assume first that

$$\phi \equiv (\phi_U) : S \equiv \big(S(U), \rho_V^U\big) \longrightarrow T \equiv \big(T(U), \tau_V^U\big)$$

is an A-morphism of A-presheaves over *any* topological space X. The morphism ϕ induces corresponding cochain morphisms

(1.6.22) $$\phi_{\mathcal{U},q} : C^q(\mathcal{U}, S) \longrightarrow C^q(\mathcal{U}, T); \qquad q \in \mathbb{Z}_0^+,$$

(actually morphisms of $A(X)$-modules) given by

(1.6.22′) $$\big(\phi_{\mathcal{U},q}(f)\big)_{\alpha_0 \ldots \alpha_q} := \phi_{U_{\alpha_0 \ldots \alpha_q}}\big(f_{\alpha_0 \ldots \alpha_q}\big)$$

for every $f \in C^q(\mathcal{U}, S)$ and $(\alpha_0, \ldots, \alpha_q) \in I^{q+1}$. It is easily verified that the morphisms (1.6.22) commute with the coboundary operators, as pictured in the next diagram.

$$\begin{array}{ccc} C^q(\mathcal{U}, S) & \xrightarrow{\ \phi_{\mathcal{U},q}\ } & C^q(\mathcal{U}, T) \\ \Big\downarrow{\scriptstyle \delta^q} & & \Big\downarrow{\scriptstyle \delta^q} \\ C^{q+1}(\mathcal{U}, S) & \xrightarrow{\ \phi_{\mathcal{U},q+1}\ } & C^{q+1}(\mathcal{U}, T) \end{array}$$

Diagram 1.12

Hence, they define (by passage to the quotients) the morphisms

$$\phi^*_{\mathcal{U},q} : \check{H}^q(\mathcal{U}, S) \longrightarrow \check{H}^q(\mathcal{U}, T), \qquad q \in \mathbb{Z}_0^+.$$

Moreover, for every open refinement $\mathcal{V}$ of $\mathcal{U}$, we obtain the following commutative diagram.

$$\begin{array}{ccc} \check{H}^q(\mathcal{U}, S) & \xrightarrow{\phi^*_{\mathcal{U},q}} & \check{H}^q(\mathcal{U}, T) \\ {\scriptstyle t^{\mathcal{U}}_{\mathcal{V}}}\downarrow & & \downarrow{\scriptstyle t^{\mathcal{U}}_{\mathcal{V}}} \\ \check{H}^q(\mathcal{V}, S) & \xrightarrow{\phi^*_{\mathcal{V},q}} & \check{H}^q(\mathcal{V}, T) \end{array}$$

Diagram 1.13

The vertical morphisms are defined analogously to (1.6.16). For simplicity we have used the same symbol, although they refer to modules derived from cochains with values in different kinds of sections. Therefore, for each $q \in \mathbb{Z}_0^+$, $(\phi^*_{\mathcal{U},q})$ is a morphism of inductive systems (with respect to $\mathcal{U}$) yielding the $A(X)$-morphism of cohomology modules

(1.6.23) $$\phi^*_q := \varinjlim_{\mathcal{U}} \phi^*_{\mathcal{U},q} \;:\; \check{H}^q(X, S) \longrightarrow \check{H}^q(X, T).$$

Usually, we simply write ϕ^* instead of ϕ^*_q, unless we want to explicitly mention the dimension of the cohomology modules involved.

An important consequence of the previous considerations is that:

A short exact sequence of A-modules over any topological space X

(1.6.24) $$0 \longrightarrow R \xrightarrow{\psi} S \xrightarrow{\phi} T \longrightarrow 0$$

induces the long exact (Čech) cohomology sequence

(1.6.25) $$\begin{aligned} 0 \longrightarrow \check{H}^0(X,R) &\xrightarrow{\psi^*} \check{H}^0(X,S) \xrightarrow{\phi^*} \check{H}^0(X,T) \xrightarrow{\delta^*} \check{H}^1(X,R) \\ &\xrightarrow{\psi^*} \check{H}^1(X,S) \xrightarrow{\phi^*} \check{H}^1(X,T) \xrightarrow{\delta^*} \check{H}^2(X,R) \longrightarrow \dots \\ \check{H}^q(X,R) &\xrightarrow{\psi^*} \check{H}^q(X,S) \xrightarrow{\phi^*} \check{H}^q(X,T) \xrightarrow{\delta^*} \check{H}^{q+1}(X,R) \longrightarrow \cdots \end{aligned}$$

The ***connecting morphisms*** or ***Bockstein operators*** δ^* will be defined in a moment. Once again we write δ^* instead of δ^*_q, for all $q \in \mathbb{Z}_0^+$.

Note. Another common notation for the connecting morphisms δ^* is ∂. However, the latter symbol is reserved for a completely different use from Chapter 3 onwards.

Let us now explain how δ^* is defined by the so-called *chasing diagram routine* . For convenience, we consider the diagram

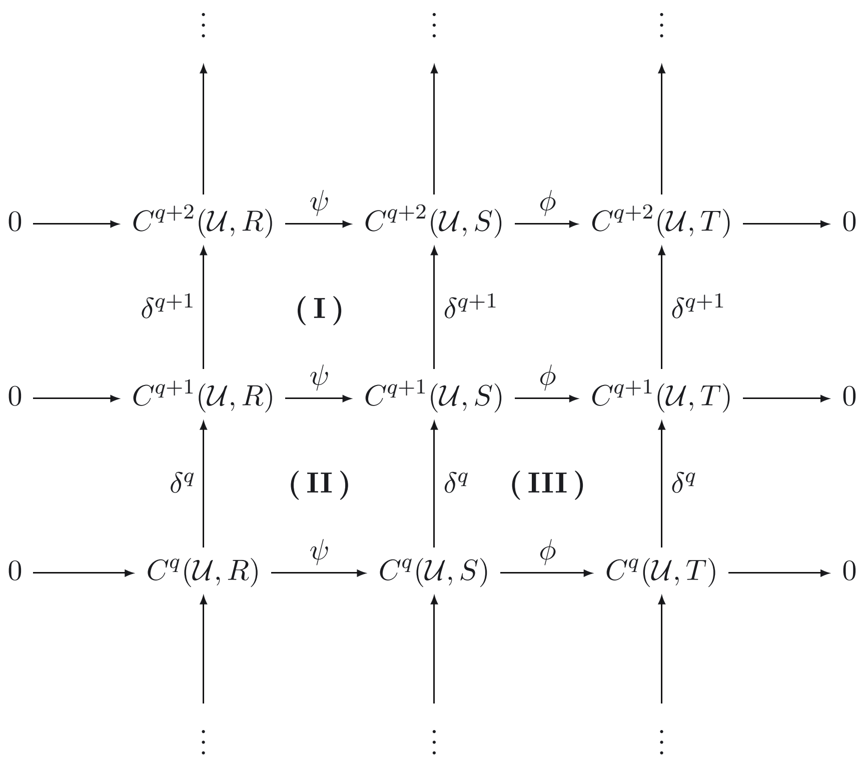


Diagram 1.14

consisting of commutative squares (in virtue of Diagram 1.12) and horizontal exact sequences (in virtue of equalities (1.6.24) and (1.6.22)). For simplicity, the morphisms between cochains in the horizontal sequences of the diagram, induced by the morphisms ϕ and ψ of (1.6.24), are denoted by the same symbols instead of $\phi_{\mathcal{U},q}$, $\psi_{\mathcal{U},q}$ etc., as in (1.6.22).

Let $c \in \check{H}^q(X, T)$ be an arbitrary cohomology class. Then there is an open covering $\mathcal{U}$ of X and a q-cocycle $f \in \check{Z}^q(\mathcal{U}, T)$ such that $c = [f] = t_{\mathcal{U}}([f]_{\mathcal{U}})$ (by the analog of (1.6.19) for the presheaf T). Because of the exactness of the bottom sequence of Diagram 1.14, there exists a cochain

$g \in C^q(\mathcal{U}, S)$ with $\phi(g) \equiv \phi_{\mathcal{U},q}(g) = f$. By the commutativity of sub-diagram (III), we have that

$$\phi_{\mathcal{U},q+1}(\delta^q(g)) = \delta^q(\phi_{\mathcal{U},q}(g)) = \delta^q(f) = 0,$$

thus $\delta^q(g) \in \ker \phi_{\mathcal{U},q+1} = \operatorname{im} \psi_{\mathcal{U},q+1}$. Hence, there exists an $h \in C^{q+1}(\mathcal{U}, R)$ satisfying $\psi_{\mathcal{U},q+1}(h) = \delta^q(g)$. This equality, combined with the commutative sub-diagram (I) and (1.6.6), yields

$$\psi_{\mathcal{U},q+2}(\delta^{q+1}(h)) = \delta^{q+1}(\psi_{\mathcal{U},q+1}(h)) = \delta^{q+1}(\delta^q(g)) = 0.$$

But $\psi_{\mathcal{U},q+2}$ is injective, thus $\delta^{q+1}(h)$=0, i.e., $h \in \check{Z}^{q+1}(\mathcal{U}, R)$. Therefore, the previous constructions allow one to define δ^* by setting

$$\delta^*(c) \equiv \delta^*_q([f]) := [h].$$

In a more detailed form, we can equivalently write

$$\delta^*_q([f]) = [\psi^{-1}_{\mathcal{U},q+1}(\delta^q(g))],$$

for an arbitrary $g \in C^q(\mathcal{U}, \mathcal{S})$ with $\phi_{\mathcal{U},q}(g) = f$.

Of course, we must show that the previous definition is independent of the choice of g. Indeed, assume that $\bar{g} \in C^q(\mathcal{U}, S)$ is a cochain with $\phi_{\mathcal{U},q}(\bar{g}) = f$. As before, we find a cocycle $\bar{h} \in \check{Z}^{q+1}(\mathcal{U}, R)$ such that $\psi_{\mathcal{U},q+1}(\bar{h}) = \delta^q(\bar{g})$. As a result,

$$\psi_{\mathcal{U},q+1}(h - \bar{h}) = \delta^q(g - \bar{g}). \tag{1.6.26}$$

On the other hand, $\phi_{\mathcal{U},q}(g - \bar{g}) = 0$, or $(g - \bar{g}) \in \ker \phi_{\mathcal{U},q} = \operatorname{im} \psi_{\mathcal{U},q}$. Thus, there is a $k \in C^q(\mathcal{U}, R)$ with $\psi_{\mathcal{U},q}(k) = g - \bar{g}$. Applying δ^q to both sides of the last equality, we obtain $\delta^q(\psi_{\mathcal{U},q}(k)) = \delta^q(g - \bar{g})$, which, by (1.6.26) and the commutativity of sub-diagram (II), yields $\psi_{\mathcal{U},q+1}(\delta^q(k)) = \psi_{\mathcal{U},q+1}(h - \bar{h})$. The injectivity of $\psi_{\mathcal{U},q+1}$ now implies that $\delta^q(k) = h - \bar{h}$. In other words, $h - \bar{h} \in \check{B}^{q+1}(\mathcal{U}, R)$ from which it follows that $[h] = [\bar{h}]$, as required.

We now return to the case of $\mathcal{A}$-modules and consider the short exact sequence

$$0 \longrightarrow \mathcal{R} \xrightarrow{\psi} \mathcal{S} \xrightarrow{\phi} \mathcal{T} \longrightarrow 0. \tag{1.6.27}$$

From the comments at the end of Section 1.5, it is clear that in the induced sequences of cochains

$$0 \longrightarrow C^q(\mathcal{U}, \mathcal{R}) \xrightarrow{\psi_{\mathcal{U},q}} C^q(\mathcal{U}, \mathcal{S}) \xrightarrow{\phi_{\mathcal{U},q}} C^q(\mathcal{U}, \mathcal{T})$$

$\phi_{\mathcal{U},q}$ is not surjective. The cochain morphism $\phi_{\mathcal{U},q}(f)$ in the preceding sequence of cochains is determined by

$$(\phi_{\mathcal{U},q}(f))_{\alpha_0\ldots\alpha_q} := \bar{\phi}_{U_{\alpha_0\ldots\alpha_q}} \circ f_{\alpha_0\ldots\alpha_q}$$

(compare with the case of presheaves, where the analogous expression is given by (1.6.22′)). Therefore, the foregoing method of deriving the long cohomology sequence (1.6.25) cannot be directly applied to the case of sheaves. We overcome this shortcoming by defining the ***liftable cochains***

$$\bar{C}^q(\mathcal{U},\mathcal{T}) := \phi_{\mathcal{U},q}\left(C^q(\mathcal{U},\mathcal{S})\right) \subseteq C^q(\mathcal{U},\mathcal{T}),$$

yielding in turn the exact sequences

$$0 \longrightarrow C^q(\mathcal{U},\mathcal{R}) \xrightarrow{\psi_{\mathcal{U},q}} C^q(\mathcal{U},\mathcal{S}) \xrightarrow{\phi_{\mathcal{U},q}} \bar{C}^q(\mathcal{U},\mathcal{T}) \longrightarrow 0,$$

for every $q \in \mathbb{Z}_0^+$. Moreover, restricting the coboundary operator δ^q ($q \in \mathbb{Z}_0^+$) to $\bar{C}^q(\mathcal{U},\mathcal{T})$, we get the complex

$$\cdots \longrightarrow \bar{C}^{q-1}(\mathcal{U},\mathcal{T}) \xrightarrow{\delta^{q-1}} \bar{C}^q(\mathcal{U},\mathcal{T}) \xrightarrow{\delta^q} \bar{C}^{q+1}(\mathcal{U},\mathcal{T}) \longrightarrow \cdots,$$

from which, analogously to (1.6.8a-b), (1.6.9) and (1.6.17), we define the $\mathcal{A}(X)$-modules

$$\bar{Z}^q(\mathcal{U},\mathcal{T}), \quad \bar{B}^q(\mathcal{U},\mathcal{T}), \quad \bar{H}^q(\mathcal{U},\mathcal{T}),$$

also called modules of ***liftable q-cocycles***, ***q-coboundaries***, and ***q-cohomology***, respectively. Accordingly, we define the ***liftable (Čech) cohomology*** $\bar{H}^*(X,\mathcal{T})$ of X with coefficients in $\mathcal{T}$. Then, working as in the case of the presheaf cohomology long exact sequence, we obtain the long exact sequence in liftable Čech cohomology

$$\begin{aligned} 0 \longrightarrow &\check{H}^0(X,\mathcal{R}) \xrightarrow{\psi^*} \check{H}^0(X,\mathcal{S}) \xrightarrow{\phi^*} \bar{H}^0(X,\mathcal{T}) \xrightarrow{\delta^*} \check{H}^1(X,\mathcal{R}) \\ &\xrightarrow{\psi^*} \check{H}^1(X,\mathcal{S}) \xrightarrow{\phi^*} \bar{H}^1(X,\mathcal{T}) \xrightarrow{\delta^*} \check{H}^2(X,\mathcal{R}) \longrightarrow \ldots \\ &\check{H}^q(X,\mathcal{R}) \xrightarrow{\psi^*} \check{H}^q(X,\mathcal{S}) \xrightarrow{\phi^*} \bar{H}^q(X,\mathcal{T}) \xrightarrow{\delta^*} \check{H}^{q+1}(X,\mathcal{R}) \longrightarrow \cdots \end{aligned} \tag{1.6.28}$$

Note that, in the present case, Diagram 1.14 needs to be replaced by Diagram 1.15 on the next page.

$$
\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & C^{q+2}(\mathcal{U},\mathcal{R}) & \xrightarrow{\psi} & C^{q+2}(\mathcal{U},\mathcal{S}) & \xrightarrow{\phi} & \bar{C}^{q+2}(\mathcal{U},\mathcal{T}) & \longrightarrow & 0 \\
 & & \delta^{q+1}\uparrow & \textbf{(I)} & \uparrow\delta^{q+1} & & \uparrow\delta^{q+1} & & \\
0 & \longrightarrow & C^{q+1}(\mathcal{U},\mathcal{R}) & \xrightarrow{\psi} & C^{q+1}(\mathcal{U},\mathcal{S}) & \xrightarrow{\phi} & \bar{C}^{q+1}(\mathcal{U},\mathcal{T}) & \longrightarrow & 0 \\
 & & \delta^{q}\uparrow & \textbf{(II)} & \uparrow\delta^{q} & \textbf{(III)} & \uparrow\delta^{q} & & \\
0 & \longrightarrow & C^{q}(\mathcal{U},\mathcal{R}) & \xrightarrow{\psi} & C^{q}(\mathcal{U},\mathcal{S}) & \xrightarrow{\phi} & \bar{C}^{q}(\mathcal{U},\mathcal{T}) & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}
$$

Diagram 1.15

At this stage the assumption that X is a (Hausdorff) paracompact space (see (1.6.21)) enters the scene. Namely, one can prove that:

> *If X is a (Hausdorff) paracompact space, then there exists an isomorphism of $\mathcal{A}(X)$-modules*

(1.6.29)
$$\bar{H}^q(X,\mathcal{T}) \xrightarrow{\simeq} \check{H}^q(X,\mathcal{T});$$

> *thus, the short exact sequence of $\mathcal{A}$-modules (1.6.27) induces the long exact cohomology sequence*

(1.6.30)
$$
\begin{aligned}
0 \longrightarrow \check{H}^0(X,\mathcal{R}) &\xrightarrow{\psi^*} \check{H}^0(X,\mathcal{S}) \xrightarrow{\phi^*} \check{H}^0(X,\mathcal{T}) \xrightarrow{\delta^*} \check{H}^1(X,\mathcal{R}) \\
&\xrightarrow{\psi^*} \check{H}^1(X,\mathcal{S}) \xrightarrow{\phi^*} \check{H}^1(X,\mathcal{T}) \xrightarrow{\delta^*} \check{H}^2(X,\mathcal{R}) \longrightarrow \dots \\
\check{H}^q(X,\mathcal{R}) &\xrightarrow{\psi^*} \check{H}^q(X,\mathcal{S}) \xrightarrow{\phi^*} \check{H}^q(X,\mathcal{T}) \xrightarrow{\delta^*} \check{H}^{q+1}(X,\mathcal{R}) \longrightarrow \cdots
\end{aligned}
$$

A key factor to the proof of (1.6.29) and (1.6.30) is the following result:

(1.6.31) *Let* $\mathcal{S} \xrightarrow{\phi} \mathcal{T} \longrightarrow 0$ *be an exact sequence of* $\mathcal{A}$*-modules over a (Hausdorff) paracompact space* X*, and let* $\mathcal{U}$ *be an open covering of* X*. Then, for any* q*-cochain* $f \in C^q(\mathcal{U}, \mathcal{T})$*,* $q \in \mathbb{Z}_0^+$*, there is an open refinement* $\mathcal{V}$ *of* $\mathcal{U}$ *with a refining map* $\tau : J \to I$ *such that* $\tau_q(f) \in \phi_{\mathcal{V},q}\,(C^q(\mathcal{V}, \mathcal{S})) =: \bar{C}^q(\mathcal{V}, \mathcal{T})$*.*

Stated otherwise, the previous result asserts that any q-cochain f, as above, is ***refinement liftable***. Hence, there exists a q-cochain $g \in C^q(\mathcal{V}, \mathcal{S})$ such that $\phi_{\mathcal{V},q}(g) = \tau_q(f)$.

The proof of the previous statement can be found, in one form or another, in most of the references cited in the introduction of the present chapter. A particularly detailed proof is given in Mallios [62, Vol. I, Lemma 5.2]. The reader may also consult the same source (pp. 202–207) for equally detailed proofs of (1.6.29) and (1.6.30).

We close with the following fundamental fact:

(1.6.32) *Over a (Hausdorff) paracompact space* X*, all the cohomology theories with coefficients in* $\mathcal{A}$*-modules coincide up to isomorphism.*

1.6.3. Resolutions and the abstract de Rham theorem

Another approach to cohomology is based on resolutions. Although it is not employed in this work, we shall describe its rudiments in order to state the theorem in the title, needed only in Section 9.5.

An (abstract) ***complex of*** $\mathcal{A}$***-modules*** over a topological space X, denoted by

$$\mathcal{C}^{\bullet} \equiv (\mathcal{C}^q, d^q)_{q\in\mathbb{Z}},$$

is a sequence of $\mathcal{A}$-modules and morphisms

$$\cdots \longrightarrow \mathcal{C}^{q-1} \xrightarrow{d^{q-1}} \mathcal{C}^q \xrightarrow{d^q} \mathcal{C}^{q+1} \longrightarrow \cdots$$

(over X), such that

$$d^q \circ d^{q-1} = 0, \qquad q \in \mathbb{Z}$$

(compare with the cochain complexes defined in Section 1.6.1). The morphisms d^q are traditionally called the *differentials* of the complex.

If $\mathcal{S}$ is an $\mathcal{A}$-module over X, then a ***resolution*** of $\mathcal{S}$ is an (abstract) complex of $\mathcal{A}$-modules of *positive degree*, i.e.,

(1.6.33)
$$0 \longrightarrow \mathcal{C}^0 \xrightarrow{d^0} \mathcal{C}^1 \xrightarrow{d^1} \mathcal{C}^2 \longrightarrow \cdots$$

such that the augmented sequence

(1.6.34) $$0 \longrightarrow \mathcal{S} \xrightarrow{\ i\ } \mathcal{C}^0 \xrightarrow{d^0} \mathcal{C}^1 \xrightarrow{d^1} \mathcal{C}^2 \longrightarrow \cdots$$

is *exact*, for some $\mathcal{A}$-morphism i.

A resolution, as above, is called ***acyclic*** if $\check{H}^p(X, \mathcal{C}^q) = 0$, for all $p > 0$ and $q \geq 0$. Here we have used the Čech cohomology defined earlier, although the definition of acyclicity remains valid for any other cohomology theory we might use.

A common example of acyclic resolution is a ***fine resolution***; that is, one whose modules $\mathcal{C}^q$, $q \in \mathbb{Z}_0^+$, are fine sheaves. More precisely, a sheaf $\mathcal{F}$ of $\mathcal{A}$-modules (over X) is said to be ***fine***, if, for every *locally finite* open covering $\mathcal{U} = (U_\alpha)_{\alpha \in I}$ of X, there exists a ***partition of unity of $\mathcal{F}$ subordinate to $\mathcal{U}$***. By the last term we mean a family of endomorphisms

$$\{f_\alpha : \mathcal{F} \to \mathcal{F} \mid \alpha \in I\},$$

with the following properties:

$$\sum_{\alpha \in I} f_\alpha = id|_{\mathcal{F}},$$

$$\operatorname{supp}(f_\alpha) := \overline{\{x \in X : \ f_\alpha|_{\mathcal{F}_x} \neq 0\}} \subseteq U_\alpha.$$

For instance, if X is a (Hausdorff) paracompact space, then the sheaf $\mathcal{C}_X^0$ of germs of continuous $\mathbb{K}$-valued functions on X is fine. Similarly, if X is a (Hausdorff) paracompact *smooth manifold*, the sheaf of germs of $\mathbb{K}$-valued smooth functions $\mathcal{C}_X^\infty$, and the sheaf differential forms Ω_X are also fine. On the contrary, constant sheaves, and the sheaf of germs of holomorphic functions over a complex analytic manifold are *not* fine.

To facilitate the concluding discussion of this subsection, we define some other important categories of sheaves.

A sheaf $\mathcal{S}$ is called ***soft*** if every section of $\mathcal{S}$ over a *closed* subset of the base X can be extended to a section over the entire X. This is equivalent to saying that the restriction map $\mathcal{S}(X) \to \mathcal{S}(B)$ is *surjective*, for every closed $B \subseteq X$. *Fine sheaves are soft.*

On the other hand, a sheaf $\mathcal{F}$ is called ***flabby*** if the restriction map $\mathcal{F}(X) \to \mathcal{F}(U)$ is *surjective*, for every *open* $U \subseteq X$. *Flabby sheaves are soft.*

Finally, an $\mathcal{A}$-module $\mathcal{E}$ is called ***injective*** if, for any exact sequence of $\mathcal{A}$-modules

$$0 \longrightarrow \mathcal{S} \xrightarrow{\ \phi\ } \mathcal{F},$$

the induced sequence

$$\mathrm{Hom}_{\mathcal{A}}(\mathcal{F},\mathcal{E}) \xrightarrow{\phi^*} \mathrm{Hom}_{\mathcal{A}}(\mathcal{S},\mathcal{E}) \longrightarrow 0$$

is also exact. The induced morphism ϕ^* is given by $\phi^*(f) := f \circ \phi$.

Now, given the resolution (1.6.34), the global section functor Γ_X induces the sequence of $\mathcal{A}(X)$-modules

$$\begin{aligned} 0 \longrightarrow \Gamma_X(\mathcal{S}) \xrightarrow{\Gamma_X(i)} \Gamma_X(\mathcal{C}^0) \xrightarrow{\Gamma_X(d^0)} \\ \Gamma_X(\mathcal{C}^1) \xrightarrow{\Gamma_X(d^1)} \Gamma_X(\mathcal{C}^2) \longrightarrow \cdots \end{aligned} \tag{1.6.35}$$

From (1.1.1′), (1.1.2) and (1.2.4b), we recall that $\Gamma_X(\mathcal{S}) = \Gamma(X,\mathcal{S}) \equiv \mathcal{S}(X)$, for any sheaf $\mathcal{S}$, and $\Gamma_X(\phi) = \bar{\phi}_X$, for any morphism ϕ.

The sequence (1.6.35) is not necessarily exact, except at $\Gamma_X(\mathcal{S})$ and $\Gamma_X(\mathcal{C}^0)$. On the other hand,

$$\Gamma_X(d^{q+1}) \circ \Gamma_X(d^q) = 0,$$

thus the $\mathcal{A}(X)$-modules $\Gamma_X(\mathcal{C}^q)$, together with the $\mathcal{A}(X)$-morphisms $\Gamma_X(d^q)$, for all $q \geq 0$, form a complex.

Taking into account the previous considerations, the so-called ***abstract de Rham theorem*** is stated as follows:

> *If the sequence (1.6.34) is an acyclic resolution of $\mathcal{S}$ over a (Hausdorff) paracompact space X, then the following isomorphisms of $\mathcal{A}(X)$-modules hold true:*

$$\begin{aligned} \check{H}^0(X,\mathcal{S}) &\cong \ker \Gamma_X(i), \\ \check{H}^q(X,\mathcal{S}) &\cong \ker \Gamma_X(d^q)/\mathrm{im}\, \Gamma_X(d^{q-1}), \quad q \geq 1. \end{aligned} \tag{1.6.36}$$

To close, we mention that another way of building up a cohomology theory of a space X with coefficients in a sheaf $\mathcal{S}$ is by associating $\mathcal{S}$ with a certain *acyclic* resolution. Then the cohomology modules (groups) are obtained as quotients of (abstract) cocycles by coboundaries, by means of derived functors (like Γ_X). For instance, we can associate $\mathcal{S}$ with ***injective resolutions*** (see, e.g., Brylinski [17], Mallios [62, Vol. I]), ***fine*** or ***soft resolutions*** (see, e.g., Warner [140], Wells [142]), as well as ***flabby*** ones (see, e.g., Bredon [16], Godement [33]). In all these cases, the isomorphisms (1.6.36) are still valid for the corresponding –via resolutions– cohomology groups $H^q(X,\mathcal{S})$.

1.6.4. The 1st cohomology set

Unlike the Čech cohomology with coefficients in an $\mathcal{A}$-module or in a sheaf of abelian groups, in the *non-abelian* case we cannot define cohomology groups of dimension greater than 1.

Let X be a fixed topological space and $\mathcal{G}$ a sheaf of (not necessarily abelian) *groups*. As in Subsection 1.6.1, given an open covering $\mathcal{U} = \{U_\alpha \,|\, \alpha \in I\}$ of X, we define the *groups* of 0 and 1-***cochains*** $C^0(\mathcal{U},\mathcal{G})$ and $C^1(\mathcal{U},\mathcal{G})$ respectively.

A 1-cochain $f \equiv (f_{\alpha\beta}) \in C^1(\mathcal{U},\mathcal{G})$ is called a ***1-cocycle*** if the condition

$$f_{\alpha\beta} \cdot f_{\beta\gamma} = f_{\alpha\gamma} \tag{1.6.37}$$

holds for all $\alpha, \beta, \gamma \in I$ with $U_{\alpha\beta\gamma} \neq \emptyset$. The set of $\mathcal{G}$-valued 1-cocycles over $\mathcal{U}$ is denoted, as usual, by $Z^1(\mathcal{U},\mathcal{G})$.

Two 1-cocycles $f, f' \in Z^1(\mathcal{U},\mathcal{G})$ are said to be ***cohomologous*** (or ***equivalent***), if there is a 0-cochain $h \equiv (h_\alpha) \in C^0(\mathcal{U},\mathcal{G})$, satisfying

$$f'_{\alpha\beta} = h_\alpha \cdot f_{\alpha\beta} \cdot h_\beta^{-1}, \tag{1.6.38}$$

for all $\alpha, \beta \in I$ with $U_{\alpha\beta} \neq \emptyset$. The corresponding quotient is, by definition, the ***1st cohomology set of $\mathcal{U}$ with coefficients in $\mathcal{G}$***

$$H^1(\mathcal{U},\mathcal{G}), \tag{1.6.39}$$

whose elements are denoted by $[f]_{\mathcal{U}} \equiv [(f_{\alpha\beta})]_{\mathcal{U}}$.

The previous set is equipped with the particular element $1_{\mathcal{U}}$, determined by the class of the trivial 1-cocycle $(f_{\alpha\beta}) = (\mathbf{1}|_{U_{\alpha\beta}})$, i.e.,

$$1_{\mathcal{U}} := [(\mathbf{1}|_{U_{\alpha\beta}})]_{\mathcal{U}}$$

(recall that $\mathbf{1}$ is the unit section of $\mathcal{G}$; see (1.1.5)).

If we take an open refinement $\mathcal{V} = (V_i)_{i\in J}$ of $\mathcal{U}$ and any refining map $\tau : J \to I$, as in the abelian case (see also (1.6.12)), we define the cochain maps

$$\tau_0 : C^0(\mathcal{U},\mathcal{G}) \longrightarrow C^0(\mathcal{V},\mathcal{G}) : f \equiv (f_\alpha) \mapsto \big(\tau_0(f)_i\big) := \big(f_{\tau(i)}\big|_{V_i}\big),$$
$$\tau_1 : C^1(\mathcal{U},\mathcal{G}) \longrightarrow C^1(\mathcal{V},\mathcal{G}) : f \equiv (f_{\alpha\beta}) \mapsto \big(\tau_1(f)_{ij}\big) := \big(f_{\tau(i)\tau(j)}\big|_{V_{ij}}\big).$$

The maps τ_0, τ_1 induce the morphism

$$t^{\mathcal{U}}_{\mathcal{V}} \equiv \tau_1^* : H^1(\mathcal{U},\mathcal{G}) \longrightarrow H^1(\mathcal{V},\mathcal{G}) : [f]_{\mathcal{U}} \mapsto [\tau_1(f)]_{\mathcal{V}}, \tag{1.6.40}$$

which is independent of the choice of τ (compare with (1.6.13) and (1.6.16)). This is quite straightforward now. Indeed, assume that $\bar{\tau} : J \to I$ is another refining map. Then, for every $f \in Z^1(\mathcal{U}, \mathcal{G})$ and $i, j \in J$, the cocycle condition (1.6.37) implies that

$$\bar{\tau}_1(f)_{ij} = h_i \cdot \tau_1(f)_{ij} \cdot h_j^{-1},$$

with $h_i := f_{\bar{\tau}(i)\tau(i)}$. Hence, $[\bar{\tau}_1(f)]_{\mathcal{V}} = [\tau_1(f)]_{\mathcal{V}}$, which proves the assertion.

Moreover, analogously to (1.6.20), it can be shown that (see also Mallios [62, Vol. I, pp. 183 and 274])

(1.6.41) *the morphisms* $t^{\mathcal{U}}_{\mathcal{V}}$ *are injective.*

As a result, we obtain the inductive system $\big(H^1(\mathcal{U}, \mathcal{G}),\, t^{\mathcal{U}}_{\mathcal{V}}\big)$, whose inductive limit (as $\mathcal{U}$ is running the set of all proper coverings of X) is the ***1st cohomology set of*** X ***with coefficients in*** $\mathcal{G}$; that is,

$$H^1(X, \mathcal{G}) := \varinjlim_{\mathcal{U}} H^1(\mathcal{U}, \mathcal{G}). \tag{1.6.42}$$

For every open covering $\mathcal{U}$ of X, there is a ***canonical map***

$$t_{\mathcal{U}} : H^1(\mathcal{U}, \mathcal{G}) \longrightarrow H^1(X, \mathcal{G}),$$

thus we define the cohomology classe

$$[(f_{\alpha\beta})] := t_{\mathcal{U}}\big([(f_{\alpha\beta})]_{\mathcal{U}}\big). \tag{1.6.43}$$

An immediate consequence of (1.6.41) is that

(1.6.44) *the canonical maps* $t_{\mathcal{U}}$ *are injective, for every* $\mathcal{U}$*.*

The 1st cohomology set is equipped with ***the distinguished element***

$$1 := t_{\mathcal{U}}(1_{\mathcal{U}}) \in H^1(X, \mathcal{G}).$$

It is independent of the choice of the element $1_{\mathcal{U}}$ corresponding to a particular covering $\mathcal{U}$. This follows from the definition of the trivial cocycle, and Diagram 1.11 adapted to the present data.

1.6.5. Čech hypercohomology

We outline the construction of certain hypercohomology groups, which will be applied in Theorem 6.7.2. For complete details we refer to Brylinski [17], Mallios [62, Vol. I] and their references on the subject. Here we mainly follow the terminology and notations of [62].

With this future application in mind, we consider a complex of $\mathcal{A}$-modules of positive degree (see the beginning of Subsection 1.6.3)

$$\mathcal{E}^{\bullet} = \big(\mathcal{E}^m,\, d \equiv \{d^m\}\big)_{m\in\mathbb{Z}_0^+}.$$

For convenience we also set $\mathcal{E}^m = 0$, for every $m < 0$.

Let $\mathcal{U}$ be an open covering of X. Fixing, for a moment, an $m \in \mathbb{Z}_0^+$, we may consider the chain complex (see also (1.6.7))

$$\check{C}^{\bullet}(\mathcal{U},\mathcal{E}^m) = \big(C^n(\mathcal{U},\mathcal{E}^m),\, \delta \equiv \{\delta^{n,m}\}\big)_{n\in\mathbb{Z}_0^+},$$

where each $\delta^{n,m} : C^n(\mathcal{U},\mathcal{E}^m) \to C^{n+1}(\mathcal{U},\mathcal{E}^m)$ is the usual coboundary operator. Varying now *both* n and m, we form a ***double complex*** of $\mathcal{A}$-modules

$$C^{\bullet}(\mathcal{U},\mathcal{E}^{\bullet},\delta,d) = \big(\{C^n(\mathcal{U},\mathcal{E}^m)\}_{(n,m)\in\mathbb{Z}_0^+\times\mathbb{Z}_0^+},\, \delta,\, d\big).$$

$$\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \\
 & & \uparrow{\scriptstyle d^{n,m}} & & \uparrow{\scriptstyle d^{n+1,m}} & & \\
\cdots & \xrightarrow{\delta^{n-1,m}} & C^n(\mathcal{U},\mathcal{E}^m) & \xrightarrow{\delta^{n,m}} & C^{n+1}(\mathcal{U},\mathcal{E}^m) & \xrightarrow{\delta^{n+1,m}} & \cdots \\
 & & \uparrow{\scriptstyle d^{n,m-1}} & (\mathbf{I}) & \uparrow{\scriptstyle d^{n+1,m-1}} & & \\
\cdots & \xrightarrow{\delta^{n-1,m-1}} & C^n(\mathcal{U},\mathcal{E}^{m-1}) & \xrightarrow{\delta^{n,m-1}} & C^{n+1}(\mathcal{U},\mathcal{E}^{m-1}) & \xrightarrow{\delta^{n+1,m-1}} & \cdots \\
 & & \uparrow{\scriptstyle d^{n,m-2}} & & \uparrow{\scriptstyle d^{n+1,m-2}} & & \\
 & & \vdots & & \vdots & &
\end{array}$$

Diagram 1.16

The previous diagram illustrates the double complex just described. By definition, all the sub-diagrams like (I) are assumed to be commutative. The vertical operators between various cochains are induced by the corresponding differentials of the given complex $\mathcal{E}$.

A double complex gives rise to an ordinary complex of $\mathcal{A}$-modules

$$\big\{\mathsf{tot}\big(C^{\bullet}(\mathcal{U},\mathcal{E}^{\bullet})\big), D\big\} := (\mathcal{S}^p, D^p)_{p\in\mathbb{Z}_0^+},$$

whose elements are defined by the relations:

$$\mathcal{S}^p := \bigoplus_{n+m=p} C^n(\mathcal{U},\mathcal{E}^m),$$

$$D^p := \sum_{n+m=p} \delta^{n,m} + (-1)^n d^{n,m} \ : \mathcal{S}^p \longrightarrow \mathcal{S}^{p+1},$$

for every $p \in \mathbb{Z}_0^+$.

The ***p-th Čech hypercohomology group*** $\check{\boldsymbol{H}}^p(\mathcal{U},\mathcal{E}^{\bullet})$ ***of*** $\mathcal{U}$ ***with coefficients in the complex*** $\mathcal{E}^{\bullet}$ is defined to be the p-th cohomology group of the (total) complex $\mathsf{tot}\big(C^{\bullet}(\mathcal{U},\mathcal{E}^{\bullet})\big)$. In other words,

$$\check{\boldsymbol{H}}^p(\mathcal{U},\mathcal{E}^{\bullet}) = \ker D^p \big/ \mathrm{im}\, D^{p-1}, \qquad p \in \mathbb{Z}_0^+.$$

As in the case of the ordinary Čech cohomology, the ***p-dimensional Čech hypercohomology group of*** X ***with coefficients in*** $\mathcal{E}^{\bullet}$ is

$$\check{\boldsymbol{H}}^p(X,\mathcal{E}^{\bullet}) := \varinjlim_{\mathcal{U}} \check{\boldsymbol{H}}^p(\mathcal{U},\mathcal{E}^{\bullet}).$$

The inductive limit is taken with respect to all the proper open coverings of the space X.

Chapter 2

The category of differential triads

In physics there is an urgent necessity to base some geometric models of physical phenomena on sufficiently non smooth generalizations of the differential manifold concept.

M. Heller [42, p. 12]

This chapter introduces the notion of an *algebraized space*, which will be the base space of all the sheaves considered in the remainder of this work. An algebraized space, together with an appropriate *differential (Leibniz) operator,* determines a *differential triad*. The latter lies in the foundations of the abstract differential geometry alluded to in the preface.

These notions, originally due to A. Mallios, have been defined in [62]. We supplement his treatment by showing that differential triads form a category containing, as subcategories, the smooth manifolds and –after a suitable sheafification– the differential spaces in the sense of R. Sikorski [113] (see also the seminal ideas of his earlier work [112]). Subspaces, quotients, infinite products, projective and inductive limits also exist in this category.

Finally, a convenient notion of *abstract differentiation* for maps between two topological spaces is defined, provided that at least one of the spaces bears a differential triad. This notion of differentiability (having nothing to do with ordinary calculus) implies that *every continuous map is differentiable in abstracto.*

The categorical results of this chapter are due to M. Papatriantafillou and are taken from [97], [99] and [100]. With the exception of Sections 2.1 and 2.5, the remainder of this chapter can be omitted on a first reading.

2.1. Differential triads

Before proceeding to the fundamental definitions, we note that, in order to keep track of ordinary differential geometry and to clarify its analogy with the present abstraction,

> we deliberately use the terms `differential, differentiable` etc., although there is not any kind of differentiation, in the usual sense of calculus, involved.

Following [62, Chapter II, Scholium 1.2] we start with:

2.1.1 Definition. An ***algebraized space*** is a pair

$$(X, \mathcal{A}), \tag{2.1.1}$$

where $X \equiv (X, \mathfrak{T}_X)$ is a topological space and $\mathcal{A}$ a sheaf of *unital commutative associative* $\mathbb{K}$-*algebras* over X, with $\mathbb{K} = \mathbb{R}, \mathbb{C}$.

Completing the terminology of Definition 2.1.1, X is called the ***base space*** and $\mathcal{A}$ the ***structure sheaf*** of the algebraized space.

However, in order to develop abstract differential geometry, we need to complement the algebraized space by some *algebraic* differential (or derivation). This leads to the following:

2.1.2 Definition. A ***differential triad*** is a triplet $(\mathcal{A}, d, \Omega)$, where Ω is an $\mathcal{A}$-*module* over X, and

$$d : \mathcal{A} \longrightarrow \Omega \tag{2.1.2}$$

is a $\mathbb{K}$-linear morphism satisfying the ***Leibniz condition***

$$d(a \cdot b) = a \cdot db + b \cdot da, \quad (a, b) \in \mathcal{A} \times_X \mathcal{A}. \tag{2.1.3}$$

The previous definition means that d is an ***Ω-valued derivation of** $\mathcal{A}$*. Condition (2.1.3) can be written in the equivalent form

$$d(s \cdot t) = s \cdot dt + t \cdot ds, \tag{2.1.3$'$}$$

for any (local) sections $s, t \in \mathcal{A}(U)$ and every (open) $U \in \mathfrak{T}_X$, as explained in (1.2.15$'$). We clarify that the operator d in (2.1.3$'$) is in fact the induced morphism $\bar{d}_U$ between the corresponding presheaves of sections, according to our convention (1.1.3).

Note. In Mallios [62] d is denoted by ∂. As already mentioned in the note on p. 42, ∂ is reserved for another use in this work.

In order to state a simple consequence of Definition 2.1.2, we need to add a few remarks on the constant sheaf $\mathbb{K}_X$, $\mathbb{K} = \mathbb{R}, \mathbb{C}$ (see Subsection 1.3.1). By the very construction, $\mathbb{K} \equiv \mathbb{K}_X$ is naturally imbedded in $\mathcal{A}$ by means of the 1-1 morphism

$$i : \mathbb{K}_X \hookrightarrow \mathcal{A} \, : \, (x, k) \mapsto k \cdot e_x = k \cdot 1(x). \tag{2.1.4}$$

Since, $(\mathbb{K}_X)_x = \{x\} \times \mathbb{K} \cong \mathbb{K}$, identifying $k \in \mathbb{K}$ with (x, k), for arbitrary $x \in X$, we obtain:

2.1.3 Proposition. *The differential d satisfies the following equalities:*

$$de_x = 0 \quad \textit{and} \quad dk = 0,$$

for every $x \in X$ and $k \in \mathbb{K}$. Similar equalities hold for the unit section $1 \in \mathcal{A}(X)$ and any section $k \in \mathbb{K}_X(X)$.

Proof. Applying (2.1.3), we have that

$$de_x = d(e_x \cdot e_x) = e_x \cdot de_x + e_x \cdot de_x = de_x + de_x$$

which yields the first equality of the statement. The second is a result of the $\mathbb{K}$-linearity of d, namely

$$dk = d(k \cdot e_x) = k \cdot de_x = 0 \qquad \square$$

2.1.4 Examples.

(a) Differential triads from smooth manifolds

Let X be a *real* C^∞ *(smooth) manifold.* If $\mathcal{A} := \mathcal{C}^\infty_X$ is the sheaf of germs of real-valued smooth functions on X, $\Omega := \Omega^1_X$ the sheaf of germs of (smooth) differential 1-forms on X, and d the morphism induced by the sheafification of the ordinary differentiation of smooth functions, then

$$(\mathcal{C}^\infty_X,\, d,\, \Omega^1_X)$$

is the *standard differential triad* associated with the smooth manifold X. In the notations of Subsection 1.2.2,

$$\begin{aligned}\mathcal{C}^\infty_X &= \mathbf{S}\big(U \longmapsto C^\infty(U,\mathbb{R})\big),\\ \Omega^1_X &= \mathbf{S}\big(U \longmapsto \Lambda^1(U,\mathbb{R})\big),\\ d &= \mathbf{S}\big(U \longmapsto d_U\big),\end{aligned}$$

for all U running the topology $\mathfrak{T}_X$. Here $\Lambda^1(U,\mathbb{R})$ is the $C^\infty(U,\mathbb{R})$-module of real-valued differential 1-forms on U, and

$$d_U : C^\infty(U,\mathbb{R}) \longrightarrow \Lambda^1(U,\mathbb{R}) : f \to df \equiv Tf,$$

where the last differential is the ordinary differential of smooth functions.

By the completeness of the presheaves involved above, we have that

$$\mathcal{A}(U) = \mathcal{C}^\infty_X(U) \cong C^\infty(U,\mathbb{R}),\;\; \Omega(U) = \Omega^1_X(U) \cong \Lambda^1(U,\mathbb{R}),\;\; \bar{d}_U \equiv d_U.$$

Differential triads of the previous type are obtained from finite and infinite-dimensional manifolds. The latter include Banach manifolds (see, e.g., Bourbaki [13], Lang [54]) and manifolds with other infinite-dimensional models (i.e., topological vector spaces) equipped with an appropriate differentiation theory, allowing to define the usual differential mechanism on the corresponding manifolds. In this respect we refer, e.g., to Galanis [30], Omori [85], Papaghiuc [88] (for Fréchet manifolds), Kriegl-Michor [52] (for manifolds modelled on convenient locally convex spaces), Leslie [55], Papaghiuc [87], Papatriantafillou [94] (for manifolds modelled on arbitrary topological vector spaces), Papatriantafillou [95] (for manifolds modelled on projective finitely generated modules over a commutative locally m-convex algebra with unit). For a systematic treatment of the general differentiation theory we refer to Averbukh-Smolyanov [7], [8]. Another valuable source of information, containing a very extensive and annotated bibliography, is Ver Eecke [135], [136].

Similar considerations are valid in the case of complex manifolds by taking holomorphic functions and forms.

(b) Kähler's differential

In the sheaf-theoretic framework, one may wonder whether a given arbitrary algebraized space $(X, \mathcal{A})$ can be completed to a differential triad. The sheafification of W. Kähler's theory of differentials guarantees the existence of an $\mathcal{A}$-module Ω and of an operator d so that $(\mathcal{A}, d, \Omega)$ is a differential triad.

Kähler's theory is based on an nice algebraic construction, whose main idea is the following: We start with a unital commutative algebra $\mathbb{A}$ (even less: a unital commutative ring!) and consider the multiplication morphism

$$\mu : \mathbb{A} \otimes \mathbb{A} \ni a \otimes b \mapsto \mu(a \otimes b) := a \cdot b \in \mathbb{A}.$$

Then $I := \ker \mu$ and the vector space I^2 (generated by $I \cdot I$) are ideals of $\mathbb{A} \otimes \mathbb{A}$ with $I^2 \subset I$. Setting

$$\Omega := I/I^2,$$

one verifies that Ω is an $\mathbb{A}$-module. The ***1st-order Kähler derivation*** or ***differential*** $d : \mathbb{A} \to \Omega$ is defined by

$$da := (1 \otimes a - a \otimes 1) + I^2, \qquad a \in \mathbb{A}.$$

It has the following *universal property:* For any other derivation $\bar{d} : \mathbb{A} \to \bar{\Omega}$, there exists a (unique) morphism of $\mathbb{A}$-modules $f : \Omega \to \bar{\Omega}$ such that the following diagram is commutative.

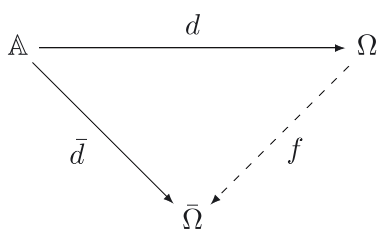


DIAGRAM 2.1

For details we refer to Bourbaki [14, Chap. III, p. 132], as well as to Mallios [62, Section XI.5] in the case of a *topological algebra*.

Now, given an arbitrary sheaf of unital commutative associative algebras $\mathcal{A}$, we construct a differential triad $(\mathcal{A}, d, \Omega)$ by the sheafification process. More precisely, to each $\mathcal{A}(U)$, $U \subseteq X$ open, we associate an $\mathcal{A}(U)$-module

$\Omega(U)$ and a differential $d_U : \mathcal{A}(U) \longrightarrow \Omega(U)$ as above. We obtain a presheaf of modules and a presheaf morphism, respectively, generating in turn an $\mathcal{A}$-module Ω and a differential d (see also Mallios [61] and [62, Section XI.6]).

(c) Differential triads from the derivations of $\mathcal{A}$

Let $(X, \mathcal{A})$ be a given algebraized space. An ***$\mathcal{A}$-valued derivation of*** $\mathcal{A}$ is a $\mathbb{K}$-linear morphism $\xi : \mathcal{A} \to \mathcal{A}$ satisfying the Leibniz condition

$$\xi(a \cdot b) = a \cdot \xi(b) + \xi(a) \cdot b, \qquad (a, b) \in \mathcal{A} \times_X \mathcal{A}.$$

Assume that $\mathcal{A}$ admits non-trivial derivations. For every $U \in \mathfrak{T}_X$, we denote by $\mathfrak{D}_U$ the set of $\mathcal{A}|_U$-valued derivations of $\mathcal{A}|_U$; i.e.,

$$\mathfrak{D}_U := \big\{\xi : \mathcal{A}|_U \to \mathcal{A}|_U \text{ derivations}\big\} \subseteq \mathrm{Hom}_{\mathcal{A}|_U}(\mathcal{A}|_U, \mathcal{A}|_U).$$

Then $\big(\mathfrak{D}_U, \rho^U_V\big)$ is a complete presheaf, where $\rho^U_V(\xi)$ is the restriction of the derivation ξ of $\mathcal{A}|_U$ to $\mathcal{A}|_V$, for every open $V \subseteq U$. If $\mathfrak{D}$ is the sheaf generated by $\big(\mathfrak{D}_U, \rho^U_V\big)$, we set

$$\Omega := \mathfrak{D}^* = \mathcal{H}om_{\mathcal{A}}(\mathfrak{D}, \mathcal{A})$$

(see Subsection 1.3.5).

We shall construct a differential $d : \mathcal{A} \to \Omega$. To this end, we consider the morphism of presheaves

$$\big\{d_U : \mathcal{A}(U) \longrightarrow \mathrm{Hom}_{\mathcal{A}|_U}(\mathfrak{D}|_U, \mathcal{A}|_U) \,|\, U \in \mathfrak{T}_X\big\},$$

where d_U is defined as follows: If $\alpha \in \mathcal{A}(U)$, then the morphism $d_U(\alpha) \in \mathrm{Hom}_{\mathcal{A}|_U}(\mathfrak{D}|_U, \mathcal{A}|_U)$ is generated by the presheaf morphism

$$\big\{d_U(\alpha)_V : \mathfrak{D}(V) \to \mathcal{A}(V) \,|\, V \subseteq U \text{ open}\big\}$$

with

$$d_U(\alpha)_V(\xi) := \xi(\alpha|_V), \qquad \xi \in \mathfrak{D}(V).$$

Varying V in U we get an element $d_U(\alpha) \in \Omega(U)$ and then, varying U in $\mathfrak{T}_X$, we obtain the desired d. The $\mathbb{K}$-linearity and the Leibniz condition follow from the analogous properties of each $d_U(\alpha)_V$. Therefore, the triplet $(\mathcal{A}, d, \Omega)$ thus produced is a differential triad.

We illustrate the previous general construction in the next example.

(d) A concretization of Example (c)

Let $X := \mathbb{E}$ be any *topological vector space* equipped with a method of differentiation such as Gâteaux, Hadamard, or any other appropriate one (the existence of a directional derivative would also suffice in this discussion). Fix a vector $v \in \mathbb{E}$. Then, for every open $U \subseteq \mathbb{E}$ and every smooth map $\alpha : U \to \mathbb{R}$, we set

$$\xi_U^v(\alpha)(x) := (D\alpha(x))(v); \qquad x \in U,$$

(the previous derivative can be replaced by $D_v\alpha(x)$, if we are given a directional derivative). Clearly, the map

$$\xi_U^v : C^\infty(U, \mathbb{R}) \longrightarrow C^\infty(U, \mathbb{R})$$

determines a derivation. Varying U in the topology of X, we obtain the derivation $\xi^v : \mathcal{A} \to \mathcal{A}$ where $\mathcal{A} := \mathcal{C}^\infty_{\mathbb{E}}$ is the sheaf of germs of smooth functions on $\mathbb{E}$, with respect to the chosen differentiation method. If

$$\Omega := \{\xi^v \mid v \in \mathbb{E}\}^*,$$

then, as in the preceding Example (c), we define a differential triad $(\mathcal{A}, d, \Omega)$ associated with $(\mathbb{E}, \mathcal{A})$.

Note that in this case, Ω does not necessarily coincide with the $\mathcal{A}$-module $\Omega^1_{\mathbb{E}}$ of 1-forms on $\mathbb{E}$.

(e) Generalized structures and differential triads

Differentiable spaces in the sense of M. Mostow (see [79]) provide differential triads as before. Likewise, we obtain a differential triad from a *differential space* in the sense of R. Sikorski ([113]). This is accomplished by associating, via the Gel'fand's representation, an appropriate sheaf of function algebras to the (functional) structure algebra of the differential space. Details can be found in Heller [42].

For other relevant examples, including triads obtained from the *general spaces* of J. Smith ([114]), *V-manifolds* of I. Satake ([107]) and *supermanifolds* (see, e.g., Bartocci-Bruzzo and Hermández-Ruipérez [9], though this case is within a *graded* framework), the reader is referred to Mallios [62, Vol. II, Chapters 10, 11].

(f) Differential triads from algebras of generalized functions

A rather surprising example of a differential triad is obtained from the sheaf of E. E. Rosinger's *nowhere dense differential algebras of generalized functions*. These algebras contain the Schwartz distributions and provide global solutions for arbitrary analytic nonlinear PDEs. The significance of this example lies in the fact that, taking as structure sheaf the above functions, we can reproduce a great deal of the classical theory of manifolds in a *highly singular* space. Details are given in Mallios-Rosinger [71] and (along with applications to general relativity) [72].

2.2. Morphisms of differential triads

In this section we prove that differential triads form a category, by defining an appropriate notion of morphisms between them. The definition of a morphism is not obvious, as the reader will soon witness, and is certainly more complicated than that of a morphism of differentiable manifolds. Despite this complexity, the category of differential triads is more advantageous than differentiable manifolds. This will become clear in Section 2.4.

Let $X \equiv (X, \mathfrak{T}_X)$, $Y \equiv (Y, \mathfrak{T}_Y)$ be two topological spaces and $f : X \to Y$ a continuous map. We assume that $(\mathcal{A}, d, \Omega)$ is a differential triad over X. By the general discussion of Subsection 1.4.2, we see that the push-out of $\mathcal{A}$

$$f_*(\mathcal{A}) \equiv (f_*(\mathcal{A}), Y, \pi_*)$$

is a sheaf of unital commutative associative $\mathbb{K}$-algebras over Y. We recall that

$$f_*(\mathcal{A}) := \mathbf{S}\left(V \longmapsto \mathcal{A}(f^{-1}(V))\right);$$

thus, by the completeness of the presheaf involved, $f_*(\mathcal{A})(V) \cong \mathcal{A}(f^{-1}(V))$. Similarly,

$$f_*(\Omega) := \mathbf{S}\left(V \longmapsto \Omega(f^{-1}(V))\right)$$

is an $f_*(\mathcal{A})$-module. On the other hand, the sheaf-morphism

$$f_*(d) : f_*(\mathcal{A}) \longrightarrow f_*(\Omega),$$

being the sheafification of the induced morphisms of sections

$$\bar{d}_{f^{-1}(V)} : \mathcal{A}(f^{-1}(V)) \longrightarrow \Omega(f^{-1}(V)); \qquad V \in \mathfrak{T}_Y,$$

is $\mathbb{K}$-linear and satisfies the Leibniz condition. Therefore, we obtain:

2.2.1 Lemma. *Let X, Y be topological spaces and $f : X \to Y$ a continuous map. If $(\mathcal{A}, d, \Omega)$ is a differential triad over X, then the push-out*

$$(f_*(\mathcal{A}), f_*(d), f_*(\Omega))$$

of $(\mathcal{A}, d, \Omega)$ by f is a differential triad over Y.

Before defining morphisms of differential triads, we recall that a ***morphism of unital algebras*** $h : \mathbb{A} \to \mathbb{B}$ is a morphism of algebras *preserving the units.* In the same way, a ***morphism*** $h : \mathcal{A} \to \mathcal{B}$ ***of sheaves of unital algebras*** is a morphism of sheaves of algebras also preserving the units. This means that the following equivalent conditions hold true:

$h(1_{\mathcal{A}}) = 1_{\mathcal{B}},$ if $1_{\mathcal{A}}$, $1_{\mathcal{B}}$ are the unit sections of $\mathcal{A}$ and $\mathcal{B}$, respectively;

$h(1_x) = 1_x,$ if 1_x denotes (for convenience) the unit element of both $\mathcal{A}_x$ and $\mathcal{B}_x$, for all $x \in X$.

2.2.2 Definition. Let $(\mathcal{A}_X, d_X, \Omega_X)$, $(\mathcal{A}_Y, d_Y, \Omega_Y)$ be differential triads over the respective topological spaces $X \equiv (X, \mathfrak{T}_X)$, $Y \equiv (Y, \mathfrak{T}_Y)$. A ***morphism of differential triads*** between $(\mathcal{A}_X, d_X, \Omega_X)$ and $(\mathcal{A}_Y, d_Y, \Omega_Y)$ is a triplet $(f, f_{\mathcal{A}}, f_{\Omega})$, where

(MDT. 1) $f : X \to Y$ is a continuous map.

(MDT. 2) $f_{\mathcal{A}} : \mathcal{A}_Y \to f_*(\mathcal{A}_X)$, shown in Diagram 2.2, is a morphism of sheaves of unital commutative associative $\mathbb{K}$-algebras over Y.

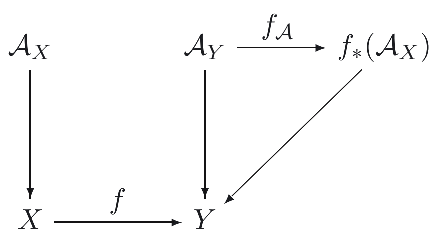


DIAGRAM 2.2

(MDT. 3) $f_{\Omega} : \Omega_Y \to f_*(\Omega_X)$ is a morphism of sheaves of $\mathbb{K}$-vector spaces over Y, compatible with the respective module structures; that is,

$$f_{\Omega}(a \cdot w) = f_{\mathcal{A}}(a) \cdot f_{\Omega}(w), \qquad (a, w) \in \mathcal{A}_Y \times_Y \Omega_Y. \tag{2.2.1}$$

(MDT. 4) The following equality holds true

$$f_*(d_X) \circ f_{\mathcal{A}} = f_{\Omega} \circ d_Y, \tag{2.2.2}$$

or, equivalently, the next diagram is commutative.

$$\begin{array}{ccc} \mathcal{A}_Y & \xrightarrow{f_{\mathcal{A}}} & f_*(\mathcal{A}_X) \\ {\scriptstyle d_Y}\downarrow & & \downarrow{\scriptstyle f_*(d_X)} \\ \Omega_Y & \xrightarrow{f_{\Omega}} & f_*(\Omega_X) \end{array}$$

DIAGRAM 2.3

The next obvious result proves that the set of endomorphisms of a differential triad is not empty.

2.2.3 Proposition. *For every differential triad* $(\mathcal{A}, d, \Omega)$, *over any topological space* X, $(id_X, id_{\mathcal{A}}, id_{\Omega})$ *is a morphism of differential triads.*

As a matter of fact, the previous morphism is an ***identity morphism*** in the *category of differential triads*, as it will be clarified shortly (see also Corollary 2.2.5). On the other hand, the following result defines the composition law for the morphisms of differential triads, formalized in (2.2.7) below.

2.2.4 Proposition. *Let* $(\mathcal{A}_I, d_I, \Omega_I)$ *be differential triads over the respective topological spaces* $(I, \mathfrak{T}_I)$, $I = X, Y, Z$. *Given two morphisms of differential triads*

$$(f, f_{\mathcal{A}}, f_{\Omega}):\ (\mathcal{A}_X, d_X, \Omega_X) \longrightarrow (\mathcal{A}_Y, d_Y, \Omega_Y),$$
$$(g, g_{\mathcal{A}}, g_{\Omega}):\ (\mathcal{A}_Y, d_Y, \Omega_Y) \longrightarrow (\mathcal{A}_Z, d_Z, \Omega_Z),$$

we define the morphisms

$$(g \circ f)_{\mathcal{A}} := g_*(f_{\mathcal{A}}) \circ g_{\mathcal{A}}, \tag{2.2.3}$$

$$(g \circ f)_{\Omega} := g_*(f_{\Omega}) \circ g_{\Omega}. \tag{2.2.4}$$

Then the triplet $\big(g \circ f, (g \circ f)_{\mathcal{A}}, (g \circ f)_{\Omega}\big)$ *is a morphism of differential triads from* $(\mathcal{A}_X, d_X, \Omega_X)$ *into* $(\mathcal{A}_Z, d_Z, \Omega_Z)$.

Proof. The construction of $(g \circ f)_{\mathcal{A}}$ is shown in the diagram on the next page. It is clearly a morphism of sheaves of unital commutative associative $\mathbb{K}$-algebras as being the composite of two morphisms of algebra sheaves of the said type. Similarly, $(g \circ f)_{\Omega}$ is a morphism of sheaves of $\mathbb{K}$-vector spaces.

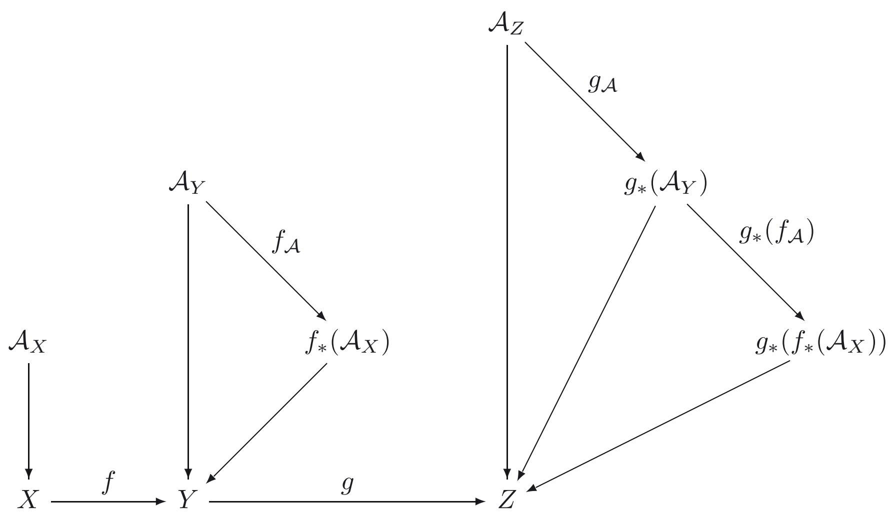


Diagram 2.4

To prove that $(g \circ f)_\Omega$ satisfies the analog of (2.2.1), we first observe that

$$g_*(f_\Omega)(a \cdot w) = g_*(f_{\mathcal{A}})(a) \cdot g_*(f_\Omega)(w), \qquad (a, w) \in \mathcal{A}_Z \times_Z \Omega_Z. \tag{2.2.5}$$

This is easily checked by applying (2.2.1) to the local sections generating the push-out. Hence, for every (a, w) as before, equalities (2.2.1) and (2.2.5), together with (2.2.3) and (2.2.4′), imply that

$$\begin{aligned} (g \circ f)_\Omega(a \cdot w) &= g_*(f_\Omega)\big(g_{\mathcal{A}}(a) \cdot g_\Omega(w)\big) \\ &= g_*(f_{\mathcal{A}})\big(g_{\mathcal{A}}(a)\big) \cdot g_*(f_\Omega)\big(g_\Omega(w)\big) \\ &= \big(g_*(f_{\mathcal{A}}) \circ g_{\mathcal{A}}\big)(a) \cdot \big(g_*(f_\Omega) \circ g_\Omega\big)(w) \\ &= (g \circ f)_{\mathcal{A}}(a) \cdot (g \circ f)_\Omega(w), \end{aligned}$$

thus proving condition (MDT. 3) of Definition 2.2.2.

We now verify (MDT. 4). To this end, we apply g_* to both sides of (2.2.2), thus the functoriality of the push-out implies that

$$g_*(f_*(d_X)) \circ g_*(f_{\mathcal{A}}) = g_*(f_\Omega) \circ g_*(d_Y),$$

and, composing with $g_{\mathcal{A}}$,

$$(g \circ f)_*(d_X) \circ g_*(f_{\mathcal{A}}) \circ g_{\mathcal{A}} = g_*(f_\Omega) \circ g_*(d_Y) \circ g_{\mathcal{A}}. \tag{2.2.6}$$

On the other hand, the analog of (2.2.2) for $(g, g_{\mathcal{A}}, g_{\Omega})$, namely

$$g_*(d_Y) \circ g_{\mathcal{A}} = g_{\Omega} \circ d_Z,$$

substituted in (2.2.6) yields

$$(g \circ f)_*(d_X) \circ g_*(f_{\mathcal{A}}) \circ g_{\mathcal{A}} = g_*(f_{\Omega}) \circ g_{\Omega} \circ d_Z.$$

Hence, in virtue of (2.2.3) and (2.2.4), the last equality leads to

$$(g \circ f)_*(d_X) \circ (g \circ f)_{\mathcal{A}} = (g \circ f)_{\Omega} \circ d_Z,$$

which concludes the proof. □

Given two morphisms $(f, f_{\mathcal{A}}, f_{\Omega})$ and $(g, g_{\mathcal{A}}, g_{\Omega})$ as in Proposition 2.2.4, we define their composition in the obvious way; that is,

$$(g, g_{\mathcal{A}}, g_{\Omega}) \circ (f, f_{\mathcal{A}}, f_{\Omega}) := \big(g \circ f, (g \circ f)_{\mathcal{A}}, (g \circ f)_{\Omega}\big), \tag{2.2.7}$$

where the right-hand side is determined by (2.2.3) and (2.2.4). This is precisely the ***composition law*** for the morphisms of differential triads alluded to in the comments preceding the statement of Proposition 2.2.4.

2.2.5 Corollary. *The following assertions are true:*

i) If $(f, f_{\mathcal{A}}, f_{\Omega}) : (\mathcal{A}_X, d_X, \Omega_X) \longrightarrow (\mathcal{A}_Y, d_Y, \Omega_Y)$ *is a morphism of differential triads, then*

$$(f, f_{\mathcal{A}}, f_{\Omega}) \circ (id_X, id_{\mathcal{A}_X}, id_{\Omega_X}) = (f, f_{\mathcal{A}}, f_{\Omega}),$$
$$(id_Y, id_{\mathcal{A}_Y}, id_{\Omega_Y}) \circ (f, f_{\mathcal{A}}, f_{\Omega}) = (f, f_{\mathcal{A}}, f_{\Omega}).$$

ii) The composition law, defined by (2.2.7), is associative.

Proof. The equalities of the first assertion are direct consequences of the definitions. For the second it suffices to show that

$$((h \circ g) \circ f)_{\mathcal{X}} = (h \circ (g \circ f))_{\mathcal{X}}; \qquad \mathcal{X} = \mathcal{A}, \Omega,$$

for any morphisms $(f, f_{\mathcal{A}}, f_{\Omega})$, $(g, g_{\mathcal{A}}, g_{\Omega})$ as in Proposition 2.2.4, and any morphism of differential triads $(h, h_{\mathcal{A}}, h_{\Omega}) : (\mathcal{A}_Z, d_Z, \Omega_Z) \to (\mathcal{A}_W, d_W, \Omega_W)$, where W is a topological space. Indeed, since h_* is a functor,

$$\begin{aligned}
((h \circ g) \circ f)_{\mathcal{X}} &= (h \circ g)_*(f_{\mathcal{X}}) \circ (h \circ g)_{\mathcal{X}} \\
&= h_*\left(g_*(f_{\mathcal{X}})\right) \circ \left(h_*(g_{\mathcal{X}}) \circ h_{\mathcal{X}}\right) \\
&= \big(h_*\left(g_*(f_{\mathcal{X}})\right) \circ h_*(g_{\mathcal{X}})\big) \circ h_{\mathcal{X}} \\
&= h_*\big(g_*(f_{\mathcal{X}}) \circ g_{\mathcal{X}}\big) \circ h_{\mathcal{X}} \\
&= h_*\big((g \circ f)_{\mathcal{X}}\big) \circ h_{\mathcal{X}} \\
&= (h \circ (g \circ f))_{\mathcal{X}}.
\end{aligned}$$

□

As already mentioned after Proposition 2.2.3, Corollary 2.2.5 implies that $(id_X, id_{\mathcal{A}}, id_{\Omega})$ is an identity for the composition law, while the latter is also associative. Therefore, we have proved that the differential triads and their morphisms satisfy the axioms of a category.

For the sake of completeness, we record the previous results in the following statement.

2.2.6 Theorem. *The differential triads and their morphisms, together with the composition law (2.2.7), form a category, denoted by* $\mathcal{DT}$.

The category $\mathcal{DT}$ contains in a natural manner the category of smooth manifolds. In fact, if the latter is denoted by $\mathcal{DM}$, we prove the following:

2.2.7 Theorem. *There exists an imbedding* $F : \mathcal{DM} \hookrightarrow \mathcal{DT}$.

Proof. Let X be a C^∞-manifold and let $(\mathcal{A}_X, d_X, \Omega_X) := (\mathcal{C}^\infty_X,\, d,\, \Omega^1_X)$ be the standard differential triad of X, defined in Example 2.1.4(a). Then we set $F(X) := (\mathcal{A}_X, d_X, \Omega_X)$. Now, if $f : X \to Y$ is a C^∞-map, then we define the morphism of differential triads $F(f) := (f, f_{\mathcal{A}}, f_{\Omega})$ as follows: Since $\mathcal{A}_Y := \mathcal{C}^\infty_Y$ is generated by the presheaf $V \mapsto C^\infty(V, \mathbb{R})$, whereas $f_*(\mathcal{A}_X) = f_*(\mathcal{C}^\infty_X)$ is generated by $V \mapsto C^\infty(f^{-1}(V), \mathbb{R})$ (with V running in $\mathfrak{T}_Y$), $f_{\mathcal{A}}$ is defined to be the morphism generated by the morphism of presheaves

$$\left\{f_{\mathcal{A},V} : C^\infty(V,\mathbb{R}) \longrightarrow C^\infty(f^{-1}(V),\mathbb{R}) :\ \alpha \mapsto \alpha \circ f\right\}_{V\in\mathfrak{T}_Y}.$$

Similarly, f_Ω is the morphism generated by

$$\left\{f_{\Omega,V} : \Lambda^1(V,\mathbb{R}) \longrightarrow \Lambda^1(f^{-1}(V),\mathbb{R}) :\ \omega \mapsto \omega \circ df\right\}_{V\in\mathfrak{T}_Y},$$

where the last df is the ordinary differential of the smooth map f.

Conditions (MDT. 2) and (MDT. 3) of Definition 2.2.2 follow directly from the corresponding properties of the presheaf morphisms $(f_{\mathcal{A},V})$ and $(f_{\Omega,V})$. Condition (MDT. 4) is verified locally by taking into account the properties of the ordinary differential of smooth maps. Hence, $F(f)$ is a morphism of differential triads.

Moreover, in virtue of (2.2.7), we see that $F(g \circ f) = F(g) \circ F(f)$, thus F is a covariant functor between the aforementioned categories. Finally, $F(f) = F(g)$ yields $f = g$, for any smooth maps in $\mathcal{DM}$. This completes the proof. □

2.3. Products of differential triads

Continuing the investigation of the category $\mathcal{DT}$, we shall show the existence of (finite) ***products*** therein.

According to the theory of categories, the product of two differential triads $(\mathcal{A}_X, d_X, \Omega_X)$ and $(\mathcal{A}_Y, d_Y, \Omega_Y)$, over the respective topological spaces X and Y, should be a differential triad $(\mathcal{A}_P, d_P, \Omega_P)$, over some topological space P, together with two morphisms of differential triads, called ***projections***,

$$(p, p_{\mathcal{A}}, p_{\Omega}) : (\mathcal{A}_P, d_P, \Omega_P) \longrightarrow (\mathcal{A}_X, d_X, \Omega_X),$$
$$(q, q_{\mathcal{A}}, q_{\Omega}) : (\mathcal{A}_P, d_P, \Omega_P) \longrightarrow (\mathcal{A}_Y, d_Y, \Omega_Y),$$

satisfying the following ***universal property***: if $(\mathcal{A}_Z, d_Z, \Omega_Z)$ is any differential triad and

$$(f, f_{\mathcal{A}}, f_{\Omega}) : (\mathcal{A}_Z, d_Z, \Omega_Z) \longrightarrow (\mathcal{A}_X, d_X, \Omega_X),$$
$$(g, g_{\mathcal{A}}, g_{\Omega}) : (\mathcal{A}_Z, d_Z, \Omega_Z) \longrightarrow (\mathcal{A}_Y, d_Y, \Omega_Y)$$

are morphisms of differential triads, then there exists a *unique* morphism of differential triads

$$(h, h_{\mathcal{A}}, h_{\Omega}) : (\mathcal{A}_Z, d_Z, \Omega_Z) \rightarrow (\mathcal{A}_P, d_P, \Omega_P)$$

such that

$$(p, p_{\mathcal{A}}, p_{\Omega}) \circ (h, h_{\mathcal{A}}, h_{\Omega}) = (f, f_{\mathcal{A}}, f_{\Omega}),$$
$$(q, q_{\mathcal{A}}, q_{\Omega}) \circ (h, h_{\mathcal{A}}, h_{\Omega}) = (g, g_{\mathcal{A}}, g_{\Omega}).$$

The universal property of the product is shown in the typical Diagram 2.5 on the next page.

For the construction of the product, we first take

$$P := X \times Y.$$

Moreover, motivated by the classical formula (see Mallios [57, p. 490])

$$\mathcal{C}^{\infty}(X \times Y) = \mathcal{C}^{\infty}(X) \otimes \mathcal{C}^{\infty}(Y),$$

if X and Y are smooth manifolds, we consider the presheaf

$$U \times V \longmapsto \mathcal{A}_X(U) \otimes \mathcal{A}_Y(V), \qquad U \times V \in \mathfrak{T}_X \times \mathfrak{T}_Y \tag{2.3.1}$$

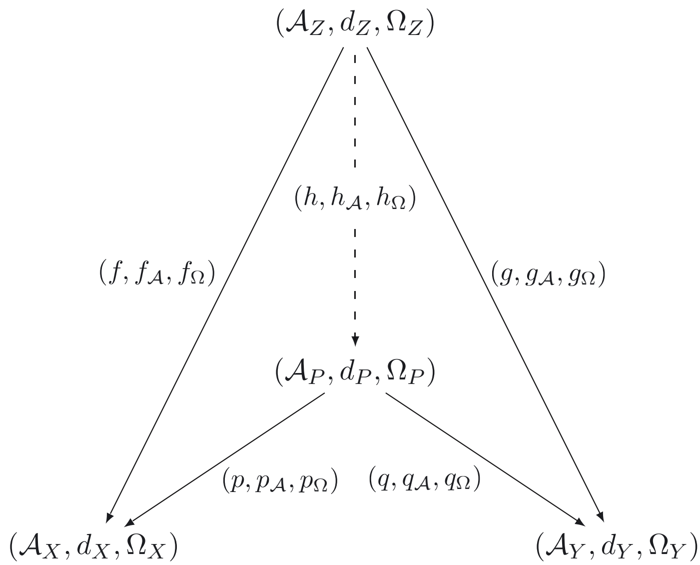


Diagram 2.5

with restriction maps $\rho^U_{U'} \otimes \rho^V_{V'}$, where $\rho^U_{U'}$ and $\rho^V_{V'}$ are the restriction maps of $\mathcal{A}_X$ and $\mathcal{A}_Y$, respectively, with $U' \subseteq U$ and $V' \subseteq V$. *The tensor product in (2.3.1) is taken with respect to* $\mathbb{K}$.

There is no difficulty in verifying that (2.3.1) determines a presheaf of unital commutative associative $\mathbb{K}$-algebras, whose multiplication is defined (on decomposable elements) by

$$(\alpha \otimes \beta) \cdot (\gamma \otimes \delta) := \alpha\gamma \otimes \beta\delta, \tag{2.3.2}$$

for every $\alpha, \gamma \in \mathcal{A}_X(U)$; $\beta, \delta \in \mathcal{A}_Y(V)$, and every $U \in \mathfrak{T}_X$, $V \in \mathfrak{T}_Y$. The sheaf generated by (2.3.1) is denoted by $\mathcal{A}_{X\times Y}$, i.e.,

$$\mathcal{A}_{X\times Y} := \mathbf{S}\big(U \times V \longmapsto \mathcal{A}_X(U) \otimes \mathcal{A}_Y(V)\big).$$

It is a sheaf of unital commutative associative $\mathbb{K}$-algebras over $X \times Y$. For each pair $(U, V) \in \mathfrak{T}_X \times \mathfrak{T}_Y$, we denote by

$$\rho_{UV} : \mathcal{A}_X(U) \otimes \mathcal{A}_Y(V) \longrightarrow \mathcal{A}_{X\times Y}(U \times V) \tag{2.3.3}$$

the canonical map (algebra morphism) of sections. Note that the unit section of $\mathcal{A}_{X\times Y}$ is $1_{X\times Y} = \rho_{XY}(1_X \otimes 1_Y)$, if 1_X and 1_Y are the unit sections of $\mathcal{A}_X$ and $\mathcal{A}_Y$, respectively.

We also consider the presheaf

$$U \times V \longmapsto \big(\mathcal{A}_X(U) \otimes \Omega_Y(V)\big) \times \big(\Omega_X(U) \otimes \mathcal{A}_Y(V)\big), \tag{2.3.4}$$

with $U \times V \in \mathfrak{T}_X \times \mathfrak{T}_Y$ and the obvious restrictions, defined analogously to (2.3.1). This is a presheaf of $\{\mathcal{A}_X(U) \otimes \mathcal{A}_Y(V)\}_{U,V}$–modules with respect to the scalar multiplication

$$(\alpha \otimes \beta) \cdot (\gamma \otimes \varphi, \omega \otimes \delta) := (\alpha\gamma \otimes \beta\varphi, \alpha\omega \otimes \beta\delta), \tag{2.3.5}$$

for every $\alpha, \gamma \in \mathcal{A}_X(U)$; $\beta, \delta \in \mathcal{A}_Y(V)$, $\omega \in \Omega_X(U)$, $\varphi \in \Omega_Y(V)$, and every $U \in \mathfrak{T}_X$, $V \in \mathfrak{T}_Y$. Then we set

$$\Omega_{X\times Y} := \mathbf{S}\big(U \times V \longmapsto \big(\mathcal{A}_X(U) \otimes \Omega_Y(V)\big) \times \big(\Omega_X(U) \otimes \mathcal{A}_Y(V)\big)\big).$$

In virtue of (2.3.5), $\Omega_{X\times Y}$ is an $\mathcal{A}_{X\times Y}$-module over $X \times Y$. We have the corresponding canonical maps

$$\tau_{UV} : \big(\mathcal{A}_X(U) \otimes \Omega_Y(V)\big) \times \big(\Omega_X(U) \otimes \mathcal{A}_Y(V)\big) \longrightarrow \Omega_{X\times Y}(U \times V), \tag{2.3.6}$$

which are morphisms of modules. Applying Diagram 1.7 to the case of the scalar multiplication (2.3.5), we see that

$$\tau_{UV}\big((\alpha \otimes \beta) \cdot (\gamma \otimes \varphi, \omega \otimes \delta)\big) = \rho_{UV}(\alpha \otimes \beta) \cdot \tau_{UV}(\gamma \otimes \varphi, \omega \otimes \delta), \tag{2.3.6$'$}$$

for all $\alpha, \beta, \gamma, \delta, \varphi, \omega$ as in (2.3.5).

Finally, we denote by

$$d_{X\times Y} : \mathcal{A}_{X\times Y} \longrightarrow \Omega_{X\times Y}$$

the morphism generated by the presheaf morphism $(d_{U\times V})$, for all $U \in \mathfrak{T}_X$ and $V \in \mathfrak{T}_Y$, where

$$d_{U\times V} : \mathcal{A}_X(U)\otimes\mathcal{A}_Y(V) \longrightarrow \big(\mathcal{A}_X(U)\otimes\Omega_Y(V)\big)\times\big(\Omega_X(U)\otimes\mathcal{A}_Y(V)\big) \tag{2.3.7}$$

is given by

$$d_{U\times V}(\alpha \otimes \beta) = \big(\alpha \otimes d_Y\beta, (d_X\alpha) \otimes \beta\big), \tag{2.3.8}$$

for every $\alpha \in \mathcal{A}_X(U)$ and $\beta \in \mathcal{A}_Y(V)$. Here we have applied convention (1.1.3) for the differentials d_X and d_Y. Equality (2.3.8) is defined for arbitrary elements by an obvious $\mathbb{K}$-linear extension. It is clear that $d_{X\times Y}$ is a $\mathbb{K}$-linear morphism.

With the previous notations in mind, we obtain the first result towards the main goal of this section.

2.3.1 Proposition. *Let $(\mathcal{A}_X, d_X, \Omega_X)$ and $(\mathcal{A}_Y, d_Y, \Omega_Y)$ be two differential triads over X and Y, respectively. Then $(\mathcal{A}_{X\times Y}, d_{X\times Y}, \Omega_{X\times Y})$ is a differential triad over $X \times Y$.*

Proof. After the preceding preliminary discussion, it remains to show that $d_{X\times Y}$ satisfies the Leibniz condition. Therefore, it suffices to verify it for each morphism $d_{U\times V}$, on decomposable elements. Indeed, for any $\alpha, \gamma \in \mathcal{A}_X(U)$ and $\beta, \delta \in \mathcal{A}_Y(V)$, equalities (2.3.2), (2.3.8) and (2.3.5), along with (2.1.3′), yield

$$\begin{aligned}
d_{U\times V}\big((\alpha\otimes\beta)\cdot(\gamma\otimes\delta)\big) &= \big(\alpha\gamma\otimes d_Y(\beta\delta),\, d_X(\alpha\gamma)\otimes\beta\delta\big)\\
&= \big(\alpha\gamma\otimes(\beta\, d_Y\delta+\delta\, d_Y\beta),\, (\alpha\, d_X\gamma+\gamma\, d_X\alpha)\otimes\beta\delta\big)\\
&= \big(\alpha\gamma\otimes\beta\, d_Y\delta+\alpha\gamma\otimes\delta\, d_Y\beta,\, \alpha(d_X\gamma)\otimes\beta\delta+\gamma(d_X\alpha)\otimes\beta\delta\big)\\
&= \big(\alpha\gamma\otimes\beta\, d_Y\delta, \alpha(d_X\gamma)\otimes\beta\delta\big)+\big(\gamma\alpha\otimes\delta\, d_Y\beta,\, \gamma(d_X\alpha)\otimes\delta\beta\big)\\
&= (\alpha\otimes\beta)\cdot\big(\gamma\otimes d_Y\delta,\, (d_X\gamma)\otimes\delta\big)+(\gamma\otimes\delta)\cdot\big(\alpha\otimes d_Y\beta,\, (d_X\alpha)\otimes\beta\big)\\
&= (\alpha\otimes\beta)\cdot d_{U\times V}(\gamma\otimes\delta)+(\gamma\otimes\delta)\cdot d_{U\times V}(\alpha\otimes\beta).
\end{aligned}$$

This completes the proof. □

The triad $(\mathcal{A}_{X\times Y}, d_{X\times Y}, \Omega_{X\times Y})$, being a candidate for the product structure sought, will now be provided with two appropriate morphisms playing the rôle of projections. So, if

$$p : X \times Y \longrightarrow X$$

is the ordinary projection to the first factor, we construct the morphism $(p, p_{\mathcal{A}}, p_\Omega)$ in the following way. First we consider the presheaf

$$U \longmapsto \mathcal{A}_X(U)\otimes\mathcal{A}_Y(Y); \qquad U \in \mathfrak{T}_X,$$

with restriction maps $(\rho^U_{U'}\otimes 1_Y)$. We also consider the presheaf morphism

$$(2.3.9) \qquad \big\{\tilde{p}_{\mathcal{A},U} : \mathcal{A}_X(U) \longmapsto \mathcal{A}_X(U)\otimes\mathcal{A}_Y(Y) : \alpha\mapsto\alpha\otimes 1_Y \,|\, U\in\mathfrak{T}_X\big\}.$$

Since the sheaf $p_*(\mathcal{A}_{X\times Y})$ is generated by the presheaf

$$U \longmapsto \mathcal{A}_{X\times Y}(p^{-1}(U)) = \mathcal{A}_{X\times Y}(U\times Y),$$

the restriction maps (ρ_{UY}) (see (2.3.3)) can be composed with those of (2.3.9), as shown in the next diagram.

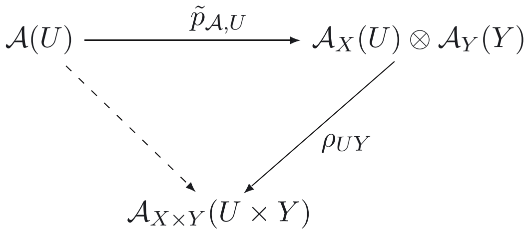


Diagram 2.6

The desired morphism

$$p_{\mathcal{A}} : \mathcal{A}_X \to p_*(\mathcal{A}_{X\times Y})$$

is generated by the presheaf morphism $\left(\rho_{UY} \circ \tilde{p}_{\mathcal{A},U}\right)_{U\in\mathfrak{T}_X}$; that is,

$$p_{\mathcal{A}} := \mathbf{S}\big(\big\{\rho_{UY} \circ \tilde{p}_{\mathcal{A},U} \,\big|\, U \in \mathfrak{T}_X\big\}\big). \tag{2.3.10}$$

Analogously, we define

$$p_{\Omega} : \Omega_X \to p_*(\Omega_{X\times Y})$$

to be the morphism of sheaves (see also (2.3.6))

$$p_{\Omega} := \mathbf{S}\big(\big\{\tau_{UY} \circ \tilde{p}_{\Omega,U} \,\big|\, U \in \mathfrak{T}_X\big\}\big), \tag{2.3.11}$$

where

$$\begin{aligned} \tilde{p}_{\Omega,U} :\ \Omega_X(U) &\longrightarrow \big(\mathcal{A}_X(U) \otimes \Omega_Y(Y)\big) \times \big(\Omega_X(U) \otimes \mathcal{A}_Y(Y)\big) : \\ \omega &\longmapsto (0, \omega \otimes 1_Y). \end{aligned} \tag{2.3.12}$$

The morphisms (2.3.11) and (2.3.12) are morphisms of modules. Note that, in virtue of (1.4.9),

$$\mathcal{A}_{X\times Y}(U \times Y) = \mathcal{A}_{X\times Y}(p^{-1}(U)) \cong p_*(\mathcal{A}_{X\times Y})(U),$$
$$\Omega_{X\times Y}(U \times Y) = \Omega_{X\times Y}(p^{-1}(U)) \cong p_*(\Omega_{X\times Y})(U),$$

for every open $U \subseteq X$.

If $q : X \times Y \to Y$ is the projection to the second factor, we define the triplet $(q, q_{\mathcal{A}}, q_{\Omega})$ in a similar manner. More precisely, we set

$$q_{\mathcal{A}} := \mathbf{S}\left(\{\rho_{XV} \circ \tilde{q}_{\mathcal{A},V} \,\big|\, V \in \mathfrak{T}_Y\}\right), \tag{2.3.13}$$

$$q_{\Omega} := \mathbf{S}\left(\{\tau_{XV} \circ \tilde{q}_{\Omega,V} \mid V \in \mathfrak{T}_Y\}\right), \tag{2.3.14}$$

where

$$\tilde{q}_{\mathcal{A},V} : \mathcal{A}_Y(V) \longmapsto \mathcal{A}_X(X) \otimes \mathcal{A}_Y(V) : \beta \mapsto 1_X \otimes \beta; \quad V \in \mathfrak{T}_Y, \tag{2.3.15}$$

and

$$\begin{aligned} \tilde{q}_{\Omega,V} : \Omega_Y(V) &\longrightarrow \big(\mathcal{A}_X(X) \otimes \Omega_Y(V)\big) \times \big(\Omega_X(X) \otimes \mathcal{A}_Y(V)\big) : \\ \omega &\longmapsto (1_X \otimes \omega, 0). \end{aligned} \tag{2.3.16}$$

Clearly, for every open $V \subseteq Y$,

$$\mathcal{A}_{X\times Y}(X \times V) = \mathcal{A}_{X\times Y}(q^{-1}(V)) \cong q_*(\mathcal{A}_{X\times Y})(V),$$
$$\Omega_{X\times Y}(X \times V) = \Omega_{X\times Y}(q^{-1}(V)) \cong q_*(\Omega_{X\times Y})(V).$$

2.3.2 Proposition. *With the previous notations, the triplets of maps*

$$(p, p_{\mathcal{A}}, p_\Omega) : (\mathcal{A}_{X\times Y}, d_{X\times Y}, \Omega_{X\times Y}) \longrightarrow (\mathcal{A}_X, d_X, \Omega_X)$$
$$(q, q_{\mathcal{A}}, q_\Omega) : (\mathcal{A}_{X\times Y}, d_{X\times Y}, \Omega_{X\times Y}) \longrightarrow (\mathcal{A}_Y, d_Y, \Omega_Y)$$

are morphisms of differential triads.

Proof. We prove the assertion for $(p, p_{\mathcal{A}}, p_\Omega)$. The map $p_{\mathcal{A}}$ is a morphism of sheaves of commutative associative $\mathbb{K}$-algebras by its very construction. It preserves the units, since ρ_{UY} does so and

$$(\rho_{UY} \circ \tilde{p}_{\mathcal{A},U})(1_U) = \rho_{UY}(1_U \otimes 1_Y) = 1_{U\times Y} \in \mathcal{A}_{X\times Y}(U \times Y),$$

where, obviously, $1_U = 1_X|_U$ (recall that the unit section of $\mathcal{A}_{X\times Y}$ is given in the comments following (2.3.3)). Hence, we have shown condition (MDT. 2) of Definition 2.2.2.

On the other hand, in virtue of (2.3.6′), equalities (2.3.5), (2.3.9) and (2.3.12) imply that, for every $\alpha \in \mathcal{A}_X(U)$ and $\omega \in \Omega_X(U)$,

$$\begin{aligned} (\tau_{UY} \circ \tilde{p}_{\Omega,U})(\alpha \cdot \omega) &= \tau_{UY}(0, \alpha\omega \otimes 1_Y) = \tau_{UY}\big((\alpha \otimes 1_Y) \cdot (0, \omega \otimes 1_Y)\big) \\ &= \tau_{UY}\big(\tilde{p}_{\mathcal{A},U}(\alpha) \cdot \tilde{p}_{\Omega,U}(\omega)\big) = (\rho_{UY} \circ \tilde{p}_{\mathcal{A},U})(\alpha) \cdot (\tau_{UY} \circ \tilde{p}_{\Omega,U}(\omega)). \end{aligned}$$

Hence, by sheafification,

$$p_\Omega(a \cdot w) = p_{\mathcal{A}}(a) \cdot p_\Omega(w), \qquad (a, w) \in \mathcal{A}_{X\times Y} \times_{X\times Y} \Omega_{X\times Y},$$

thus proving (MDT. 3).

Finally, we verify the analog of (2.2.2) for the present data; that is, the commutativity of the next diagram.

$$
\begin{array}{ccc}
\mathcal{A}_X & \xrightarrow{\;p_{\mathcal{A}}\;} & p_*(\mathcal{A}_{X\times Y}) \\
{\scriptstyle d_X}\big\downarrow & & \big\downarrow{\scriptstyle p_*(d_{X\times Y})} \\
\Omega_X & \xrightarrow{\;p_{\Omega}\;} & p_*(\Omega_{X\times Y})
\end{array}
$$

Diagram 2.7

Applying (for clarity) the full notation (1.1.2) for the induced morphisms of sections, we observe that d_X can be thought of as generated by $\{(\overline{d_X})_U \mid U \in \mathfrak{T}_X\}$, while $p_*(d_{X\times Y})$ is generated by

$$\big\{(\overline{d_{X\times Y}})_{p^{-1}(U)} = (\overline{d_{X\times Y}})_{U\times Y} \mid U \in \mathfrak{T}_X\big\}.$$

Thus, taking also into account the definition of the sheaves and morphisms of Diagram 2.7, it suffices to show that

$$(\overline{d_{X\times Y}})_{U\times Y} \circ (\rho_{UY} \circ \tilde{p}_{\mathcal{A},U}) = (\tau_{UY} \circ \tilde{p}_{\Omega,U}) \circ (\overline{d_X})_U,$$

for every open $U \subseteq X$. This amounts to the commutativity of the outer part of the following diagram, where (due to size restrictions) the extreme down-arrows denote in fact the aforementioned induced morphisms of sections.

$$
\begin{array}{ccccc}
\mathcal{A}_X(U) & \xrightarrow{\;\tilde{p}_{\mathcal{A},U}\;} & \mathcal{A}_X(U)\otimes\mathcal{A}_Y(Y) & \xrightarrow{\;\rho_{UY}\;} & \mathcal{A}_{X\times Y}(U\times Y) \\
{\scriptstyle d_X}\big\downarrow & (\mathbf{I}) & \vdots\,{\scriptstyle d_{U\times Y}} & (\mathbf{II}) & \big\downarrow{\scriptstyle d_{X\times Y}} \\
\Omega_X(U) & \xrightarrow{\;\tilde{p}_{\Omega,U}\;} & (\mathcal{A}_X(U)\otimes\Omega_Y(Y))\times(\Omega_X(U)\otimes\mathcal{A}_Y(Y)) & \xrightarrow{\;\tau_{UY}\;} & \Omega_{X\times Y}(U\times Y)
\end{array}
$$

Diagram 2.8

The desired commutativity is now a consequence of the commutativity of the individual sub-diagrams (I) and (II). The first of them is checked as follows: For any $\alpha \in \mathcal{A}_X(U)$, equalities (2.3.9), (2.3.8), together with Proposition 2.1.3 and (2.3.12), yield

$$\begin{gathered}(d_{U\times Y}\circ \tilde{p}_{\mathcal{A},U})\,(\alpha) = d_{U\times Y}(\alpha\otimes 1_Y) = \\ \big(\alpha\otimes d_Y 1_Y, (d_X\alpha)\otimes 1_Y\big) = \\ \big(0, (d_X\alpha)\otimes 1_Y\big) = (\tilde{p}_{\Omega,U}\circ d_X)(\alpha).\end{gathered}$$

Sub-diagram (II) is merely the commutative Diagram 1.7 (adapted to our data), relating the presheaf morphism, generating a sheaf morphism, and the presheaf morphism of sections, induced by the latter. Thus we obtain (MDT. 4), by which we conclude the proof. □

For immediate use in the next theorem we need:

2.3.3 Lemma. *The canonical morphisms ρ_{UV} and τ_{UV}, defined by (2.3.3) and (2.3.6) respectively, satisfy the following equalities:*

$$\begin{aligned}\rho_{UV}(\alpha\otimes\beta) &= \rho_{UY}(\alpha\otimes 1_Y)|_{U\times V}\cdot\rho_{XV}(1_X\otimes\beta)|_{U\times V},\\ \tau_{UV}(\alpha\otimes\varphi,\omega\otimes\beta) &= \rho_{UY}(\alpha\otimes 1_Y)|_{U\times V}\cdot\tau_{XV}(1_X\otimes\varphi,0)|_{U\times V}\\ &\quad + \rho_{XV}(1_X\otimes\beta)|_{U\times V}\cdot\tau_{UY}(0,\omega\otimes 1_Y)|_{U\times V},\end{aligned}$$

for every $\alpha\in\mathcal{A}_X(U)$, $\beta\in\mathcal{A}_Y(V)$, $\omega\in\Omega_X(U)$, $\varphi\in\Omega_Y(V)$, and every open $U\in\mathfrak{T}_X$, $V\in\mathfrak{T}_Y$.

Proof. In virtue of (2.3.2), we see that the *algebra morphism* ρ_{UV} gives

$$\begin{aligned}\rho_{UV}(\alpha\otimes\beta) &= \rho_{UV}\big((\alpha\otimes 1_Y|_V)\cdot(1_X|_U\otimes\beta)\big)\\ &= \rho_{UV}(\alpha\otimes 1_Y|_V)\cdot\rho_{UV}(1_X|_U\otimes\beta)\\ &= \rho_{UY}(\alpha\otimes 1_Y)|_{U\times V}\cdot\rho_{XV}(1_X\otimes\beta)|_{U\times V}.\end{aligned}$$

Similarly, applying (2.3.5) and (2.3.6′), we check that the *morphism of modules* τ_{UV} yields

$$\begin{aligned}\tau_{UV}(\alpha\otimes\varphi,\omega\otimes\beta) &= \tau_{UV}\big((\alpha\otimes 1_Y|_V)\cdot(1_X|_U\otimes\varphi,0)\\ &\quad + (1_X|_U\otimes\beta)\cdot(0,\omega\otimes 1_Y|_V)\big)\\ &= \rho_{UV}(\alpha\otimes 1_Y|_V)\cdot\tau_{UV}(1_X|_U\otimes\varphi,0)\\ &\quad + \rho_{UV}(1_X|_U\otimes\beta)\cdot\tau_{UV}(0,\omega\otimes 1_Y|_V)\\ &= \rho_{UY}(\alpha\otimes 1_Y)|_{U\times V}\cdot\tau_{XV}(1_X\otimes\varphi,0)|_{U\times V}\\ &\quad + \rho_{XV}(1_X\otimes\beta)|_{U\times V}\cdot\tau_{UY}(0,\omega\otimes 1_Y)|_{U\times V}.\end{aligned}$$ □

We now prove the main result of the present section.

2.3.4 Theorem. *If* $(\mathcal{A}_X, d_X, \Omega_X)$ *and* $(\mathcal{A}_Y, d_Y, \Omega_Y)$ *are differential triads over* X *and* Y, *respectively, then the triplet*

$$\big((\mathcal{A}_{X\times Y}, d_{X\times Y}, \Omega_{X\times Y}),\ (p, p_{\mathcal{A}}, p_{\Omega}),\ (q, q_{\mathcal{A}}, q_{\Omega})\big)$$

is the ***product*** *of* $(\mathcal{A}_X, d_X, \Omega_X)$ *and* $(\mathcal{A}_Y, d_Y, \Omega_Y)$ *in the category of differential triads* $\mathcal{DT}$.

Proof. We shall prove the universal property of the product, explained in the introductory discussion of this section and Diagram 2.5. Thus, we assume that $(\mathcal{A}_Z, d_Z, \Omega_Z)$ is a differential triad, over a topological space Z, and $(f, f_{\mathcal{A}}, f_{\Omega})$, $(g, g_{\mathcal{A}}, g_{\Omega})$ are two morphisms of differential triads, where $f : Z \to X$ and $g : Z \to Y$ are continuous maps. We consider the pair

$$h := (f, g) : Z \to X \times Y.$$

If $U \in \mathfrak{T}_X$, $V \in \mathfrak{T}_Y$, and $W := f^{-1}(U) \cap g^{-1}(V)$, then

$$\begin{aligned} h_*(\mathcal{A}_Z)(U \times V) &= \mathcal{A}_Z(h^{-1}(U \times V)) = \mathcal{A}_Z(W), \\ h_*(\Omega_Z)(U \times V) &= \Omega_Z(h^{-1}(U \times V)) = \Omega_Z(W), \end{aligned}$$

within the isomorphism (1.4.9). Furthermore, we denote by

$$h_{\mathcal{A}} : \mathcal{A}_{X\times Y} \to h_*(\mathcal{A}_Z), \quad h_{\Omega} : \Omega_{X\times Y} \to h_*(\Omega_Z)$$

the morphisms generated by the respective presheaf morphisms

$$\begin{aligned} h_{\mathcal{A},U\times V} : \mathcal{A}_X(U) \otimes \mathcal{A}_Y(V) &\longrightarrow h_*(\mathcal{A}_Z)(U \times V) = \mathcal{A}_Z(W) : \\ \alpha \otimes \beta &\longmapsto f_{\mathcal{A}}(\alpha)|_W \cdot g_{\mathcal{A}}(\beta)|_W, \end{aligned} \tag{2.3.17}$$

and

$$\begin{aligned} h_{\Omega,U\times V} : \big(\mathcal{A}_X(U) \otimes \Omega_Y(V)\big) \times \big(\Omega_X(U) \otimes \mathcal{A}_Y(V)\big) &\longrightarrow \Omega_Z(W) : \\ (\alpha \otimes \varphi, \omega \otimes \beta) &\longmapsto f_{\mathcal{A}}(\alpha)|_W \cdot g_{\Omega}(\varphi)|_W \\ &\qquad + g_{\mathcal{A}}(\beta)|_W \cdot f_{\Omega}(\omega)|_W. \end{aligned} \tag{2.3.18}$$

We verify that $(h, h_{\mathcal{A}}, h_{\Omega})$ is a morphism of differential triads. Firstly, $h_{\mathcal{A}}$ is a morphism of sheaves of commutative associative algebras because the respective presheaf morphisms (2.3.17) have the analogous property. Secondly, since $f_{\mathcal{A}}$ and $g_{\mathcal{A}}$ preserve the units, so does $h_{\mathcal{A}}$, i.e.,

$$h_{\mathcal{A},U\times V}(1_U \otimes 1_V) = f_{\mathcal{A}}(1_U)|_W \cdot g_{\mathcal{A}}(1_V)|_W = 1_W \in \mathcal{A}_Z(W).$$

On the other hand, for every $(\alpha \otimes \beta) \in \mathcal{A}_X(U) \otimes \mathcal{A}_Y(V)$ and

$$(\gamma \otimes \varphi, \omega \otimes \delta) \in \big(\mathcal{A}_X(U) \otimes \Omega_Y(V)\big) \times \big(\Omega_X(U) \otimes \mathcal{A}_Y(V)\big),$$

applying (2.3.5), (2.3.18), (2.2.1) for f_Ω and g_Ω, and (2.3.17), we obtain

$$\begin{aligned} h_{\Omega,U\times V}\big((\alpha \otimes \beta) \cdot (\gamma \otimes \varphi, \omega \otimes \delta)\big) &= h_{\Omega,U\times V}(\alpha\gamma \otimes \beta\varphi, \alpha\omega \otimes \beta\delta) \\ &= f_{\mathcal{A}}(\alpha\gamma)|_W \cdot g_\Omega(\beta\varphi)|_W + g_{\mathcal{A}}(\beta\delta)|_W \cdot f_\Omega(\alpha\omega)|_W \\ &= \big(f_{\mathcal{A}}(\alpha)|_W \cdot g_{\mathcal{A}}(\beta)|_W\big) \cdot \big(f_{\mathcal{A}}(\gamma)|_W \cdot g_\Omega(\varphi)|_W \\ &\qquad\qquad + g_{\mathcal{A}}(\delta)|_W \cdot f_\Omega(\omega)|_W\big) \\ &= h_{\mathcal{A},U\times V}(\alpha \otimes \beta) \cdot h_{\Omega,U\times V}(\gamma \otimes \varphi, \omega \otimes \delta), \end{aligned}$$

from which it follows that

$$h_\Omega(a \cdot w) = h_{\mathcal{A}}(a) \cdot h_\Omega(w); \qquad (a, w) \in \mathcal{A}_{X\times Y} \times_{X\times Y} \Omega_{X\times Y},$$

that is, (2.2.1) is fulfilled.

Finally, to prove the analog of (2.2.2), namely

$$h_*(d_Z) \circ h_{\mathcal{A}} = h_\Omega \circ d_{X\times Y},$$

it suffices to work on the generating presheaves and morphism. Equivalently, one has to show that the diagram

$$\begin{array}{ccc} \mathcal{A}_X(U) \otimes \mathcal{A}_Y(Y) & \xrightarrow{\;h_{\mathcal{A},U\times V}\;} & \mathcal{A}_Z(W) \\ \Big\downarrow{\scriptstyle d_{U\times V}} & & \Big\downarrow{\scriptstyle d_Z} \\ \big(\mathcal{A}_X(U) \otimes \Omega_Y(V)\big) \times \big(\Omega_X(U) \otimes \mathcal{A}_Y(V)\big) & \xrightarrow{\;h_{\Omega,U\times V}\;} & \Omega_Z(W) \end{array}$$

DIAGRAM 2.9

is commutative, for all U, V, with d_Z denoting in fact the induced morphism of sections $(\overline{d_Z})_W$.

Now, for every $\alpha \otimes \beta \in \mathcal{A}_X(U) \otimes \mathcal{A}_Y(Y)$, (2.3.17) and the Leibniz condition imply that

$$(2.3.19) \qquad \begin{aligned} &(d_Z \circ h_{\mathcal{A},U\times V})(\alpha \otimes \beta) = d_Z\big(f_{\mathcal{A}}(\alpha)|_W \cdot g_{\mathcal{A}}(\beta)|_W\big) = \\ &f_{\mathcal{A}}(\alpha)|_W \cdot d_Z\big(g_{\mathcal{A}}(\beta)|_W\big) + g_{\mathcal{A}}(\beta)|_W \cdot d_Z\big(f_{\mathcal{A}}(\alpha)|_W\big). \end{aligned}$$

Likewise, (2.3.8) and (2.3.18) give

$$(2.3.20) \qquad \begin{aligned} &(h_{\Omega,U\times V} \circ d_{U\times V})(\alpha \otimes \beta) = h_{\Omega,U\times V}(\alpha \otimes d_Y\beta,\, (d_X\alpha)\otimes\beta) = \\ &f_{\mathcal{A}}(\alpha)|_W \cdot g_\Omega(d_Y\beta)|_W + g_{\mathcal{A}}(\beta)|_W \cdot f_\Omega(d_X\alpha)|_W. \end{aligned}$$

Since, by the analog of (2.2.2) and Diagram 1.8, $f_\Omega(d_X\alpha) = d_Z(f_{\mathcal{A}}(\alpha))$ holds for every $\alpha \in \mathcal{A}_X(U)$, it follows that

$$f_\Omega(d_X\alpha)|_W = d_Z(f_{\mathcal{A}}(\alpha))|_W = d_Z\big(f_{\mathcal{A}}(\alpha)|_W\big).$$

By the same token,

$$g_\Omega(d_Y\beta)|_W = d_Z\big(g_\Omega(\beta)|_W\big).$$

The last two equalities imply that (2.3.19) and (2.3.20) coincide, thus Diagram 2.9 is commutative and all the conditions of Definition 2.2.2 are fulfilled; that is, $(h, h_{\mathcal{A}}, h_\Omega)$ is indeed a morphism of differential triads.

We show that $(h, h_{\mathcal{A}}, h_\Omega)$ satisfies the relations

$$(2.3.21) \qquad (p, p_{\mathcal{A}}, p_\Omega) \circ (h, h_{\mathcal{A}}, h_\Omega) = (f, f_{\mathcal{A}}, f_\Omega),$$

$$(2.3.22) \qquad (q, q_{\mathcal{A}}, q_\Omega) \circ (h, h_{\mathcal{A}}, h_\Omega) = (g, g_{\mathcal{A}}, g_\Omega).$$

For the first, according to (2.2.7), (2.2.3) and (2.2.4), we have to prove that

$$(2.3.23) \qquad p_*(h_{\mathcal{A}}) \circ p_{\mathcal{A}} = f_{\mathcal{A}}, \qquad p_*(h_\Omega) \circ p_\Omega = f_\Omega.$$

Again, we work on the generating presheaf morphisms. We recall that $p_{\mathcal{A}}$ is defined by (2.3.10), while $p_*(h_{\mathcal{A}})$ is generated by the presheaf morphism of induced sections (in full notation)

$$(\overline{h_{\mathcal{A}}})_{p^{-1}(U)} = (\overline{h_{\mathcal{A}}})_{U\times Y} : \mathcal{A}_{X\times Y}(U \times Y) \longrightarrow \mathcal{A}_Z(h^{-1}(U \times Y)),$$

for all $U \in \mathfrak{T}_X$. As a result, to show the first of (2.3.23), it suffices to verify that

$$(2.3.24) \qquad (\overline{h_{\mathcal{A}}})_{U\times Y} \circ \rho_{UY} \circ \tilde{p}_{\mathcal{A},U} = (\overline{f_{\mathcal{A}}})_U \,:\, \mathcal{A}_X(U) \longrightarrow \mathcal{A}_Z(f^{-1}(U)).$$

Observe that now $h^{-1}(U \times Y) = f^{-1}(U)$, whence the range of (2.3.24). However, since $\mathcal{A}_Z$ can be identified with the sheaf of germs of its sections (see (1.2.14) and the ensuing discussion), the corresponding Diagram 1.7 takes the following form (see also (2.3.3))

$$\begin{array}{ccc}
\mathcal{A}_X(U) \otimes \mathcal{A}_Y(Y) & \xrightarrow{\;h_{\mathcal{A},U\times Y}\;} & \mathcal{A}_Z(f^{-1}) \\
\rho_{UY} \Big\downarrow & & \Big\downarrow id \\
\mathcal{A}_{X\times Y}(U \times Y) & \xrightarrow{\;(\overline{h_{\mathcal{A}}})_{U\times Y}\;} & \mathcal{A}_Z(f^{-1})
\end{array}$$

Diagram 2.10

where the identity plays the rôle of the canonical morphism of sections under the aforementioned identification. Therefore, (2.3.24) reduces to

$$h_{\mathcal{A},U\times Y} \circ \tilde{p}_{\mathcal{A},U} = (\overline{f_{\mathcal{A}}})_U.$$

The previous equality is true, because (2.3.9) and (2.3.17) give that

$$(h_{\mathcal{A},U\times Y} \circ \tilde{p}_{\mathcal{A},U})(\alpha) = h_{\mathcal{A},U\times Y}(\alpha \otimes 1_Y) =$$
$$f_{\mathcal{A}}(\alpha) \cdot g_{\mathcal{A}}(1_Y) = f_{\mathcal{A}}(\alpha) \equiv (\overline{f_{\mathcal{A}}})_U(\alpha),$$

for every $\alpha \in \mathcal{A}_X(U)$.

Similarly, for every $\omega \in \Omega_X(U)$, (2.3.12) and (2.3.18) yield

$$(h_{\Omega,U\times Y} \circ \tilde{p}_{\Omega,U})(\omega) = h_{\Omega,U\times Y}(0, \omega \otimes 1_Y) =$$
$$g_{\mathcal{A}}(1_Y) \cdot f_{\Omega}(\omega) = f_{\Omega}(\omega) \equiv (\overline{f_{\Omega}})_U(\omega).$$

Hence, $(\overline{h_\Omega})_{U\times Y} \circ \tau_{UY} \circ \tilde{p}_{\Omega,U} = (\overline{f_\Omega})_U$, for every $U \in \mathfrak{T}_X$ (see also (2.3.6)). The preceding equality proves the second one of (2.3.23) and, consequently, (2.3.21). Equality (2.3.22) is obtained analogously.

The last matter remaining to be shown is that $(h, h_{\mathcal{A}}, h_\Omega)$ is the unique morphism satisfying (2.3.21) and (2.3.22). To this end, assume that

$$(h', h'_{\mathcal{A}}, h'_\Omega) : (\mathcal{A}_Z, d_Z, \Omega_Z) \longrightarrow (\mathcal{A}_{X\times Y}, d_{X\times Y}, \Omega_{X\times Y})$$

is another morphism satisfying the properties of $(h, h_{\mathcal{A}}, h_\Omega)$. Then $h' = (f, g) = h$. Moreover, by (2.3.21) and (2.2.7), $(p \circ h')_{\mathcal{A}} = f_{\mathcal{A}} = (p \circ h)_{\mathcal{A}}$, or

(2.3.25) $$p_*(h'_{\mathcal{A}}) \circ p_{\mathcal{A}} = p_*(h_{\mathcal{A}}) \circ p_{\mathcal{A}}.$$

Analogously, we find that

$$q_*(h'_{\mathcal{A}}) \circ q_{\mathcal{A}} = q_*(h_{\mathcal{A}}) \circ q_{\mathcal{A}}. \tag{2.3.26}$$

The morphism $p_*(h'_{\mathcal{A}})$ is generated by the presheaf morphism

$$(\overline{h'_{\mathcal{A}}})_{p^{-1}(U)} = (\overline{h'_{\mathcal{A}}})_{U\times Y} : \mathcal{A}_{X\times Y}(U \times Y) \longrightarrow \mathcal{A}_Z(h^{-1}(U \times Y)),$$

for all $U \in \mathfrak{T}_X$. Therefore, (2.3.10) and (2.3.25) imply that

$$(\overline{h'_{\mathcal{A}}})_{p^{-1}(U)} \circ \rho_{UY} \circ \tilde{p}_{U,\mathcal{A}} = (\overline{h_{\mathcal{A}}})_{p^{-1}(U)} \circ \rho_{UY} \circ \tilde{p}_{U,\mathcal{A}}. \tag{2.3.27}$$

To show now that $h'_{\mathcal{A}} = h_{\mathcal{A}} : \mathcal{A}_{X\times Y} \to h'_*(\mathcal{A}_Z) = h_*(\mathcal{A}_Z)$, we take an arbitrary $u \in (\mathcal{A}_{X\times Y})_{(x,y)}$. Then there is some $\sigma \in \mathcal{A}_X(U) \otimes \mathcal{A}_Y(V)$, with $U \in \mathcal{N}(x)$, $V \in \mathcal{N}(y)$, and $u = [\sigma]_{(x,y)} = \rho_{UV}(\sigma)(x,y)$. For simplicity we can take $\sigma = \alpha \otimes \beta$ (the general case is worked out similarly, by using combinations of decomposable tensors and taking into account that all the maps involved are $\mathbb{K}$-linear). Therefore, in virtue of Lemma 2.3.3, equalities (2.3.9), (2.3.27), and omitting the restrictions, we obtain

$$\begin{gathered}
h'_{\mathcal{A}}(u) = h'_{\mathcal{A}}(\rho_{UV}(\alpha \otimes \beta)(x,y)) = \\
h'_{\mathcal{A}}\big(\rho_{UY}(\alpha \otimes 1_Y)(x,y) \cdot \rho_{XV}(1_X \otimes \beta)(x,y)\big) = \\
h'_{\mathcal{A}}\big(\rho_{UY}(\alpha \otimes 1_Y)(x,y)\big) \cdot h'_{\mathcal{A}}\big(\rho_{XV}(1_X \otimes \beta)(x,y)\big) = \\
\big[(\overline{h'_{\mathcal{A}}})_{U\times Y}\big(\rho_{UY}(\alpha \otimes 1_Y)\big)\big](x,y) \cdot \big[(\overline{h'_{\mathcal{A}}})_{X\times V}\big(\rho_{XV}(1_X \otimes \beta)\big)\big](x,y) = \\
\big[\big((\overline{h'_{\mathcal{A}}})_{U\times Y} \circ \rho_{UY} \circ \tilde{p}_{U,\mathcal{A}}\big)(\alpha)\big](x,y) \cdot \big[\big((\overline{h'_{\mathcal{A}}})_{X\times V} \circ \rho_{XV} \circ \tilde{q}_{\mathcal{A},V}\big)(\beta)\big](x,y) = \\
\big[\big((\overline{h_{\mathcal{A}}})_{U\times Y} \circ \rho_{UY} \circ \tilde{p}_{U,\mathcal{A}}\big)(\alpha)\big](x,y) \cdot \big[\big((\overline{h_{\mathcal{A}}})_{X\times V} \circ \rho_{XV} \circ \tilde{q}_{\mathcal{A},V}\big)(\beta)\big](x,y) = \\
\big[(\overline{h_{\mathcal{A}}})_{U\times Y}\big(\rho_{UY}(\alpha \otimes 1_Y)\big)\big](x,y) \cdot \big[(\overline{h_{\mathcal{A}}})_{X\times V}\big(\rho_{XV}(1_X \otimes \beta)\big)\big](x,y) = \\
h_{\mathcal{A}}\big(\rho_{UY}(\alpha \otimes 1_Y)(x,y) \cdot \rho_{XV}(1_X \otimes \beta)(x,y)\big) = \\
h_{\mathcal{A}}(\rho_{UV}(\alpha \otimes \beta)(x,y)) = h_{\mathcal{A}}(u),
\end{gathered}$$

thus proving equality $h'_{\mathcal{A}} = h_{\mathcal{A}}$. The proof of $h'_{\Omega} = h_{\Omega}$ is similar, though a bit more complicated. This completes the proof of the theorem. □

2.4. Abstract differentiability

We are now in a position to give a notion of differentiability, extending, within our abstract framework, the classical differentiability of maps between differentiable manifolds. Although the related results are not needed in the main part of this work, we include some of them in order to illustrate the potentiality of the present abstract approach and its advantages over ordinary smooth manifolds.

With the notations of Definition 2.2.2, we have the following basic definition.

2.4.1 Definition. Let X, Y be topological spaces, equipped with the differential triads $(\mathcal{A}_X, d_X, \Omega_X)$ and $(\mathcal{A}_Y, d_Y, \Omega_Y)$, respectively. A continuous map $f : X \to Y$ is said to be ***differentiable in abstracto***, if it can be completed to a morphism of differential triads $(f, f_{\mathcal{A}}, f_\Omega)$.

2.4.2 Examples. 1) The first typical example is given by the usual projections $p : X \times Y \to X$ and $q : X \times Y \to Y$, if X and Y are topological spaces equipped with differential triads as in Definition 2.4.1. In virtue of Proposition 2.3.2, the projections $(p, p_{\mathcal{A}}, p_\Omega)$ and $(q, q_{\mathcal{A}}, q_\Omega)$ are morphisms of differential triads, thus *p and q are differentiable maps in abstracto*. The same is true for the map $h : Z \to X \times Y$, used in the proof of the universal property of the product of two differential triads (see Theorem 2.3.4, as well as Diagram 2.5 with $P = X \times Y$).

2) Ordinary differentiable maps between manifolds are differentiable in abstracto. Indeed, as we have seen in the proof of Theorem 2.2.7, any smooth map f between two C^∞-manifolds is always completed to a morphism $(f, f_{\mathcal{A}}, f_\Omega)$ between the differential triads induced by the corresponding manifolds.

More examples are given below, together with some important consequences. All of them arise from the possibility of deriving differential triads by pushing out or pulling back given differential triads, by means of *arbitrary continuous maps.* There is no analogous procedure applicable to the case of differentiable manifolds.

2.4.3 Theorem. *Let $f : X \to Y$ be a continuous map and $(\mathcal{A}_X, d_X, \Omega_X)$ a differential triad over X. Then f is differentiable in abstracto, with respect to $(\mathcal{A}_X, d_X, \Omega_X)$ and the push-out triad $\big(f_*(\mathcal{A}_X), f_*(d_X), f_*(\Omega_X)\big)$.*

Proof. We already know that $\big(f_*(\mathcal{A}_X), f_*(d_X), f_*(\Omega_X)\big)$ is a differential triad over Y (see Lemma 2.2.1). Therefore, by Definition 2.2.2,

$$(f, id_{f_*(\mathcal{A})}, id_{f_*(\Omega)}) : (\mathcal{A}_X, d_X, \Omega_X) \longrightarrow \big(f_*(\mathcal{A}_X), f_*(d_X), f_*(\Omega_X)\big)$$

is a morphism of differentiable triads, and f is differentiable in abstracto. $\square$

2.4.4 Corollary. *Let X be a topological space, endowed with an equivalence relation "$\sim$". If $(\mathcal{A}_X, d_X, \Omega_X)$ is a differential triad over X, then the quotient space $Q := X/_\sim$ is provided with a differential triad so that the canonical map $q : X \to Q$ is differentiable in abstracto.*

Note. As a consequence of the previous results, a number of spaces, obtained by operations on manifolds, acquire a sort of "differential" structure (viz. differential triad), although there is no manifold structure in the usual sense. For instance, given any manifold X and any continuous action $\phi : G \times X \to X$ of a topological group G on X, the orbit space X/G is endowed with a differential triad, and the canonical map $X \to X/G$ is differentiable in abstracto.

2.4.5 Lemma. *Let $f : X \to Y$ be a continuous map and $(\mathcal{A}_Y, d_Y, \Omega_Y)$ a differential triad over Y. Then the pull-back triad*

$$\big(f^*(\mathcal{A}_Y), f^*(d_Y), f^*(\Omega_Y)\big)$$

of $(\mathcal{A}_Y, d_Y, \Omega_Y)$ by f is a differential triad over X.

Proof. Although the proof is straightforward, let us verify the Leibniz condition for $f^*(d_Y)$, as an example of application of (1.4.5) to the algebraic operations involved here.

For any $x \in X$ and arbitrary pairs $(x,a), (x,b) \in f^*(\mathcal{A})_x = \{x\} \times \mathcal{A}_{f(x)}$, we see that

$$\begin{gathered} f^*(d_Y)((x,a)\cdot(x,b)) = f^*(d_Y)(x,ab) = (x, d_Y(ab)) = \\ (x, a\,d_Y b + b\,d_Y a) = (x, a\,d_Y b) + (x, b\,d_Y a) = \\ (x,a)\cdot(x, d_Y b) + (x,b)\cdot(x, d_Y a) = \\ (x,a)\cdot f^*(d_Y)(x,b) + (x,b)\cdot f^*(d_Y)(x,a), \end{gathered}$$

as required. □

The dual of Theorem 2.4.3 is given by

2.4.6 Theorem. *Let $f : X \to Y$ be a continuous map and $(\mathcal{A}_Y, d_Y, \Omega_Y)$ a differential triad over Y. Then f is differentiable in abstracto, with respect to $\big(f^*(\mathcal{A}_Y), f^*(d_Y), f^*(\Omega_Y)\big)$ and $(\mathcal{A}_Y, d_Y, \Omega_Y)$.*

Proof. As opposed to Theorem 2.4.3, the proof of the present statement is more complicated, involving technicalities of the theory of categories. Since this approach is beyond the scope of this book, we omit the proof and refer to Papatriantafillou [99, Theorem 3.4] for full details. □

2.4.7 Corollary. *Let $(\mathcal{A}_X, d_X, \Omega_X)$ be a differential triad over a topological space X and let S be an* arbitrary *subset of X. Then the restriction of $(\mathcal{A}_X, d_X, \Omega_X)$ to S*

$$(\mathcal{A}_S, d_S, \Omega_S) := \big(\mathcal{A}_X|_S,\, d_X|_S,\, \Omega_X|_S\big)$$

is a differential triad and the canonical injection $i : S \hookrightarrow X$ is differentiable in abstracto.

Proof. This is an immediate consequence of the fact that

$$(\mathcal{A}_S, d_S, \Omega_S) \equiv \big(i^*(\mathcal{A}_X), i^*(d_X), i^*(\Omega_X)\big). \qquad \square$$

Note. The previous result means that *all* the subsets of a topological space, when the latter is equipped with a differential triad, also admit a differential triad. This is not true in the category of differentiable manifolds.

The following important property of continuous functions, on spaces endowed with a differential triads, is not shared by smooth manifolds.

2.4.8 Corollary. *Let X be a topological space. Then every continuous function $X \to \mathbb{R}$ is differentiable in abstracto in a natural way; that is, by considering the standard differential triad of $\mathbb{R}$ (see Example 2.1.4(a)) and its pull-back (by f) on X.*

2.4.9 Remarks. 1) The push-out and the pull-back of differential triads have corresponding *universal properties.* Details are given in Papatriantafillou [99].

2) The category of differential triads has also *projective* and *inductive limits*, a property not shared by ordinary smooth manifolds. This is fully explained in Papatriantafillou [98]

2.5. The de Rham complex

In many cases (as in, e.g, Chapters 8 and 9, dealing with the curvature of connections and the Chern-Weil theory, respectively) it is necessary to extend the differential $d : \mathcal{A} \to \Omega$ of a differential triad to a sequence of modules and differentials of higher order. This leads to the abstract analog of the de Rham complex, whose exactness (cf. de Rham's theorem) is one of the cornerstones of ordinary differential geometry.

Let $(\mathcal{A}, d, \Omega)$ be a differential triad over a fixed topological space $X \equiv (X, \mathfrak{T}_X)$. For reasons that will soon become clear, we set $\Omega^1 := \Omega$.

By the construction of Subsection 1.3.4, we obtain the ***p-th exterior power*** of the $\mathcal{A}$-module Ω^1, namely

$$\Omega^p \equiv \bigwedge^p \Omega^1 := \underbrace{\Omega^1 \bigwedge_{\mathcal{A}} \cdots \bigwedge_{\mathcal{A}} \Omega^1}_{p\text{-factors}}, \qquad p \geq 2. \tag{2.5.1}$$

We agree that

$$\Omega^0 = \mathcal{A} \quad \text{and} \quad \Omega^1 = \bigwedge^1 \Omega = \Omega.$$

Inspired by the classical case of ordinary differential forms, we call Ω^p the ***sheaf of p-forms*** and its sections ***p-forms.***

We recall that the $\mathcal{A}$-module (2.5.1) is generated by the presheaf of $\mathcal{A}(U)$-modules

$$U \longmapsto \bigwedge^p(\Omega^1(U)) := \Omega^1(U) \bigwedge_{\mathcal{A}(U)} \cdots \bigwedge_{\mathcal{A}(U)} \Omega^1(U), \tag{2.5.2}$$

(p factors) with U running the topology $\mathfrak{T}_X$. The restriction maps of this presheaf are determined by

$$s_1 \wedge \cdots \wedge s_p \longmapsto s_1|_V \wedge \cdots \wedge s_p|_V, \tag{2.5.3}$$

for every $s_i \in \Omega^1(U)$ $(i = 1, \ldots, p)$ and every open $V \subseteq U$. The previous expression is extended to non decomposable elements in the usual way. Since the presheaf (2.5.2) is not necessarily complete, we have that, in general,

$$\Omega^p(U) := \left(\bigwedge^p \Omega^1\right)(U) \neq \bigwedge^p(\Omega(U)). \tag{2.5.4}$$

As in the ordinary case, we define the ***(graded) exterior algebra*** of Ω

$$\Omega^\bullet \equiv \bigwedge \Omega := \bigoplus_{p=0}^{\infty} \bigwedge^p \Omega^1, \tag{2.5.5}$$

whose exterior product

$$\wedge : \Omega^p \times_X \Omega^q \longrightarrow \Omega^p \bigwedge \Omega^q \equiv \Omega^{p+q} \tag{2.5.6}$$

is generated by the (local) exterior products

$$\wedge_U : \bigwedge^p(\Omega^1(U)) \times \bigwedge^q(\Omega^1(U)) \longrightarrow \bigwedge^{p+q}(\Omega^1(U)),$$

the latter being determined (on decomposable elements) by

$$(s_1 \wedge \ldots \wedge s_p, t_1 \wedge \ldots \wedge t_q) \longmapsto s_1 \wedge \ldots \wedge s_p \wedge t_1 \wedge \ldots \wedge t_q.$$

The correspondence $U \mapsto \wedge_U$ is a morphism of presheaves, thus (2.5.6) can be defined. We note that the presheaf $U \longmapsto \bigwedge^p(\Omega^1(U)) \times \bigwedge^q(\Omega^1(U))$ generates $\Omega^p \times_X \Omega^q$, according to the comments of Subsection 1.3.6.

An immediate consequence of the definitions is the equality

$$a \wedge b = (-1)^{p \cdot q} b \wedge a, \qquad (a, b) \in \Omega^p \times_X \Omega^q. \tag{2.5.7}$$

Again for reasons that will be clear shortly, we set $d^0 := d : \mathcal{A} \to \Omega$. Now, in addition to d^0, we *assume* the existence of a $\mathbb{K}$-*linear morphism*, called the ***1st exterior derivation***

$$d^1 : \Omega^1 \longrightarrow \Omega^2,$$

satisfying the following conditions:

$$d^1 \circ d^0 = d^1 \circ d = 0, \tag{2.5.8}$$

$$d^1(a \cdot w) = (d^0 a) \wedge w + a \cdot d^1 w; \qquad (a, w) \in \mathcal{A} \times_X \Omega^1. \tag{2.5.9}$$

Next we define the $\mathbb{K}$-*linear morphism* (***2nd exterior derivation***)

$$d^2 : \Omega^2 \longrightarrow \Omega^3,$$

by setting (stalk-wise)

$$d^2(u \wedge v) := (d^1 u) \wedge v - u \wedge (d^1 v), \tag{2.5.10}$$

for every $u \wedge v \in \Omega^2$ with $(u, v) \in \Omega^1 \times_X \Omega^1$. The definition is extended to arbitrary (non decomposable) elements by

$$d^2\Big(\sum_{i,j} a_{ij} \cdot (u_i \wedge v_j)\Big) := \sum_{i,j} a_{ij} \cdot d^2(u_i \wedge v_j).$$

In addition, the derivation d^2 is *assumed* to satisfy

$$d^2 \circ d^1 = 0. \tag{2.5.11}$$

For every $p \geq 3$, the ***p-th exterior derivation*** is defined to be the $\mathbb{K}$-linear morphism

$$d^p : \Omega^p \longrightarrow \Omega^{p+1}$$

determined by

$$d^p(u_1 \wedge \cdots \wedge u_p) := \sum_{i=1}^{p} (-1)^{i+1} u_1 \wedge \cdots \wedge (d^1 u_i) \wedge \cdots \wedge u_p. \tag{2.5.12}$$

This extends to arbitrary elements by

$$d^p\Big(\sum a_{1_i,\ldots,p_i} \cdot (w_{1_i} \wedge \cdots \wedge w_{p_i})\Big) := \sum a_{1_i,\ldots,p_i}\, d^p(w_{1_i} \wedge \cdots \wedge w_{p_i}). \tag{2.5.13}$$

2.5.1 Lemma. *The morphisms d^p satisfy equality*

$$d^{p+q}(u \wedge v) = (d^p u) \wedge v + (-1)^p u \wedge (d^q v), \tag{2.5.14}$$

for every $(u, v) \in \Omega^p \times_X \Omega^q$.

Proof. We first take two decomposable elements $u = u_1 \wedge \ldots \wedge u_p$ and $v = v_1 \wedge \ldots \wedge v_p$. Setting $w_i = u_i$, for $i = 1, \ldots, p$, and $w_{p+j} = v_j$, for $j = 1, \ldots, q$, we check that

$$\begin{aligned}
&d^{p+q}(u \wedge v) = d^{p+q}(w_1 \wedge \cdots \wedge w_p \wedge w_{p+1} \wedge \cdots \wedge w_{p+q}) \\
&= \sum_{i=1}^{p} (-1)^{i+1} w_1 \wedge \cdots \wedge (d^1 w_i) \wedge \cdots \wedge w_p \wedge w_{p+1} \wedge \cdots \wedge w_{p+q} \\
&\quad + \sum_{j=1}^{q} (-1)^{p+j+1} w_1 \wedge \cdots \wedge \omega_p \wedge w_{p+1} \wedge \cdots \wedge (d^1 w_{p+j}) \wedge \cdots \wedge w_{p+q} \\
&= \Big(\sum_{i=1}^{p} (-1)^{i+1} w_1 \wedge \cdots \wedge (d^1 w_i) \wedge \cdots \wedge w_p\Big) \wedge w_{p+1} \wedge \cdots \wedge w_{p+q} + \\
&\qquad (-1)^p (w_1 \wedge \cdots \wedge \omega_p) \Big(\sum_{j=1}^{q} (-1)^{j+1} w_{p+1} \wedge \cdots \wedge (d^1 w_{p+j}) \wedge \cdots \wedge w_{p+q}\Big) \\
&= (d^p u) \wedge v + (-1)^p u \wedge (d^q v).
\end{aligned}$$

For the general case of non-decomposable elements we work similarly by applying (2.5.13). □

2.5.2 Lemma. *Equality*

$$d^{p+1} \circ d^p = 0 \tag{2.5.15}$$

is also valid, for every $p \geq 3$.

Proof. We proceed by induction. So, assuming that $d^p \circ d^{p-1} = 0$ holds true, we shall show (2.5.15). As before, it suffices to work with an element of the form $u \wedge v \in \Omega^{p-1} \bigwedge \Omega^1$. Therefore, our assumption and (2.5.11) imply that

$$\begin{aligned}(d^{p+1} \circ d^p)(u \wedge v) &= d^{p+1}\big((d^{p-1}u) \wedge v + (-1)^{p-1} u \wedge (d^1 v)\big)\\ &= d^p(d^{p-1}u) \wedge v + (-1)^p (d^{p-1}u) \wedge (d^1 v)\\ &\quad + (-1)^{p-1}(d^{p-1}u) \wedge (d^1 v) + u \wedge d^2(d^1 v)\\ &= 0.\end{aligned}$$ □

We summarize the foregoing considerations as follows:

2.5.3 Proposition. *Let $(\mathcal{A}, d, \Omega)$ be a differential triad. We assume that there exists a $\mathbb{K}$-morphism $d^1 : \Omega^1 \to \Omega^2$ satisfying equalities (2.5.8) and (2.5.9). If the $\mathbb{K}$-linear morphism $d^2 : \Omega^2 \to \Omega^3$, defined by (2.5.10), satisfies (2.5.11), then there are $\mathbb{K}$-linear morphisms $d^p : \Omega^p \to \Omega^{p+1}$ verifying (2.5.14) and (2.5.15), for all $p \geq 3$.*

Recalling that $\Omega^0 \equiv \mathcal{A}$ and letting

$$a \wedge b := a \cdot b \equiv ab; \qquad (a, b) \in \Omega^0 \times_X \Omega^0,$$
$$a \wedge w := a \cdot w \equiv aw; \qquad (a, w) \in \Omega^0 \times_X \Omega^1,$$

we see that equalities (2.1.3), (2.5.9) and (2.5.10) can be viewed as particular cases of (2.5.14).

Proposition 2.5.3 implies that the sequence

$$(2.5.16) \qquad \begin{aligned}0 \longrightarrow \mathbb{K}_X \hookrightarrow \mathcal{A} \equiv \Omega^0 \xrightarrow{d^0} \Omega^1 \xrightarrow{d^1} \Omega^2 \longrightarrow \cdots\\ \longrightarrow \Omega^p \xrightarrow{d^p} \Omega^{p+1} \xrightarrow{d^{p+1}} \cdots\end{aligned}$$

is a (differential) complex. The situation is reminiscent of the analogous complex defined in the case of differential manifolds, thus we are led to the following:

2.5.4 Definition. A complex of the form (2.5.16) is called a ***de Rham complex*** of X, *with respect to the differential triad* $(\mathcal{A}, d, \Omega)$ *and the derivations d^1 and d^2.*

Although it is customary to omit the degree of the exterior derivation and write d in place of every d^p, occasionally, for the sake of clarity, we will retain the complete notation.

A few comments are in order here. In contrast to the classical case of a differential manifold X, where one speaks of *the* de Rham complex of X, in our framework we may define more than one complexes, depending on the differential triad attached to X, as well as on the choice of the exterior derivations d^1 and d^2 (if they exist, of course), from which the rest of the operators d^p are derived. As we have already said, d^2 is also constructed from d^1 (by (2.5.10)), but it is necessary to assume (2.5.11).

Similarly, equality $\mathbb{K}_X = \ker d^0$ is not always true here and the de Rham complex (2.5.16) is not necessarily exact. Since a generalized form of de Rham's theorem has a fundamental importance, it is natural to ask when this is true in our abstract setting. Papatriantafillou [94] studies necessary and sufficient conditions ensuring the construction and exactness of abstract de Rham complexes, giving also concrete examples, related to manifolds modelled on arbitrary topological vector spaces. A further application of this approach to $\mathbb{A}$-manifolds, in the sense of Kobayashi [48], has been given in Papatriantafillou [91]. Other examples of spaces admitting an exact de Rham complex, outside the context of ordinary manifolds, can be found in Flaherty [28], Mostow [79], Smith [114], Verona [20] (in conjunction with [137]).

Before closing the present chapter, we introduce the following relevant terminology employed in Chapter 9 (see also Mallios [62, Chap. IX, Definition 3.1]).

2.5.5 Definition. A *paracompact (Hausdorff)* space X is called a ***generalized de Rham space of order*** $p \in \mathbb{Z}_0^+$ (or, a ***generalized de Rham p-space***) if, attached to X, there is an *exact sequence* of sheaves of $\mathbb{K}$-vector spaces over X

$$\begin{aligned}(2.5.17)\qquad 0 \longrightarrow \ker d^0 \hookrightarrow \mathcal{A} \equiv \Omega^0 \xrightarrow{d^0} \Omega^1 \xrightarrow{d^1} \Omega^2 \longrightarrow \cdots \\ \longrightarrow \Omega^p \xrightarrow{d^p} d^p(\Omega^p) \longrightarrow 0.\end{aligned}$$

In particular, if (2.5.17) is exact for every $p \in \mathbb{Z}_0^+$, then X is called a ***generalized de Rham space***.

A generalized de Rham space X will be simply called a ***de Rham-space*** if the sequence

$$0 \longrightarrow \mathcal{A} \xrightarrow{d^0} \Omega^1 \xrightarrow{d^1} \Omega^2 \longrightarrow \cdots \longrightarrow \Omega^p \xrightarrow{d^p} \cdots$$

is an *acyclic resolution* of $\ker d^0$ (see Subsection 1.6.3).

Note. We would like to mention here the difference between our definition of a de Rham space and that of Mallios [62, Vol. II, p. 254]. In the latter, such a space is a generalized de Rham space with $\ker d^0 = \mathbb{C}$. The last equality is not necessary in our considerations. On the other hand, the acyclicity property is essentially needed in the study of the Chern-Weil homomorphism (see Section 9.5 and op. cit., p. 262).

Chapter 3

Lie sheaves of groups

In mathematical physics and in geometry, a central role is played by the groups of automorphisms (equivalently: symmetry principles) of the various structures that arise.

C. von WESTENHOLZ [143, p. 84]

THIS chapter aims at the study of *Lie sheaves of groups*, the abstract analog of Lie groups. They are, roughly speaking, sheaves of groups, which admit a representation on a sheaf of Lie algebras, and are also equipped with a sort of logarithmic differential, referred to hereinafter as a *Maurer-Cartan differential*. Lie sheaves of groups are the structural sheaves of the (geometric) principal sheaves studied in Chapter 4. Their rôle in the theory of connections on principal sheaves is as fundamental as that played by Lie groups in principal bundles and their connections. In subsequent chapters we shall see that a great deal of the classical theory of connections extends to our abstract framework, due precisely to the aforementioned representation and differential.

The first two paragraphs center on the fundamental example of the *general linear group sheaf* $\mathcal{GL}(n, \mathcal{A})$ and its logarithmic differential, naturally derived from a differential triad. This is the motivation for the main ideas developed in the sequel. Various examples, within the abstract and classical framework, are also included. In particular, the pull-back of a Lie sheaf of groups, involving certain interesting technicalities, is carefully treated at the end of the chapter.

3.1. The matrix extension of differential triads

We fix a differential triad $(\mathcal{A}, d, \Omega)$ over a topological space $X \equiv (X, \mathfrak{T}_X)$. Our intention is to extend d to an appropriate differential on sheaves of matrices.

For a given $n \in \mathbb{N}$, and any $U \in \mathfrak{T}_X$, we denote by

$$M_n(\mathcal{A}(U)) \cong \mathcal{A}(U)^{n^2}$$

the *non-commutative* algebra of $n \times n$ matrices with entries in the (unital commutative associative) $\mathbb{K}$-algebra of sections $\mathcal{A}(U)$. Considering the restriction maps

$$\mu_V^U : M_n(\mathcal{A}(U)) \longrightarrow M_n(\mathcal{A}(V)) : \; a = (\alpha_{ij}) \mapsto a|_V = \big(\alpha_{ij}|_V\big),$$

for every open $V \subseteq U$, we see that

$$\big(M_n(\mathcal{A}(U)), \, \mu_V^U\big) \,, \tag{3.1.1}$$

with U running the topology $\mathfrak{T}_X$, is a complete presheaf. The sheaf generated by (3.1.1) is denoted by $\mathcal{M}_n(\mathcal{A})$ and called the ***matrix algebra sheaf of order n***, with respect to $\mathcal{A}$. Therefore,

$$\mathcal{M}_n(\mathcal{A}) := \mathbf{S}\big(U \longmapsto M_n(\mathcal{A}(U))\big) \tag{3.1.2}$$

and, by the completeness of the presheaf (3.1.1),

$$\mathcal{M}_n(\mathcal{A})(U) \cong M_n(\mathcal{A}(U)), \qquad U \in \mathfrak{T}_X. \tag{3.1.3}$$

By its construction, $\mathcal{M}_n(\mathcal{A})$ is a sheaf of *non-commutative algebras*, for every integer $n > 1$.

For future reference, we note that $\mathcal{M}_n(\mathcal{A})$ is also an $\mathcal{A}$-*module*. Moreover, thinking of the modules of (3.1.1) as Lie algebras with respect to the commutator of matrices, we have that $\mathcal{M}_n(\mathcal{A})$ is a sheaf of Lie algebras. Combining the last two structures, we briefly say that $\mathcal{M}_n(\mathcal{A})$ is a ***Lie algebra $\mathcal{A}$-module*** or an ***$\mathcal{A}$-module of Lie algebras***.

Analogously to (3.1.1), we consider the complete presheaf

$$U \longmapsto M_n(\Omega(U)); \qquad U \in \mathfrak{T}_X,$$

with the obvious restriction maps. The corresponding sheaf

$$\mathcal{M}_n(\Omega) := \mathbf{S}\big(U \longmapsto M_n(\Omega(U))\big), \tag{3.1.4}$$

is the ***n-th square matrix sheaf extension of*** Ω. Obviously,

$$\mathcal{M}_n(\Omega)(U) \cong M_n(\Omega(U)), \qquad U \in \mathfrak{T}_X. \tag{3.1.5}$$

We note that $\mathcal{M}_n(\Omega)$ is an $\mathcal{M}_n(\mathcal{A})$-*bimodule*, since the generating presheaf has the analogous property in a natural way. More precisely, for any open $U \subseteq X$, if $a = (\alpha_{ij}) \in M_n(\mathcal{A}(U))$ and $w = (\omega_{ij}) \in M_n(\Omega(U))$, then we define

$$a \cdot \omega := (\alpha_{ij}) \cdot (\omega_{ij}) = \Big(\sum_{k=1}^{n} \alpha_{ik} \cdot \omega_{kj}\Big) \in M_n(\Omega(U)),$$

which is meaningful because Ω is already an $\mathcal{A}$-module. A similar multiplication is defined from the right.

The sheaves (3.1.2) and (3.1.4) are nicely related as follows. For any open $U \subseteq X$, working as in the case of ordinary tensor products of vector spaces whose one factor is free, we check that (see also equality (1.3.5) and the notations of Subsection 1.3.3)

$$\begin{gathered} M_n(\Omega(U)) \cong \Omega(U)^{n^2} \cong \big(\Omega(U) \otimes_{\mathcal{A}(U)} \mathcal{A}(U)\big)^{n^2} \cong \\ \Omega(U) \otimes_{\mathcal{A}(U)} \mathcal{A}(U)^{n^2} \cong \Omega(U) \otimes_{\mathcal{A}(U)} M_n(\mathcal{A}(U)). \end{gathered}$$

Since the above isomorphisms are not canonical (being depended on the choice of bases), for convenience we single out the particular family of $\mathcal{A}(U)$-isomorphisms

$$\lambda^1_U : M_n(\Omega(U)) \longrightarrow \Omega(U) \otimes_{\mathcal{A}(U)} M_n(\mathcal{A}(U)) \tag{3.1.6}$$

given by

$$\lambda^1_U\,((\theta_{ij})) := \sum_{i,j=1}^{n} \theta_{ij} \otimes E^U_{ij}, \tag{3.1.6a}$$

where the matrices

$$E^U_{ij} = \begin{pmatrix} 0 & \dots & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & \dots & 0 \end{pmatrix}$$

form the natural basis of $M_n(\mathcal{A}(U))$. Clearly, 0 and 1 (the latter at the ij-entry) are, respectively, the zero and unit sections of $\mathcal{A}$ over U. The use of the superscript 1 in λ^1_U is dictated by the need to introduce –at a later stage– analogous morphisms for higher (exterior) powers of Ω.

The inverse of λ^1_U, denoted by μ^1_U, is determined by

$$\mu^1_U\big(\theta \otimes (a_{ij})\big) := \big(a_{ij} \cdot \theta\big), \tag{3.1.6b}$$

on decomposable tensors and extended, by $\mathcal{A}(U)$-linearity, to arbitrary elements. Therefore, the presheaf isomorphism (λ^1_U) generates an $\mathcal{A}$-isomorphism

$$\lambda^1 : \mathcal{M}_n(\Omega) \xrightarrow{\simeq} \Omega \otimes_{\mathcal{A}} \mathcal{M}_n(\mathcal{A}). \tag{3.1.7}$$

The inverse of λ^1 is denoted by μ^1. The situation is reminiscent of the case of ordinary matrix-valued 1-forms on a differential manifold.

Regarding the sections of the tensor product figuring in (3.1.7), we remark that

$$\begin{aligned} (\Omega \otimes_{\mathcal{A}} \mathcal{M}_n(\mathcal{A}))(U) &\overset{(3.1.7)}{\cong} \mathcal{M}_n(\Omega)(U) \\ &\overset{(3.1.5)}{\cong} M_n(\Omega(U)) \\ &\overset{(3.1.6)}{\cong} \Omega(U) \otimes_{\mathcal{A}(U)} M_n(\mathcal{A}(U)) \\ &\overset{(3.1.3)}{\cong} \Omega(U) \otimes_{\mathcal{A}(U)} \mathcal{M}_n(\mathcal{A})(U); \end{aligned}$$

that is, we obtain the identification

$$(\Omega \otimes_{\mathcal{A}} \mathcal{M}_n(\mathcal{A}))(U) \cong \Omega(U) \otimes_{\mathcal{A}(U)} \mathcal{M}_n(\mathcal{A})(U), \tag{3.1.8}$$

for every $U \in \mathfrak{T}_X$. In general, this is not true for arbitrary $\mathcal{A}$-modules, by the very construction of the tensor product of sheaves, unless one of the factors is free, or a vector sheaf (see Section 5.1) and U is an open set over which the latter is free. The isomorphism (3.1.17) also yields

$$\mathcal{M}_n(\Omega)(U) \cong (\Omega \otimes_{\mathcal{A}} \mathcal{M}_n(\mathcal{A}))(U). \tag{3.1.9}$$

We can now extend the differential d of the given differential triad $(\mathcal{A}, d, \Omega)$ to an appropriate differential on $\mathcal{M}_n(\mathcal{A})$, denoted (for simplicity) by the *same* symbol. To this end, for any $a = (\alpha_{ij}) \in M_n(\mathcal{A}(U))$, we define the matrix

$$d_U a := (da_{ij}) \equiv \left(\bar{d}_U a_{ij}\right) \in M_n(\Omega(U)). \tag{3.1.10}$$

(Warning: The differential d_U should not be confused with the induced morphism of sections $\bar{d}_U : \mathcal{A}(U) \to \Omega(U)$ in the entries of the image matrix.) Thus $\{d_U : M_n(\mathcal{A}(U)) \to M_n(\Omega(U)) \,|\, U \in \mathfrak{T}_X\}$ is a presheaf morphism, generating the $\mathbb{K}$-linear morphism

$$d : \mathcal{M}_n(\mathcal{A}) \longrightarrow \mathcal{M}_n(\Omega) \cong \Omega \otimes_{\mathcal{A}} \mathcal{M}_n(\mathcal{A}), \tag{3.1.11}$$

called the ***n-th square matrix sheaf extension of*** d.

Since, for any $U \in \mathfrak{T}_X$ and matrices $a = (\alpha_{ij})$, $b = (\beta_{ij}) \in M_n(\mathcal{A}(U))$, the evaluation of the initial d at the general element of the matrix product $a \cdot b$ yields

$$d\Big(\sum_{k=1}^{n} \alpha_{ik} \cdot \beta_{kj}\Big) = \sum_{k=1}^{n} \big(d(\alpha_{ik}) \cdot \beta_{kj} + \alpha_{ik} \cdot d\beta_{kj}\big),$$

we have the (local) matrix Leibniz condition

$$d_U(a \cdot b) = (d_U a) \cdot b + a \cdot d_U b; \qquad a, b \in M_n(\mathcal{A}(U)), \tag{3.1.12}$$

whose sheafification yields the Leibniz condition

$$d(u \cdot v) = (du) \cdot v + u \cdot dv, \tag{3.1.12$'$}$$

for every $(u, v) \in \mathcal{M}_n(\mathcal{A}) \times_X \mathcal{M}_n(\mathcal{A})$.

Of course, in all the previous formulas one should bear in mind the *non-commutativity* of $M_n(\mathcal{A}(U))$ and $\mathcal{M}_n(\mathcal{A})$.

In conclusion, $d : \mathcal{A} \to \Omega$ induces the (non-commutative) *derivation* $d : \mathcal{M}_n(\mathcal{A}) \to \mathcal{M}_n(\Omega)$ of $\mathcal{M}_n(\mathcal{A})$ with values in $\mathcal{M}_n(\Omega) \cong \Omega \otimes_{\mathcal{A}} \mathcal{M}_n(\mathcal{A})$. Thus, enlarging Definition 2.1.2 so that sheaves of non-commutative algebras are also included, we can state the following result, whose proof is contained in the preceding arguments.

3.1.1 Proposition. $(\mathcal{M}_n(\mathcal{A}), d, \mathcal{M}_n(\Omega))$ *is a differential triad, with respect to the matrix differential (3.1.11) and the (non-commutative) matrix algebra sheaf* $\mathcal{M}_n(\mathcal{A})$.

The differential triad $(\mathcal{M}_n(\mathcal{A}), d, \mathcal{M}_n(\Omega))$ is called the ***n-th square matrix sheaf extension of*** $(\mathcal{A}, d, \Omega)$.

It is sometimes desirable to "differentiate" arbitrary (i.e., not necessarily square) matrices. Following the previous pattern, we define the ***matrix sheaf extension of*** d to be the morphism (retaining the same symbol)

$$d : \mathcal{M}_{m\times n}(\mathcal{A}) \longrightarrow \mathcal{M}_{m\times n}(\Omega) \cong \Omega \otimes_{\mathcal{A}} \mathcal{M}_{m\times n}(\mathcal{A}), \tag{3.1.13}$$

generated by the morphisms

$$d_U : M_{m\times n}(\mathcal{A}(U)) \longrightarrow M_{m\times n}(\Omega(U)) : (\alpha_{ij}) \mapsto (d\alpha_{ij}) \equiv (\bar{d}_U \alpha_{ij}),$$

where the given quantities have a meaning analogous to that of their square matrix counterparts. However, (3.1.13) is *only* a $\mathbb{K}$-linear morphism, the Leibniz condition being nonsensical as $M_{m\times n}(\mathcal{A}(U))$ is not an algebra.

3.2. The logarithmic differential

We fix again a differential triad $(\mathcal{A}, d, \Omega)$ over X. For every open $U \subseteq X$, $\mathcal{A}(U)^\bullet$ denotes the ***group of units*** (alias ***invertible elements***) of the algebra $\mathcal{A}(U)$. Thus, as in (1.1.4), for each $s \in \mathcal{A}(U)^\bullet$ one defines the section $s^{-1} \in \mathcal{A}(U)$ with $s^{-1}(x) := s(x)^{-1}$, $x \in U$.

Since the sections of $\mathcal{A}$ form the complete presheaf $(\mathcal{A}(U), \rho^U_V)$, the assignment $U \longmapsto \mathcal{A}(U)^\bullet$, together with the restrictions of ρ^U_V to $\mathcal{A}(U)^\bullet$, also determines a complete presheaf, as is routinely checked. Therefore, by sheafification, we obtain the *sheaf of groups*

$$\mathcal{A}^\bullet := \mathbf{S}\big(U \longmapsto \mathcal{A}(U)^\bullet\big), \tag{3.2.1}$$

called the ***sheaf of units*** of $\mathcal{A}$. By the very construction,

$$\mathcal{A}^\bullet(U) \cong \mathcal{A}(U)^\bullet, \qquad U \in \mathfrak{T}_X. \tag{3.2.2}$$

We easily verify that the stalk of $\mathcal{A}^\bullet$ over $x \in X$ coincides with the group of units of $\mathcal{A}_x$, thus we write

$$\mathcal{A}^\bullet_x := (\mathcal{A}^\bullet)_x = (\mathcal{A}_x)^\bullet, \qquad x \in X.$$

We define the ***logarithmic differential*** of $\mathcal{A}^{\bullet}$ to be the morphism (of sheaves of sets)

$$\widetilde{\partial} : \mathcal{A}^{\bullet} \longrightarrow \Omega \cong \Omega \otimes_{\mathcal{A}} \mathcal{A}, \tag{3.2.3}$$

generated by the presheaf morphism

$$\{\widetilde{\partial}_U : \mathcal{A}(U)^{\bullet} \longrightarrow \Omega(U) \,|\, U \in \mathfrak{T}_X\},$$

where each $\widetilde{\partial}_U$ is given by

$$\widetilde{\partial}_U(s) := s^{-1} \cdot ds \equiv s^{-1} \cdot \bar{d}_U s, \qquad s \in \mathcal{A}^{\bullet}(U). \tag{3.2.4}$$

The tensor product in the target of (3.2.3) is the forerunner of the more general logarithmic differential given in (3.2.10) and Definition 3.3.2 in the next section.

Observing that, if $a \in \mathcal{A}^{\bullet}_x$ is represented by some $s \in \mathcal{A}(U)^{\bullet}$, i.e., $a = [s]_x$, then $a^{-1} = [s^{-1}]_x$, we prove the stalk-wise analog of (3.2.4), namely

$$\widetilde{\partial}(a) = a^{-1} \cdot da, \qquad a \in \mathcal{A}^{\bullet}. \tag{3.2.4$'$}$$

Indeed, by (1.2.17), the induced morphisms of sections $\overline{(\widetilde{\partial})}_U$ coincide with the generating morphisms $\widetilde{\partial}_U$, thus

$$\widetilde{\partial}(a) = \widetilde{\partial}([s]_x) = [\widetilde{\partial}_U(s)]_x =$$
$$[s^{-1} \cdot \bar{d}_U s]_x = [s^{-1}]_x \cdot [\bar{d}_U s]_x = a^{-1} \cdot da.$$

On the other hand, by (3.2.2), (3.2.4), the commutativity of $\mathcal{A}$ and $\mathcal{A}^{\bullet}$, as well as convention (1.1.3), we straightforwardly see that

$$\widetilde{\partial}(s \cdot t) = \widetilde{\partial}(s) + \widetilde{\partial}(t); \qquad s, t \in \mathcal{A}^{\bullet}(U), \tag{3.2.5}$$

whose stalk-wise equivalent is

$$\widetilde{\partial}(a \cdot b) = \widetilde{\partial}(a) + \widetilde{\partial}(b); \qquad a, b \in \mathcal{A}^{\bullet} \times_X \mathcal{A}^{\bullet}. \tag{3.2.5$'$}$$

The (equivalent) equalities (3.2.5) and (3.2.5$'$) may be thought of as the *Leibniz condition* of $\widetilde{\partial}$.

Inspired by the foregoing discussion and that of Section 3.1, we now proceed to the matrix extension of $\widetilde{\partial}$, which will motivate the general setting of Section 3.3. First we construct the matrix analog of $\mathcal{A}^{\bullet}$; that is, the

general linear group sheaf (of order n), denoted by $\mathcal{GL}(n, \mathcal{A})$. It is the sheaf of groups generated by the *complete* presheaf

$$U \longmapsto \mathrm{GL}(n, \mathcal{A}(U)),$$

with U running in $\mathfrak{T}_X$; in other words,

$$\mathcal{GL}(n, \mathcal{A}) := \mathbf{S}\big(U \longmapsto \mathrm{GL}(n, \mathcal{A}(U))\big). \tag{3.2.6}$$

As usual, $\mathrm{GL}(n, \mathcal{A}(U))$ is the ***general linear group*** with coefficients in $\mathcal{A}(U)$, i.e., the group of *invertible* $n \times n$ matrices with entries in $\mathcal{A}(U)$, so

$$\mathcal{GL}(n, \mathcal{A})(U) \cong \mathrm{GL}(n, \mathcal{A}(U)) = M_n(\mathcal{A}(U))^{\bullet}, \tag{3.2.7}$$

for all $U \in \mathfrak{T}_X$. Therefore, by (3.1.1) – (3.1.3), and in analogy to $\mathcal{A}^{\bullet}$,

$$\mathcal{GL}(n, \mathcal{A}) = \mathcal{M}_n(\mathcal{A})^{\bullet}, \tag{3.2.8}$$

i.e., $\mathcal{GL}(n, \mathcal{A})$ *is the sheaf of units of the matrix algebra* $\mathcal{M}_n(\mathcal{A})$.

Next, generalizing (3.2.4) and retaining, for convenience, the same symbols, we set

$$\widetilde{\partial}_U(a) := a^{-1} \cdot d_U a; \quad a \in \mathrm{GL}(n, \mathcal{A}(U)), \tag{3.2.9}$$

where d_U is given by (3.1.10). Running U in $\mathfrak{T}_X$, (3.2.9) generates a morphism of sheaves of sets (see also (3.1.17))

$$\widetilde{\partial} : \mathcal{GL}(n, \mathcal{A}) \longrightarrow \mathcal{M}_n(\Omega) \cong \Omega \otimes_{\mathcal{A}} \mathcal{M}_n(\mathcal{A}). \tag{3.2.10}$$

This is, by definition, the ***logarithmic differential of*** $\mathcal{GL}(n, \mathcal{A})$.

For any $a, b \in \mathrm{GL}(n, \mathcal{A}(U))$, equalities (3.1.12) and (3.2.9) imply that

$$\begin{aligned} \widetilde{\partial}_U(a \cdot b) &= (a \cdot b)^{-1} \cdot d_U(a \cdot b) \\ &= b^{-1} \cdot a^{-1} \cdot (d_U a) \cdot b + b^{-1} \cdot d_U b \\ &= b^{-1} \cdot \widetilde{\partial}_U(a) \cdot b + \widetilde{\partial}_U(b), \end{aligned} \tag{3.2.11}$$

which is the *Leibniz condition of* $\widetilde{\partial}$, expressed in terms of local sections.

Let us elaborate a little more on (3.2.11), in order to describe its stalkwise version, thus paving the way to the considerations of Section 3.3: Fixing a $U \in \mathfrak{T}_X$ and a matrix $b \in \mathrm{GL}(n, \mathcal{A}(U))$, we define the automorphism (of a Lie algebra $\mathcal{A}|_U$-module)

$$\mathrm{Ad}_U(b) : \mathcal{M}_n(\mathcal{A})|_U \longrightarrow \mathcal{M}_n(\mathcal{A})|_U,$$

generated by the automorphisms of $M_n(\mathcal{A}(V))$

$$\mathrm{ad}(b|_V) : a \longmapsto b|_V \cdot a \cdot b^{-1}|_V,$$

for all open $V \subseteq U$ (see also the notations preceding (3.1.1)). The Lie algebra structure of $M_n(\mathcal{A}(V))$ is provided by the usual commutator of matrices. As a result of the completeness of the presheaves involved, the automorphisms $\mathrm{ad}(b|_V)$ identify with the automorphisms of sections induced by $\mathrm{Ad}_U(b)$, i.e.,

$$\mathrm{ad}(b|_V) \equiv \overline{\mathrm{Ad}_U(b)}_V.$$

Running $b \in \mathrm{GL}(n, \mathcal{A}(U))$, we obtain the morphism of groups

$$\mathrm{Ad}_U : \mathrm{GL}(n, \mathcal{A}(U)) \longrightarrow \mathrm{Aut}((\mathcal{M}_n(\mathcal{A})|_U)).$$

If we now allow U to vary in $\mathfrak{T}_X$, we obtain a morphism of (complete) presheaves $(\mathrm{Ad}_U)_{U \in \mathfrak{T}_X}$, generating, in turn, a morphism of sheaves of groups, denoted by

$$\mathcal{A}d : \mathcal{GL}(n, \mathcal{A}) \longrightarrow \mathcal{A}ut(\mathcal{M}_n(\mathcal{A})). \tag{3.2.12}$$

By analogy with the classical case, (3.2.12) is called the ***adjoint representation*** of $\mathcal{GL}(n, \mathcal{A})$.

We define a similar representation of $\mathcal{GL}(n, \mathcal{A})$ in $\mathcal{A}ut(\mathcal{M}_n(\Omega))$,

$$\mathcal{A}d : \mathcal{GL}(n, \mathcal{A}) \longrightarrow \mathcal{A}ut(\mathcal{M}_n(\Omega)), \tag{3.2.12$'$}$$

whose construction follows that of (3.2.12). The only difference is that, for any $b \in \mathrm{GL}(n, \mathcal{A}(U))$, the map $\mathrm{ad}(b)$ is now

$$\mathrm{ad}(b) : M_n(\Omega(U)) \longrightarrow M_n(\Omega(U)) : \omega \to b \cdot \omega \cdot b^{-1}.$$

For convenience, we use the same symbol and terminology for both (3.2.12) and (3.2.12′), their difference being clear from the context.

The representation (3.2.12′) induces a natural action of $\mathcal{GL}(n, \mathcal{A})$ on the left of $\mathcal{M}_n(\Omega)$:

$$\delta_n : \mathcal{GL}(n, \mathcal{A}) \times_X \mathcal{M}_n(\Omega) \longrightarrow \mathcal{M}_n(\Omega). \tag{3.2.13}$$

This is the morphism of sheaves generated by the local actions, for every open $U \subseteq X$,

$$\delta_{n,U} : \mathrm{GL}(n, \mathcal{A}(U)) \times M_n(\Omega(U)) \longrightarrow M_n(\Omega(U)),$$

determined by

$$\begin{aligned} \delta_{n,U}(b,\omega) &:= b\cdot\omega\cdot b^{-1} = \mathrm{ad}(b)(\omega) \\ &= \overline{\mathrm{Ad}_U(b)}_U(\omega) \equiv \mathrm{Ad}_U(b)(\omega), \end{aligned} \tag{3.2.14}$$

for every (a,ω) in the given domain.

To remind ourselves that δ_n is induced by the adjoint representation, we adopt the following notation:

$$\delta_n(g,w) =: \mathcal{A}d(g).w, \qquad (g,w)\in\mathcal{GL}(n,\mathcal{A})\times_X\mathcal{M}_n(\Omega). \tag{3.2.15}$$

The line dot on the right-hand side of (3.2.15) is set in order to distinguish this action from various multiplications of matrices introduced at earlier stages and were denoted, as usual, by a center dot. Further comments on this notation will follow shortly.

Applying the previous notations, (3.2.11) takes the equivalent form

$$\widetilde{\partial}_U(a\cdot b) = \mathrm{Ad}_U(b^{-1})(\widetilde{\partial}_U(a)) + \widetilde{\partial}_U(b), \qquad a,b\in\mathrm{GL}(n,\mathcal{A}(U)).$$

Thus, by (1.2.17) and (3.2.15), the previous equalities, for all $U\in\mathfrak{T}_X$, lead to the following fundamental property of the logarithmic differential:

$$\widetilde{\partial}(g\cdot h) = \mathcal{A}d(h^{-1}).\widetilde{\partial}(g) + \widetilde{\partial}(h), \tag{3.2.16}$$

for every $(g,h)\in\mathcal{GL}(n,\mathcal{A})\times_X\mathcal{GL}(n,\mathcal{A})$. Its section-wise analog, for each $U\in\mathfrak{T}_X$, is (3.2.11) or its equivalent form given just before (3.2.16) above. Notice that (3.2.5′) is actually (3.2.16) in the case of $\mathcal{GL}(1,\mathcal{A})\equiv\mathcal{A}^{\bullet}$.

In the previous considerations $\widetilde{\partial}$ was taken as an $\mathcal{M}_n(\Omega)$-valued morphism on $\mathcal{GL}(n,\mathcal{A})$, satisfying (3.2.16) with respect to the action (3.2.13). Let us now think of $\widetilde{\partial}$ as taking values in $\Omega\otimes_{\mathcal{A}}\mathcal{M}_n(\mathcal{A})$, after the identification (3.1.7). In this case we consider the action

$$\delta'_n : \mathcal{GL}(n,\mathcal{A})\times_X\big(\Omega\otimes_{\mathcal{A}}\mathcal{M}_n(\mathcal{A})\big) \longrightarrow \Omega\otimes_{\mathcal{A}}\mathcal{M}_n(\mathcal{A}), \tag{3.2.13′}$$

generated by the local actions (see also (3.1.3))

$$\delta'_{n,U} : \mathrm{GL}(n,\mathcal{A}(U))\times\big(\Omega(U)\otimes_{\mathcal{A}(U)}M_n(\mathcal{A}(U))\big) \longrightarrow \Omega(U)\otimes_{\mathcal{A}(U)}M_n(\mathcal{A}(U))$$

given (on decomposable tensors) by

$$\delta'_{n,U}(g,\theta\otimes a) := \theta\otimes\mathrm{Ad}_U(g)(a) = \theta\otimes(g\cdot a\cdot g^{-1}). \tag{3.2.14′}$$

We set again

$$(3.2.15') \quad \delta'_n(g,w) =: \mathcal{A}d(g).w; \qquad (g,w) \in \mathcal{GL}(n,\mathcal{A}) \times_X (\Omega \otimes_{\mathcal{A}} \mathcal{M}_n(\mathcal{A})),$$

where $\mathcal{A}d$ is meant in the sense of (3.2.12).

Equalities (3.2.15) and (3.2.15′) are consistent after the identification (3.1.7). More precisely, one obtains the commutative diagram

$$\begin{array}{ccc} \mathcal{GL}(n,\mathcal{A}) \times_X \big(\Omega \otimes_{\mathcal{A}} \mathcal{M}_n(\mathcal{A})\big) & \xrightarrow{\delta'_n} & \Omega \otimes_{\mathcal{A}} \mathcal{M}_n(\mathcal{A}) \\ {\scriptstyle 1\times\mu^1}\downarrow & & \downarrow{\scriptstyle \mu^1} \\ \mathcal{GL}(n,\mathcal{A}) \times_X \mathcal{M}_n(\Omega) & \xrightarrow{\delta_n} & \mathcal{M}_n(\Omega) \end{array}$$

DIAGRAM 3.1

where $1 = id_{\mathcal{GL}(n,\mathcal{A})}$, and μ^1 is the isomorphism generated by (μ^1_U), the latter being defined by (3.1.6b). This follows from the commutative diagram

$$\begin{array}{ccc} \mathrm{GL}(n,\mathcal{A}(U)) \times \big(\Omega(U) \otimes_{\mathcal{A}(U)} M_n(\mathcal{A}(U))\big) & \xrightarrow{\delta'_{n,U}} & \Omega(U) \otimes_{\mathcal{A}(U)} M_n(\mathcal{A}(U)) \\ {\scriptstyle 1\times\mu^1_U}\downarrow & & \downarrow{\scriptstyle \mu^1_U} \\ \mathrm{GL}(n,\mathcal{A}(U)) \times M_n(\Omega(U)) & \xrightarrow{\delta_{n,U}} & M_n(\Omega(U)) \end{array}$$

DIAGRAM 3.2

for every open $U \subseteq X$. To prove the commutativity of Diagram 3.2, it suffices to show that

$$\mu^1_U\big(\theta \otimes (g \cdot a \cdot g^{-1})\big) = g \cdot \big(a_{ij} \cdot \theta\big) \cdot g^{-1},$$

for every $g \in \mathrm{GL}(n,\mathcal{A}(U))$ and $\theta \otimes a \in \Omega(U) \otimes_{\mathcal{A}(U)} M_n(\mathcal{A}(U))$, with $a = (a_{ij})$.

Indeed, if

$$g = (g_{ij}), \quad g^{-1} = (h_{ij}), \quad g \cdot a \cdot g^{-1} = (c_{ij}),$$

equality (3.1.6b) implies that

$$\text{(A)}\quad \mu_U^1(\theta \otimes (g \cdot a \cdot g^{-1})) = \mu_U^1(\theta \otimes (c_{ij})) = (c_{ij} \cdot \theta) = \sum_{\kappa,\lambda} g_{i\lambda} \cdot a_{\lambda\kappa} \cdot h_{\kappa j} \cdot \theta.$$

On the other hand, the ij-entry of $g \cdot (a_{ij} \cdot \theta) \cdot g^{-1}$ has the form

$$\text{(B)}\qquad \sum_{\kappa,\lambda} g_{i\lambda} \cdot a_{\lambda\kappa} \cdot \theta \cdot h_{\kappa j}.$$

Since $\mathcal{A}$ (and each $\mathcal{A}(U)$) is commutative, the last term of (A) coincides with (B), thus we prove the commutativity of Diagrams 3.2 and 3.1.

As a consequence of (3.1.7), (3.2.15) and (3.2.15′), we see that the morphism $\widetilde{\partial} : \mathcal{GL}(n,\mathcal{A}) \to \Omega \otimes_{\mathcal{A}} \mathcal{M}_n(\mathcal{A})$ satisfies also (3.2.16). It is now clear that the precise meaning of the expression $\mathcal{A}d(g^{-1}).\widetilde{\partial}(w)$ in (3.2.16) depends on the interpretation of $\widetilde{\partial}$ (as a morphism with values in $\mathcal{M}_n(\Omega)$ or $\Omega \otimes_{\mathcal{A}} \mathcal{M}_n(\mathcal{A})$) and the respective action ((3.2.13) or (3.2.13′)).

Summarizing the above constructions we obtain:

3.2.1 Proposition. *Let $(\mathcal{A}, d, \Omega)$ be a given differential triad over the topological space X. Then the general linear group sheaf $\mathcal{GL}(n,\mathcal{A})$ admits the adjoint representation $\mathcal{A}d : \mathcal{GL}(n,\mathcal{A}) \to \mathcal{A}ut(\mathcal{M}_n(\mathcal{A}))$ and the logarithmic differential $\widetilde{\partial} : \mathcal{GL}(n,\mathcal{A}) \to \mathcal{M}_n(\Omega) \cong \Omega \otimes_{\mathcal{A}} \mathcal{M}_n(\mathcal{A})$ satisfying equality (3.2.16), with respect to the action of $\mathcal{GL}(n,\mathcal{A})$ on $\mathcal{M}_n(\Omega)$ and $\Omega \otimes_{\mathcal{A}} \mathcal{M}_n(\mathcal{A})$ by means of $\mathcal{A}d$.*

3.2.2 Remark. For certain calculations it would be desirable to distinguish the differential $\widetilde{\partial} : \mathcal{GL}(n,\mathcal{A}) \to \mathcal{M}_n(\Omega)$ –induced by (3.2.9)– from its counterpart with values in $\Omega \otimes_{\mathcal{A}} \mathcal{M}_n(\mathcal{A})$. In the latter case we would prefer to use the symbol ∂, thus

$$\text{(3.2.17)}\qquad \partial := \lambda^1 \circ \widetilde{\partial},$$

where λ^1 is the isomorphism (3.1.7). This conforms with the general notation of Definition 3.3.2 in the sequel (see also the discussion of Example 3.3.6(b)).

3.3. The Maurer-Cartan differential

As in the previous sections, let $(\mathcal{A}, d, \Omega)$ be a fixed differential triad over $X \equiv (X, \mathfrak{T}_X)$. We assume that $\mathcal{G}$ is a *sheaf of groups* and $\mathcal{L}$ a ***Lie algebra***

$\mathcal{A}$-module over X. From the relevant discussion of $\mathcal{M}_n(\mathcal{A})$ in Section 3.1, we recall that $\mathcal{L}$ is simultaneously a sheaf of Lie algebras and an $\mathcal{A}$-module over X.

We further assume that we are given a ***representation of $\mathcal{G}$ on $\mathcal{L}$***; i.e, a morphism of sheaves of groups

$$\rho : \mathcal{G} \longrightarrow \mathcal{A}ut(\mathcal{L}).$$

Here $\mathcal{A}ut(\mathcal{L})$ is the sheaf of germs of automorphisms of $\mathcal{L}$ with respect to both of its structures, i.e.,

$$\mathcal{A}ut(\mathcal{L}) = \mathcal{A}ut_{Lie\ alg}(\mathcal{L}) \cap \mathcal{A}ut_{\mathcal{A}}(\mathcal{L}).$$

Analogously to the case of an ordinary group action, an ***action of $\mathcal{G}$ on $\mathcal{L}$*** (from the left) is a morphism of sheaves

$$\delta : \mathcal{G} \times_X \mathcal{L} \longrightarrow \mathcal{L} : \ (g, u) \mapsto \delta(g, u) =: g.u,$$

satisfying the properties:

$$(g \cdot g').u = g.(g'.u),$$

for every $g, g' \in \mathcal{G}$ and $u \in \mathcal{L}$ over the same base point, and

$$1_x .u = u,$$

for every $u \in \mathcal{L}_x$ and every $x \in X$. The reader might have observed that, in the notation $g.u$, we have again used a line dot to distinguish the action from the multiplication of $\mathcal{G}$ (denoted by a center dot). This practice has already been applied in (3.2.15).

3.3.1 Proposition. *A representation ρ induces an action of $\mathcal{G}$ on $\mathcal{L}$, compatible with the $\mathcal{A}$-module structure of $\mathcal{L}$; that is, the following equalities hold true:*

$$g.(au + bu') = a\, g.u + b\, g.u',$$
$$g.[u, u'] = [g.u,\ g.u'],$$

for every $g \in \mathcal{G}_x$; $a, b \in \mathcal{A}_x$; $u, u' \in \mathcal{L}_x$, and every $x \in X$. Conversely, an action δ, compatible with the $\mathcal{A}$-module structure of $\mathcal{L}$, determines a representation ρ.

Recall that $\mathcal{A}\times_X\mathcal{L}\to\mathcal{L}:(a,u)\mapsto a\cdot u\equiv au$ is the "scalar" multiplication (see Subsection 1.1.2).

Proof. For any open $U\subseteq X$, we define the local action

$$\delta_U:\mathcal{G}(U)\times\mathcal{L}(U)\longrightarrow\mathcal{L}(U)\quad\text{with}\quad\delta_U(g,\ell):=\rho(g)(\ell), \tag{3.3.1}$$

where convention (1.3.1) has been repeatedly applied to ρ and $\rho(g)$. In fact, $\rho(g)$ in (3.3.1) is the isomorphism

$$\overline{\bar{\rho}_U(g)}_U:\mathcal{L}(U)\longrightarrow\mathcal{L}(U), \tag{3.3.2}$$

where $\bar{\rho}_U(g):\mathcal{G}(U)\to\mathcal{A}ut(\mathcal{L})(U)\cong\mathrm{Aut}(\mathcal{L}|_U)$. This is undoubtedly a cumbersome notation and will not be applied, unless the clarity of the arguments is at stake.

By elementary computations we verify that (3.3.1) is an action compatible with the structure of $\mathcal{L}(U)$. Therefore, varying U in the topology of X, we get a presheaf morphism (δ_U) generating the desired action.

Conversely, given δ, we construct a representation ρ as follows. For fixed $U\in\mathfrak{T}_X$ and $s\in\mathcal{G}(U)$, we define the family

$$\big\{\rho_U(s)_V:\mathcal{L}(V)\longrightarrow\mathcal{L}(V)\mid\forall\text{ open }V\subseteq U\big\}, \tag{3.3.3}$$

given by

$$\rho_U(s)_V(\ell):=\delta(s|_V,\ell)=s|_V.\,\ell,\qquad\ell\in\mathcal{L}(V).$$

For each V, the map $\rho_U(s)_V$ is an isomorphism of the structure involved, thus (3.3.3) is a presheaf morphism generating an automorphism $\rho_U(s)\in\mathrm{Aut}(\mathcal{L}|_U)$. We check that each $\rho_U:\mathcal{G}(U)\to\mathrm{Aut}(\mathcal{L}|_U)$ is a group morphism. Indeed, working section-wise, for every $s,t\in\mathcal{G}(U)$ and $\ell\in\mathcal{L}(V)$, with V any open subset of U, we have that

$$\begin{gathered}\rho_U(s\cdot t)_V(\ell)=(s\cdot t)|_V.\,\ell=s|_V.(t|_V.\,\ell)=\\ \rho_U(s)_V\left(\rho_U(t)_V(\ell)\right)=\left(\rho_U(s)_V\circ\rho_U(t)_V\right)(\ell),\end{gathered}$$

which proves the claim. Hence, varying U in $\mathfrak{T}_X$, we get a morphism of presheaves of groups (ρ_U) generating a representation ρ. □

For the sake of completeness, let us examine the morphisms of sections $\bar{\delta}_U:\mathcal{G}(U)\times\mathcal{L}(U)\to\mathcal{L}(U)$, $U\in\mathfrak{T}_X$, induced by the action δ determined by a representation, as in the first part of Proposition 3.3.1. If we set

$$g.\ell:=\bar{\delta}_U(g,\ell);\qquad(g,\ell)\in\mathcal{G}(U)\times\mathcal{L}(U),$$

we see that

$$(g.\ell)(x) = \bar{\delta}_U(g,\ell)(x) = \delta(g(x),\ell(x)) = g(x).\ell(x), \qquad x \in U.$$

However, by (1.2.17), $\bar{\delta}_U$ coincides (up to isomorphism) with δ_U, thus

$$(g.\ell)(x) = \bar{\delta}_U(g,\ell)(x) \equiv \delta_U(g,\ell)(x) = (\rho(g)(\ell))\,(x), \qquad x \in U.$$

Hence, within an isomorphism, we obtain the following equality complementing (3.3.1)

$$(3.3.1') \qquad g.\ell = \rho(g)(\ell) = \delta_U(g,\ell), \qquad (g,l) \in \mathcal{G}(U) \times \mathcal{L}(U).$$

We now wish to extend the action δ on $\mathcal{L}$, induced by ρ, to an action on $\Omega \otimes_{\mathcal{A}} \mathcal{L}$. Before proceeding, we introduce the following convenient shorthand notation, which will be used systematically throughout. Namely, we set

$$(3.3.4) \qquad \boxed{\Omega(\mathcal{L}) := \Omega \otimes_{\mathcal{A}} \mathcal{L}}$$

This is the abstract analog of the sheaf of ordinary 1-forms on a differential manifold X with values in a Lie algebra.

A representation $\rho : \mathcal{G} \to \mathcal{A}ut(\mathcal{L})$ induces an action

$$(3.3.5) \qquad \Delta : \mathcal{G} \times_X \Omega(\mathcal{L}) \longrightarrow \Omega(\mathcal{L}),$$

generated by the presheaf morphism

$$\big\{\Delta_U : \mathcal{G}(U) \times \big(\Omega(U) \otimes_{\mathcal{A}(U)} \mathcal{L}(U)\big) \longrightarrow \big(\Omega(U) \otimes_{\mathcal{A}(U)} \mathcal{L}(U)\big) \mid U \in \mathfrak{T}_X\big\}$$

defined, in turn, by (see also (3.3.1) and (3.3.2))

$$(3.3.6) \qquad \Delta_U(s,\sigma) = (1 \otimes \rho(s))(\sigma); \qquad 1 \equiv 1_{\Omega(U)},$$

for every $s \in \mathcal{G}(U)$, $\sigma \in \Omega(U) \otimes_{\mathcal{A}(U)} \mathcal{L}(U)$, and every $U \in \mathfrak{T}_X$. Notice that the presheaf $U \longmapsto \mathcal{G}(U) \times (\Omega(U) \otimes_{\mathcal{A}(U)} \mathcal{L}(U))$ generates $\mathcal{G} \times_X \Omega(\mathcal{L})$, according to Subsection 1.3.6.

In conformity with (3.2.15) and the classical case (which actually inspired the former equality), the result of the action Δ will be denoted by the following notation reminding us of the involvement of ρ:

$$(3.3.7) \qquad \boxed{\rho(g).w := \Delta(g,w)}$$

for every $(g, w) \in \mathcal{G} \times_X \Omega(\mathcal{L})$.

It is immediately checked that Δ is compatible with the $\mathcal{A}$-module structure of $\Omega(\mathcal{L})$, thus

$$\rho(g).(aw + bw') = a\,\rho(g).w + b\,\rho(g).w',$$

for every $g \in \mathcal{G}_x$; $a, b \in \mathcal{A}_x$; $w, w' \in \Omega(\mathcal{L})_x$, and every $x \in X$.

Another expression of (3.3.7), needed in various calculations, is obtained in the following way: For a given $(g, w) \in \mathcal{G}_x \times \Omega(\mathcal{L})_x$, there are sections $s \in \mathcal{G}(U)$ and $\sigma \in \Omega(U) \otimes_{\mathcal{A}(U)} \mathcal{L}(U)$ such that $g = s(x)$ and $w = [\sigma]_x = \tilde{\sigma}(x)$ (see (1.2.10)). Obviously, we can take the same open $U \subseteq X$ for both sections, by restricting the original domains to their intersection containing x. Then, applying (3.3.6), and writing $\rho(s)$ in place of (3.3.2), we have that

$$\rho(g).w = [(1 \otimes \rho(s))(\sigma)]_x = \big((1 \otimes \rho(s))(\sigma)\big)^{\sim}(x). \tag{3.3.7$'$}$$

Here, the small tilde, put as a superscript, replaces (for the convenience of typography) the wide tilde set over the entire section, whenever the latter has a long expression, i.e.,

$$\big((1 \otimes \rho(s))(\sigma)\big)^{\sim}(x) := \widetilde{\overbrace{(1 \otimes \rho(s))(\sigma)}}(x). \tag{$\Diamond$}$$

Equality (3.3.7$'$) could have been directly used as the definition of the action Δ, but in this case one should have shown that Δ is indeed a well defined (continuous) morphism.

We now come to the first fundamental definition of this section.

3.3.2 Definition. Let $\mathcal{G}$ be a sheaf of groups and ρ a representation of $\mathcal{G}$ on an $\mathcal{A}$-module of Lie algebras $\mathcal{L}$. A ***Maurer-Cartan differential*** of $\mathcal{G}$ is a morphism of sheaves of sets

$$\partial : \mathcal{G} \longrightarrow \Omega(\mathcal{L}) \equiv \Omega \otimes_{\mathcal{A}} \mathcal{L}$$

with the property

$$\partial(g \cdot h) = \rho(h^{-1}).\partial(g) + \partial(h), \tag{3.3.8}$$

for every $(g, h) \in \mathcal{G} \times_X \mathcal{G}$.

Comparing the Maurer-Cartan differential ∂ with the logarithmic differential $\widetilde{\partial}$ of $\mathcal{GL}(n, \mathcal{A})$, it is obvious that the latter terminology could have been applied to ∂. However, we adhere to the first term for the sake of distinction.

As in (3.3.7), for any $s \in \mathcal{G}(U)$ and $\theta \in \Omega(\mathcal{L})(U)$, we set

$$\rho(s).\theta := \Delta(s, \theta) \equiv \bar{\Delta}_U(s, \theta); \tag{3.3.9}$$

that is, $\rho(s).\theta$ is the section of $\Omega(\mathcal{L})$ defined by

$$\big(\rho(s).\theta\big)(x) := \big(\rho(s(x))\big).\big(\theta(x)\big), \qquad x \in U. \tag{3.3.10}$$

Then the section-wise analog of (3.3.8) is

$$\partial(s \cdot t) = \rho(t^{-1}).\partial(s) + \partial(t), \tag{3.3.11}$$

for every $s, t \in \mathcal{G}(U)$, and every $U \in \mathfrak{T}_X$. Of course (3.3.8) and (3.3.11) are equivalent, as a result of (1.2.15) and (1.2.17).

The second fundamental definition of this section is:

3.3.3 Definition. A ***Lie sheaf of groups*** is a sheaf of groups $\mathcal{G}$ equipped with a Maurer-Cartan differential $\partial : \mathcal{G} \to \Omega(\mathcal{L})$.

Since a Maurer-Cartan differential is defined with respect to a given representation ρ, the structure of a Lie sheaf of groups $\mathcal{G}$, as before, is fully declared by writing

$$\mathcal{G} \equiv (\mathcal{G}, \rho, \mathcal{L}, \partial).$$

Note. The terminology of Definitions 3.3.2 and 3.3.3 will be justified in the concluding note of Example 3.3.6(a) below.

3.3.4 Definition. An ***abelian Lie sheaf of groups*** is a Lie sheaf of groups $\mathcal{G} \equiv (\mathcal{G}, \rho, \mathcal{L}, \partial)$ where $\mathcal{G}$ is a sheaf of abelian groups and ρ the trivial representation.

By the ***trivial representation*** ρ we mean that $\rho(s) \equiv id|_{\mathcal{L}_U}$, for every $s \in \mathcal{G}(U)$ and every $U \in \mathfrak{T}_X$. Therefore, for an abelian Lie sheaf of groups, equality (3.3.11) takes the form

$$\partial(s \cdot t) = \partial(s) + \partial(t). \tag{3.3.12}$$

An analogous equality holds stalk-wise.

For an arbitrary Lie sheaf of groups, with the notations (1.1.4) and (1.1.5), we prove the following:

3.3.5 Proposition. *The Maurer-Cartan differential has the properties:*

(i) $$\partial(\mathbf{1}) = 0,$$

(ii) $$\partial(s^{-1}) = -\rho(s).\partial(s),$$

(iii) $$\partial(s) = \partial(t) \quad \Rightarrow \quad \partial(s \cdot t^{-1}) = 0,$$

for every $s, t \in \mathcal{G}(U)$ and $U \subseteq X$ open. Analogous equalities hold stalk-wise.

Proof. The first property is an immediate consequence of (3.3.8) applied to the product section $\mathbf{1} \cdot \mathbf{1}$. The second is a result of (3.3.8) and (i):

$$0 = \partial(\mathbf{1}) = \partial(s \cdot s^{-1}) = \rho(s).\partial(s) + \partial(s^{-1}).$$

Finally, (ii) implies that

$$\partial(s \cdot t^{-1}) = \rho(t).(\partial(s) - \partial(t)) = 0,$$

since $\rho(t)$ is an $\mathcal{A}(U)$-isomorphism. □

We illustrate the previous definitions by the following examples, needed in subsequent sections.

3.3.6 Examples.

(a) Lie sheaves of groups from Lie groups

Let X, $\mathcal{A} = \mathcal{C}^\infty_X$, $\Omega = \Omega^1_X$ be as in Example 2.1.4(a) and let G be a *Lie group* with corresponding *Lie algebra* $\mathbb{G}$. We denote by

$$\mathcal{C}^\infty_X(G) := \mathbf{S}\big(U \longmapsto C^\infty(U, G)\big)$$

the sheaf of germs of G-valued smooth maps on X, and by

$$\mathcal{C}^\infty_X(\mathbb{G}) := \mathbf{S}\big(U \longmapsto C^\infty(U, \mathbb{G})\big)$$

the sheaf of germs of $\mathbb{G}$-valued smooth maps on X. If $\Lambda^1(U, \mathbb{G})$ is the space of $\mathbb{G}$-valued differential 1-forms on an open $U \subseteq X$, then

$$\Omega_X(\mathbb{G}) := \mathbf{S}\left(U \longmapsto \Lambda^1(U, \mathbb{G})\right)$$

denotes the sheaf of germs of $\mathbb{G}$-valued differential 1-forms on X.

Assuming that $\dim(\mathbb{G}) = n$ and fixing a basis (γ_i) for $\mathbb{G}$, we obtain the basis (E^U_i) for $C^\infty(U, \mathbb{G})$, with regard to $C^\infty(U, \mathbb{R})$, given by $E^U_i(x) := \gamma_i$, for all $x \in U$. Moreover, we define an $\mathcal{A}(U)$-isomorphism

(3.3.13) $$\underline{\lambda}^1_U : \Lambda^1(U, \mathbb{G}) \xrightarrow{\simeq} \Lambda^1(U, \mathbb{R}) \otimes_{C^\infty(U,\mathbb{R})} C^\infty(U, \mathbb{G})$$

as follows: Since any $\omega \in \Lambda^1(U, \mathbb{G})$ determines the forms $\omega_i \in \Lambda^1(U, \mathbb{R})$ $(i = 1, \ldots, n)$ by $\omega_x(v) = \sum_{i=1}^n \omega_{i,x}(v)\gamma_i$, for every $x \in U$ and $v \in T_xX$, we may write

$$\omega = \sum_{i=1}^{n} \omega_i E_i^U,$$

thus we set

$$\underline{\lambda}_U^1(\omega) := \sum_{i=1}^{n} \omega_i \otimes E_i^U. \tag{3.3.13a}$$

The inverse of $\underline{\lambda}_U^1$, denoted by $\underline{\mu}_U^1$, is given by

$$\underline{\mu}_U^1(\theta \otimes f) := \theta \cdot f, \tag{3.3.13b}$$

where the right-hand side is the 1-form given by

$$(\theta \cdot f)_x(v) = \theta_x(v) \cdot f(x),$$

for every x and v as before (for the global form of (3.3.13) see, e.g., Greub-Halperin-Vanstone [35, Vol. I, p. 81]).

The presheaf isomorphism $(\underline{\lambda}_U^1)$, when U is running the topology of X, generates the $\mathcal{C}_X^\infty$-isomorphism

$$\underline{\lambda}^1 : \Omega_X(\mathbb{G}) \xrightarrow{\simeq} \Omega \otimes_{\mathcal{C}_X^\infty} \mathcal{C}_X^\infty(\mathbb{G}), \tag{3.3.14}$$

whose inverse is denoted by $\underline{\mu}^1$.

Note. In the previous considerations it is convenient to take as open U's the *domains of the charts* of the differential structure of X (see the final note of Subsection 1.2.2).

Now, the ordinary adjoint representation $\mathrm{Ad} : G \to \mathrm{Aut}(\mathbb{G})$ induces the ***adjoint representation*** $\mathcal{A}d : \mathcal{C}_X^\infty(G) \to \mathcal{A}ut(\mathcal{C}_X^\infty(\mathbb{G}))$ generated by the morphisms

$$\mathrm{Ad}_U : C^\infty(U, G) \longrightarrow \mathrm{Aut}\big(\mathcal{C}_X^\infty(\mathbb{G})|_U\big),$$

given, in turn, by

$$\big(\mathrm{Ad}_U(g)(f)\big)(x) := \big(\mathrm{Ad}(g(x))\big)(f(x)),$$

for every $g \in C^\infty(U, G)$, $f \in C^\infty(V, \mathbb{G})$, $x \in V$, and every open $V \subseteq U$ (see the analogous discussion about $\mathcal{GL}(n, \mathcal{A})$ in Section 3.2).

The representation $\mathcal{A}d$ determines an action

$$\delta : \mathcal{C}^\infty_X(G) \times_X \Omega_X(\mathbb{G}) \longrightarrow \Omega_X(\mathbb{G})$$

by means of the family of local actions (for all open $U \subseteq X$)

$$\begin{aligned} \delta_U : C^\infty(U,G) \times \Lambda^1(U,\mathbb{G}) &\longrightarrow \Lambda^1(U,\mathbb{G}) : \\ (g,\omega) &\longmapsto \delta_U(g,\omega) := \mathrm{Ad}(g).\omega. \end{aligned} \tag{3.3.15}$$

The right-hand side of the preceding equality represents the form defined by

$$(\mathrm{Ad}(g).\omega)_x(v) = \mathrm{Ad}(g)(\omega_x(v)); \qquad x \in U,\ v \in T_xX. \tag{3.3.15$'$}$$

We set

$$\mathcal{A}d(a).w = \delta(a,w), \qquad (a,w) \in \text{domain}(\delta).$$

Similarly, we get the action

$$\delta' : \mathcal{C}^\infty_X(G) \times_X \big(\Omega \otimes_{\mathcal{C}^\infty_X} \mathcal{C}^\infty_X(\mathbb{G})\big) \longrightarrow \Omega \otimes_{\mathcal{C}^\infty_X} \mathcal{C}^\infty_X(\mathbb{G})$$

from the morphisms

$$\begin{aligned} \delta'_U : C^\infty(U,G) \times \big(\Lambda^1(U,\mathbb{R}) \otimes_{C^\infty(U,\mathbb{R})} C^\infty(U,\mathbb{G})\big) &\longrightarrow \\ \Lambda^1(U,\mathbb{R}) \otimes_{C^\infty(U,\mathbb{R})} C^\infty(U,\mathbb{G}),& \end{aligned} \tag{3.3.16}$$

the latter being given (on decomposable elements) by

$$\delta'_U(g,\theta\otimes f) = \theta \otimes \mathrm{Ad}(g)(f);$$

thus, for every $(g,\omega) \in$ domain(δ'_U),

$$\delta'_U(g,\omega) = (1 \otimes \mathrm{Ad}(g))\,(w). \tag{3.3.16$'$}$$

We recall that $\big(\mathrm{Ad}(g)(f)\big)(x) := \big(\mathrm{Ad}(g(x))\big)(f(x))$. Again, we may set

$$\mathcal{A}d(a).w = \delta'(a,w), \qquad (a,w) \in \text{domain}(\delta').$$

The two expressions of $\mathcal{A}d(a).w$ are identified by (3.3.14). To see this, it suffices to show that

$$\underline{\mu}^1_U \circ \delta'_U = \delta_U \circ (1 \times \underline{\mu}^1_U), \tag{$\triangle$}$$

for every $U \in \mathfrak{T}_X$. Indeed, for any $g \in C^\infty(U,G)$ and any decomposable tensor

$$\theta \otimes f \in \Lambda^1(U,\mathbb{R}) \otimes_{C^\infty(U,\mathbb{R})} C^\infty(U,\mathbb{G}),$$

we have that

$$(\underline{\mu}^1_U \circ \delta'_U)(g, \theta \otimes f) = \underline{\mu}^1_U\big(\theta \otimes \mathrm{Ad}(g)(f)\big) = \theta \cdot \mathrm{Ad}(g)(f),$$

while

$$\big(\delta_U \circ (1 \times \underline{\mu}^1_U)\big)(g, \theta \otimes f) = \mathrm{Ad}(g).(\theta \cdot f).$$

We readily check that the last terms of the previous equalities coincide.

On the other hand, for any $f \in C^\infty(U,\mathbb{R})$, the ordinary ***(left) logarithmic*** or ***total differential*** of f

$$f^{-1} \cdot df \in \Lambda^1(U,\mathbb{G})$$

is given by (see, e.g., Bourbaki [15, p. 162], Kriegl-Michor [52, p. 404], Kreĭn-Yatskin [51, p. 55])

$$(f^{-1} \cdot df)_x(u) := (T_{f(x)}\lambda_{f(x)^{-1}} \circ T_x f)(u); \quad x \in U,\ u \in T_x M,$$

where λ_g is the left translation of G (by $g \in G$) and $T_x f \equiv d_x f$ the ordinary differential of f at x. Thus, in virtue of the identifications $T_e G \cong \mathbb{G}$ and (3.3.13), we define a corresponding logarithmic differential between sections

(3.3.17)
$$\begin{aligned} \partial_U : C^\infty(U,G) &\longrightarrow \Lambda^1(U,\mathbb{R}) \otimes_{C^\infty(U,\mathbb{R})} C^\infty(U,\mathbb{G}) \cong \Lambda^1(U,\mathbb{G}) \\ f &\longmapsto \partial_U(f) := \underline{\lambda}^1_U(f^{-1} \cdot df) \equiv f^{-1} \cdot df, \end{aligned}$$

for every $U \in \mathfrak{T}_X$. By an easy calculation, we verify that

$$\partial_U(f \cdot g) = \mathrm{Ad}(g^{-1}).\partial_U(f) + \partial_U(g); \qquad f, g \in C^\infty(U,G).$$

Therefore, based on (3.3.15) and (3.3.15′) (or (3.3.16) and (3.3.16′), depending on the range of ∂_U), and varying U in the topology of X, we obtain a Maurer-Cartan differential

(3.3.18)
$$\partial : \mathcal{C}^\infty_X(G) \longrightarrow \Omega \otimes_{\mathcal{C}^\infty_X} \mathcal{C}^\infty_X(\mathbb{G}) \cong \Omega_X(\mathbb{G}),$$

satisfying the equality

$$\partial(a \cdot b) = \mathcal{A}d(b^{-1}).\partial(a) + \partial(b), \qquad (a,b) \in \mathcal{C}^\infty_X(G) \times_X \mathcal{C}^\infty_X(G).$$

Let us verify in detail the previous fundamental property of ∂, when we think of it as an $\Omega_X(\mathbb{G})$-valued morphism. In this case, for a pair $(a, w) \in \mathcal{C}_X^\infty(G)_x \times \Omega_X(\mathbb{G})_x$, we can find a $U \in \mathfrak{T}_X$ and $f \in C^\infty(U, G)$, $\omega \in \Lambda^1(U, \mathbb{G})$, such that $a = \widetilde{f}(x) = [f]_x$ and $w = \widetilde{\omega}(x) = [\omega]_x$. Thus (see ($\Diamond$), p. 104)

$$\mathcal{A}d(a).w = \delta(a, w) = \delta_U(f, \omega)\widetilde{\ }(x) = (\mathrm{Ad}(f).\omega)\widetilde{\ }(x).$$

Therefore, for any (a, b) in the domain of ∂, there are $f, g \in C^\infty(U, G)$ such that $a = \widetilde{f}(x)$ and $b = \widetilde{g}(x)$, and the definition of ∂ implies that

$$\begin{aligned}\partial(a \cdot b) &= \partial\big((\widetilde{f \cdot g})(x)\big) = \big(\partial_U(f \cdot g)\big)\widetilde{\ }(x) \\ &= \big(\mathrm{Ad}(g^{-1}).\partial_U(f)\big)\widetilde{\ }(x) + \widetilde{\partial_U(g)}(x) \\ &= \mathcal{A}d\big(\widetilde{g^{-1}}(x)\big).\widetilde{\partial_U(f)}(x) + \widetilde{\partial_U(g)}(x) \\ &= \mathcal{A}d(b^{-1}).\partial(a) + \partial(b).\end{aligned}$$

A similar proof works if we interpret ∂ as an $\Omega \otimes_{\mathcal{C}_X^\infty} \mathcal{C}_X^\infty(\mathbb{G})$-valued morphism.

Summarizing, we have shown that:

Given a Lie group G, the sheafification of the ordinary operator of (left) logarithmic differential determines a Maurer-Cartan differential ∂, with respect to the sheafification $\mathcal{A}d$ of the adjoint representation of G. Therefore,

$$\mathcal{C}_X^\infty(G) \equiv \big(\mathcal{C}_X^\infty(G), \mathcal{A}d, \mathcal{C}_X^\infty(\mathbb{G}), \partial\big)$$

is a Lie sheaf of groups.

Note. The present example now explains the terminology of Definition 3.3.3: It is inspired by the classical relation

$$f^{-1} \cdot df = f^*\alpha,$$

where α is the *Maurer-Cartan form* of G.

(b) *The Lie sheaves of groups $\mathcal{A}^\bullet$ and $\mathcal{GL}(n, \mathcal{A})$*

Let $(\mathcal{A}, d, \Omega)$ be a differential triad, $\mathcal{A}^\bullet$ the (abelian) group sheaf of units (see (3.2.1)) and $\rho^\bullet : \mathcal{A}^\bullet \to \mathcal{A}ut(\mathcal{A})$ the trivial representation given by $\rho^\bullet(s) := \mathrm{id}|\mathcal{A}_U$, for every $s \in \mathcal{A}(U)$ and every $U \in \mathfrak{T}_X$.

Thinking of $\mathcal{A}$ as a sheaf of Lie algebras in a trivial way (: $[a, b] := a \cdot b - b \cdot a = 0$), the morphism $\widetilde{\partial}$, defined by (3.2.4), is a Maurer-Cartan differential. Therefore,

$$\mathcal{A}^\bullet \equiv (\mathcal{A}^\bullet, \rho^\bullet, \mathcal{A}, \widetilde{\partial})$$

is an *abelian* Lie sheaf of groups, according to Definition 3.3.4.

Similarly, in virtue of Proposition 3.2.1, its preceding discussion, and equality (3.2.17), we have that

$$\mathcal{GL}(n,\mathcal{A}) \equiv \big(\mathcal{GL}(n,\mathcal{A}), \mathcal{A}d, \mathcal{M}_n(\mathcal{A}), \partial \equiv \widetilde{\partial}\big)$$

is a (*non abelian*) Lie sheaf of groups.

The classical holomorphic group sheaf $\mathcal{GL}(n,\mathcal{O})$, as expounded in Gunning [38], can be treated analogously. It provides another example of a Lie sheaf of groups.

(c) Projective systems of Lie groups

We consider a *projective* (or *inverse*) *system of Lie groups* $\{G_i, \rho_{ij}\}$ together with the induced projective system of *Lie algebras* $\{\mathbb{G}_i \cong T_eG_i,\ r_{ij} := T_e\rho_{ij}\}$, the indices running in a directed set $(J, \leq)$. The corresponding projective limits are denoted by

$$G := \varprojlim G_i \quad \text{and} \quad \mathbb{G} := \varprojlim \mathbb{G}_i, \tag{3.3.19}$$

respectively. For the general theory of projective systems and their limits we refer to Bourbaki [12, Chap. 3], Eilenberg-Steenrod [25, Chap. 8]. We note, in passing, that G is not necessarily a Lie group (however, see Galanis [30] for conditions ensuring that G is a Fréchet-Lie group whose Lie algebra is $\mathbb{G}$).

Let X be a fixed (finite-dimensional or Banach) smooth manifold and let $\mathfrak{T}_X$ be its canonical topology (induced by the smooth structure). It is readily checked that

$$(C^\infty(U,G_i), P_{ij}) \quad \text{and} \quad (C^\infty(U,\mathbb{G}_i), R_{ij})\,,$$

where

$$\begin{aligned} P_{ij}(f) &:= \rho_{ij}\circ f; \qquad f \in C^\infty(U,G_i),\\ R_{ij}(g) &:= r_{ij}\circ g; \qquad g \in C^\infty(U,\mathbb{G}_i), \end{aligned}$$

are projective systems, so the respective projective limits

$$\varprojlim C^\infty(U,G_i) \quad \text{and} \quad \varprojlim C^\infty(U,\mathbb{G}_i)$$

exist. Therefore, running U in $\mathfrak{T}_X$, we obtain the complete presheaves

$$U \longmapsto \varprojlim C^\infty(U,G_i), \quad U \longmapsto \varprojlim C^\infty(U,\mathbb{G}_i)$$

generating, respectively, the sheaf of groups

$$\mathcal{G} := \mathbf{S}\left(U \longmapsto \varprojlim C^\infty(U, G_i)\right)$$

and the sheaf of Lie algebras

$$\mathcal{L} := \mathbf{S}\left(U \longmapsto \varprojlim C^\infty(U, \mathbb{G}_i)\right).$$

As a result,

$$\mathcal{G}(U) \cong \varprojlim C^\infty(U, G_i), \tag{3.3.20}$$

$$\mathcal{L}(U) \cong \varprojlim C^\infty(U, \mathbb{G}_i). \tag{3.3.21}$$

The individual adjoint representations $\mathrm{Ad}^i : G_i \to \mathrm{Aut}(\mathbb{G}_i)$ determine the representation $\mathrm{Ad} : G \to \mathrm{Aut}(\mathbb{G})$ with

$$\mathrm{Ad}(g) := \varprojlim(\mathrm{Ad}^i(g_i)), \tag{3.3.22}$$

for every $g = (g_i) \in G$. Hence, working as in Example (a), we obtain a representation $\mathcal{A}d : \mathcal{G} \to \mathcal{A}ut(\mathcal{L})$.

On the other hand, for a fixed $U \in \mathfrak{T}_X$, the logarithmic differentials of G_i induce the maps

$$\partial_U^i : C^\infty(U, G_i) \longrightarrow \Lambda^1(U, \mathbb{R}) \otimes_{C^\infty(U,\mathbb{R})} C^\infty(U, \mathbb{G}_i); \qquad i \in J, \tag{3.3.23}$$

providing, in turn, a morphism of projective systems $\left(\partial_U^i\right)_{i\in J}$. Hence, we obtain the (limit) map

$$\partial_U := \varprojlim \partial_U^i : \varprojlim C^\infty(U, G_i) \longrightarrow \Lambda^1(U, \mathbb{R}) \otimes_{C^\infty(U,\mathbb{R})} \left(\varprojlim C^\infty(U, \mathbb{G}_i)\right).$$

Varying now U in $\mathfrak{T}_X$, we get a presheaf morphism (∂_U) generating a Maurer-Cartan differential $\partial : \mathcal{G} \to \Omega \otimes_{\mathcal{A}} \mathcal{L}$, where $\mathcal{A} = \mathcal{C}_X^\infty$ (see Example 2.1.4(a)). We verify (3.3.8) or (3.3.11) by applying, on the level of each ∂_U, the analogous property of (3.3.23) with respect to the representation induced by (3.3.22).

We can give another useful description of $\mathcal{G}$ (analogously for $\mathcal{L}$), by inducing the following generalized notion of differentiability for G-valued maps on X. Namely, an $f : U \to G$ is said to be ***generalized smooth*** if $f_i := p_i \circ f \in C^\infty(U, G_i)$, for every $i \in J$. Here p_i is the natural projection of G onto G_i. The set of generalized smooth maps is denoted by $\underline{C}^\infty(U, G)$. When U is running $\mathfrak{T}_X$, the assignment $U \mapsto \underline{C}^\infty(U, G)$ is a (complete) presheaf generating *the sheaf of germs of G-valued generalized smooth maps on X*, denoted by $\underline{\mathcal{C}}^{\infty}_{X}(G)$. Similarly, we define the sheaf $\underline{\mathcal{C}}^{\infty}_{X}(\mathbb{G})$ of germs of $\mathbb{G}$-valued generalized smooth maps on X.

As a consequence of (3.3.20) and (3.3.21), we obtain

$$\mathcal{G}(U) \cong \varprojlim C^{\infty}(U, G_i) =: \underline{C}^{\infty}(U, G),$$
$$\mathcal{L}(U) \cong \varprojlim C^{\infty}(U, \mathbb{G}_i) =: \underline{C}^{\infty}(U, \mathbb{G}),$$

which lead to the identifications

$$\mathcal{G} \cong \underline{\mathcal{C}}^{\infty}_{X}(G) \equiv \underline{\mathcal{C}}^{\infty}_{X}(\varprojlim G_i) \quad \text{and} \quad \mathcal{L} \cong \underline{\mathcal{C}}^{\infty}_{X}(\mathbb{G}) \equiv \underline{\mathcal{C}}^{\infty}_{X}(\varprojlim \mathbb{G}_i).$$

Therefore, one infers:

If $\mathcal{A}d$ and ∂ are the morphisms induced by (3.3.22) and (3.3.23), respectively, then

$$\big(\mathcal{G} \cong \underline{\mathcal{C}}^{\infty}_{X}(\varprojlim G_i),\, \mathcal{A}d,\, \mathcal{L} \cong \underline{\mathcal{C}}^{\infty}(\varprojlim \mathbb{G}_i),\, \partial\big)$$

is a Lie sheaf of groups.

It is known that every *compact* group is the projective limit of a family of compact Lie groups (see Price [104, p. 140], Weil [141]). Hence, compact groups fit in this example.

For the sake of completeness, we sketch another description of the sheaves $\mathcal{G}$ and $\mathcal{L}$ just obtained, which can motivate a general construction of projective limits of sheaves.

For a fixed index $i \in J$, we set

$$\mathcal{G}_i := \mathbf{S}\big(U \longmapsto C^{\infty}(U, G_i)\big), \quad \mathcal{L}_i := \mathbf{S}\big(U \longmapsto C^{\infty}(U, \mathbb{G}_i)\big).$$

On the other hand, we define the sheaves

$$\varprojlim \mathcal{G}_i := \mathbf{S}\left(U \longmapsto \varprojlim \mathcal{G}_i(U)\right), \quad \varprojlim \mathcal{L}_i := \mathbf{S}\left(U \longmapsto \varprojlim \mathcal{L}_i(U)\right).$$

The completeness of the previous presheaves and (3.3.20), (3.3.21) yield

$$\big(\varprojlim \mathcal{G}_i\big)(U) \cong \varprojlim \mathcal{G}_i(U) \cong \varprojlim \left(C^{\infty}(U, G_i)\right) \cong \mathcal{G}(U),$$
$$\big(\varprojlim \mathcal{L}_i\big)(U) \cong \varprojlim \mathcal{L}_i(U) \cong \varprojlim \left(C^{\infty}(U, \mathbb{G}_i)\right) \cong \mathcal{L}(U),$$

thus leading to

$$\mathcal{G} \cong \varprojlim \mathcal{G}_i \quad \text{and} \quad \mathcal{L} \cong \varprojlim \mathcal{L}_i.$$

In the same way, we can prove that $\partial \equiv \varprojlim \partial^i$, where the Maurer-Cartan differentials $\partial^i : \mathcal{G}_i \to \Omega \otimes_{\mathcal{A}} \mathcal{L}_i$ ($\mathcal{A} = \mathcal{C}^\infty_X$) are given by

$$\partial^i := \mathbf{S}\left((\partial^i_U)_{U \in \mathfrak{T}_X}\right), \qquad i \in J.$$

Of course, the structure of $\varprojlim \mathcal{G}_i$ can be defined directly as the limit of the projective system of topological spaces $(\mathcal{G}_i)_{i \in J}$, with connecting morphisms obtained by the sheafification of the original ones. However, the previous approach seems to be convenient when we want to complete the structure of $\mathcal{G}$ to a Lie sheaf of groups.

3.4. Morphisms of Lie sheaves of groups

Roughly speaking, morphisms of Lie sheaves of groups are morphisms of groups interrelated with their representations and the Maurer-Cartan differentials. They will be applied in order to link connections on principal sheaves with different structure sheaves (Chapter 6), and to introduce connections on sheaves associated with a given principal sheaf (Chapter 7).

Let $\mathcal{G} \equiv (\mathcal{G}, \rho_{\mathcal{G}}, \mathcal{L}_{\mathcal{G}}, \partial_{\mathcal{G}})$ and $\mathcal{H} \equiv (\mathcal{H}, \rho_{\mathcal{H}}, \mathcal{L}_{\mathcal{H}}, \partial_{\mathcal{H}})$ be two Lie sheaves of groups over the same base X. Clearly, the subscripts $\mathcal{G}$ and $\mathcal{H}$ mark the components of each Lie sheaf of groups, whenever necessary for the sake of clarity.

3.4.1 Definition. A ***morphism of Lie sheaves of groups*** from $\mathcal{G}$ into $\mathcal{H}$ is a pair $(\phi, \overline{\phi})$, where $\phi : \mathcal{G} \to \mathcal{H}$ and $\overline{\phi} : \mathcal{L}_{\mathcal{G}} \to \mathcal{L}_{\mathcal{H}}$ are morphisms of the corresponding structures (i.e., groups and $\mathcal{A}$-modules of Lie algebras, respectively) such that

$$\delta_{\mathcal{H}} \circ (\phi \times \overline{\phi}) = \overline{\phi} \circ \delta_{\mathcal{G}}, \tag{3.4.1}$$

$$\partial_{\mathcal{H}} \circ \phi = (1 \otimes \overline{\phi}) \circ \partial_{\mathcal{G}}. \tag{3.4.2}$$

Here $\delta_{\mathcal{G}}$ and $\delta_{\mathcal{H}}$ are the actions of $\mathcal{G}$ and $\mathcal{H}$ on the left of $\mathcal{L}_{\mathcal{G}}$ and $\mathcal{L}_{\mathcal{H}}$, respectively, induced by the corresponding representations $\rho_{\mathcal{G}}$ and $\rho_{\mathcal{H}}$, as in Proposition 3.3.1.

Conditions (3.4.1) and (3.4.2) mean that the following diagrams are commutative.

$$\begin{CD} \mathcal{G}\times_X \mathcal{L}_{\mathcal{G}} @>{\phi\times\overline{\phi}}>> \mathcal{H}\times_X \mathcal{L}_{\mathcal{H}} \\ @V{\delta_{\mathcal{G}}}VV \textbf{(I)} @VV{\delta_{\mathcal{H}}}V \\ \mathcal{L}_{\mathcal{G}} @>{\overline{\phi}}>> \mathcal{L}_{\mathcal{H}} \end{CD} \qquad \begin{CD} \mathcal{G} @>{\phi}>> \mathcal{H} \\ @V{\partial_{\mathcal{G}}}VV \textbf{(II)} @VV{\partial_{\mathcal{H}}}V \\ \Omega\otimes_{\mathcal{A}}\mathcal{L}_{\mathcal{G}} @>{1\otimes\overline{\phi}}>> \Omega\otimes_{\mathcal{A}}\mathcal{L}_{\mathcal{H}} \end{CD}$$

Diagrams 3.3

Occasionally it is useful to express the commutativity of Diagram 3.3(I) in the following equivalent form

$$\overline{\phi}\circ\rho_{\mathcal{G}}(g) = \rho_{\mathcal{H}}(\phi(g))\circ\overline{\phi}, \tag{3.4.1$'$}$$

for every $g\in\mathcal{G}(U)$ and every open $U\subseteq X$; in other words, the diagram

$$\begin{CD} \mathcal{L}_{\mathcal{G}}(U) @>{\overline{\phi}}>> \mathcal{L}_{\mathcal{H}}(U) \\ @V{\rho_{\mathcal{G}}(g)}VV @VV{\rho_{\mathcal{H}}(\phi(g))}V \\ \mathcal{L}_{\mathcal{G}}(U) @>{\overline{\phi}}>> \mathcal{L}_{\mathcal{H}}(U) \end{CD}$$

Diagram 3.4

is commutative. Obviously, the morphisms of the last diagram are the induced morphism of sections.

Indeed, for any $g\in\mathcal{G}(U)$ and $\ell\in\mathcal{L}_{\mathcal{G}}(U)$, Diagram 3.3(I) and equality (3.3.1), along with (1.1.3), imply that

$$\overline{\phi}(\rho_{\mathcal{G}}(g)(\ell)) = \overline{\phi}(\delta_{\mathcal{G}}(g,\ell)) = \delta_{\mathcal{H}}(\phi(g),\overline{\phi}(\ell)) = \rho_{\mathcal{H}}(\phi(g))(\overline{\phi}(\ell)),$$

which yields the commutativity of Diagram 3.4. The converse follows from the same calculations and the comments (1.2.15), (1.2.15$'$).

A typical example of a morphism of Lie sheaves of groups is provided by an ordinary morphism of Lie groups $f: G\to H$. It is obtained by the sheafification process described in Example 3.3.6(a), also applied, in an obvious way, to f and the induced Lie algebra morphism

$$T_e f \equiv d_e f : \mathbb{G}\cong T_e G \longrightarrow T_e H \cong \mathbb{H}.$$

Similarly, morphisms of projective systems of Lie groups induce morphisms of Lie sheaves of groups between the corresponding projective limits, defined in Example 3.3.6(c).

3.5. The pull-back of a Lie sheaf of groups

This important construction is more complicated than the pull-back of other structures encountered in previous sections. Our results rely heavily on the material of Subsection 1.4.1.

We assume that $\mathcal{G} \equiv (\mathcal{G}, \rho, \mathcal{L}, \partial)$ is a Lie sheaf of groups over X and $f : Y \to X$ a continuous map. We intend to show that $f^*(\mathcal{G})$ is a Lie sheaf of groups.

First we need the following general lemma concerning the pull-back of tensor products.

3.5.1 Lemma. *Let $\mathcal{S}$ and $\mathcal{T}$ be two $\mathcal{A}$-modules over X. If $f : Y \to X$ is a continuous map, then there is an $f^*(\mathcal{A})$-isomorphism*

$$\boldsymbol{\tau} : f^*(\mathcal{S} \otimes_{\mathcal{A}} \mathcal{T}) \xrightarrow{\simeq} f^*(\mathcal{S}) \otimes_{f^*(\mathcal{A})} f^*(\mathcal{T})$$

between the $f^*(\mathcal{A})$-modules (over Y) representing, respectively, the domain and the range of $\boldsymbol{\tau}$.

Proof. Let $(y, z) \in f^*(\mathcal{S} \otimes_{\mathcal{A}} \mathcal{T})$ be an arbitrarily chosen element. Obviously,

$$(y, z) \in \{y\} \times (\mathcal{S} \otimes_{\mathcal{A}} \mathcal{T})_{f(y)} \cong \{y\} \times \big(\mathcal{S}_{f(y)} \otimes_{\mathcal{A}_{f(y))}} \mathcal{T}_{f(y)}\big).$$

By the construction of the tensor product (Subsection 1.3.3), there exists a $\sigma \in \mathcal{S}(U) \otimes_{\mathcal{A}(U)} \mathcal{T}(U)$ such that (see (1.2.10))

$$z = [\sigma]_{f(y)} = \widetilde{\sigma}(f(y)), \tag{3.5.1}$$

for some $U \in \mathfrak{T}_X$, with $f(y) \in U$. Then we define $\boldsymbol{\tau}$ by setting

$$\boldsymbol{\tau}(y, z) = \boldsymbol{\tau}(y, \widetilde{\sigma}(f(y))) := \big((f^*_{\mathcal{S},U} \otimes f^*_{\mathcal{T},U})(\sigma)\big)^{\sim}(y), \tag{3.5.2}$$

where $f^*_{\mathcal{S},U} : \mathcal{S}(U) \to f^*(\mathcal{S})(f^{-1}(U))$ and $f^*_{\mathcal{T},U} : \mathcal{T}(U) \to f^*(\mathcal{T})(f^{-1}(U))$ are the canonical (adjunction) maps of sections (see (1.4.2) – (1.4.3)), and

$$\big((f^*_{\mathcal{S},V} \otimes f^*_{\mathcal{T},V})(\sigma)\big)^{\sim} \in \big(f^*(\mathcal{S}) \otimes_{f^*(\mathcal{A})} f^*(\mathcal{T})\big)\,(f^{-1}(U))$$

is the (local) sheaf section induced by the presheaf "section"

$$(f^*_{\mathcal{S},U} \otimes f^*_{\mathcal{T},U})(\sigma) \in f^*(\mathcal{S})(f^{-1}(U)) \otimes_{f^*(\mathcal{A})(f^{-1}(U))} f^*(\mathcal{T})(f^{-1}(U)). \tag{3.5.3}$$

We remind the reader that the meaning of the superscript "$^{\sim}$" is explained by equality ($\lozenge$) on p. 104 and the relevant comments.

By (3.5.2), it is clear that

$$\boldsymbol{\tau}(y,z) \in \big(f^*(\mathcal{S}) \otimes_{f^*(\mathcal{A})} f^*(\mathcal{T})\big)_y \cong f^*(\mathcal{S})_y \otimes_{f^*(\mathcal{A})_y} f^*(\mathcal{T})_y.$$

Leaving aside, for the moment, the question of whether $\boldsymbol{\tau}$ is well defined (this will be answered after establishing some useful formulas), we first check that $\boldsymbol{\tau}$ *is a morphism of sheaves*. Clearly, it commutes with the respective projections. We show the continuity of $\boldsymbol{\tau}$ at an arbitrary element $(y_{\rm o}, z_{\rm o})$: As before, we can find a $\sigma_{\rm o} \in \mathcal{S}(U_{\rm o}) \otimes_{\mathcal{A}(U_{\rm o})} \mathcal{T}(U_{\rm o})$ with $\sigma_{\rm o}(f(y_{\rm o})) = z_{\rm o}$, for some open $U_{\rm o} \subseteq X$ with $f(y_{\rm o}) \in U_{\rm o}$. Fixing $\sigma_{\rm o}$ and varying $y \in f^{-1}(U_{\rm o})$, we obtain the set

$$B := f^*_{\mathcal{S}\otimes_{\mathcal{A}}\mathcal{T},U_{\rm o}}(\widetilde{\sigma}_{\rm o})\left(f^{-1}(U_{\rm o})\right) = \big\{\big(y, \widetilde{\sigma}_{\rm o}(f(y))\big) \,\big|\, y \in f^{-1}(U_{\rm o})\big\},$$

which belongs to the basis for the topology of $f^*(\mathcal{S}\otimes_{\mathcal{A}}\mathcal{T})$, in analogy to the basis $(\mathcal{B})$ of Subsection 1.4.1.

As a result, in virtue of (3.5.2), the restriction of $\boldsymbol{\tau}$ to B is given by

$$\boldsymbol{\tau}|_B = \big((f^*_{\mathcal{S},V} \otimes f^*_{\mathcal{T},V})(\sigma_{\rm o})\big)^{\sim} \circ p_1\,\big|_B,$$

where $p_1 : Y \times_X (\mathcal{S}\otimes_{\mathcal{A}}\mathcal{T}) \to X$ is the restriction (to the fiber product) of the ordinary projection to the first factor $\mathrm{pr}_1 : Y \times (\mathcal{S}\otimes_{\mathcal{A}}\mathcal{T}) \to Y$. This proves the desired continuity, thus $\boldsymbol{\tau}$ is a sheaf morphism.

Now assume that the section $\sigma \in \mathcal{S}(U) \otimes_{\mathcal{A}(U)} \mathcal{T}(U)$, considered in the beginning of the proof, is a *decomposable* tensor of the form $\sigma = s \otimes t$. Since $s \in \mathcal{S}(U)$ and $t \in \mathcal{T}(U)$ are ordinary sections of sheaves, the identifications $s \equiv \widetilde{s}$ and $t \equiv \widetilde{t}$ hold true. Moreover, the definition of the tensor product of two elements $s(x) \otimes t(x)$ implies that (see the general construction at the end of Subsection 1.2.2)

$$\widetilde{\sigma}(x) = (\widetilde{s \otimes t})(x) = \widetilde{s}(x) \otimes \widetilde{t}(x) = s(x) \otimes t(x).$$

Hence, with similar arguments, we obtain that

$$\begin{gathered}\big((f^*_{\mathcal{S},U} \otimes f^*_{\mathcal{T},U})(\sigma)\big)^{\sim}(y) = \big((f^*_{\mathcal{S},U} \otimes f^*_{\mathcal{T},U})(s\otimes t)\big)^{\sim}(y) = \\ \big(f^*_{\mathcal{S},U}(s) \otimes f^*_{\mathcal{T},U}(t)\big)^{\sim}(y) = \big(f^*_{\mathcal{S},U}(s)\big)(y) \otimes \big(f^*_{\mathcal{T},U}(t)\big)(y),\end{gathered}$$

for every $y \in f^{-1}(U)$. Therefore, taking into account (1.4.4′), the previous equalities lead to

$$\begin{gathered}\big((f^*_{\mathcal{S},U} \otimes f^*_{\mathcal{T},U})(\sigma)\big)^{\sim}(y) = \big(y, s(f(y))\big) \otimes \big(y, t(f(y))\big) = \\ f^*_{\mathcal{S},y}\big(s(f(y))\big) \otimes f^*_{\mathcal{T},y}\big(t(f(y))\big) = (f^*_{\mathcal{S},y} \otimes f^*_{\mathcal{T},y})\big(s(f(y)) \otimes t(f(y))\big) = \\ \big(f^*_{\mathcal{S},y} \otimes f^*_{\mathcal{T},y}\big)\big((\widetilde{s \otimes t})(f(y))\big) = \big(f^*_{\mathcal{S},y} \otimes f^*_{\mathcal{T},y}\big)\big(\widetilde{\sigma}(f(y))\big),\end{gathered}$$

for every $y \in f^{-1}(U)$. From the preceding equalities and (3.5.2), we see that, for a decomposable section σ as above with $\sigma(f(y)) = z$,

$$\begin{aligned} \boldsymbol{\tau}(y,z) &= \boldsymbol{\tau}(y, \widetilde{\sigma}(f(y))) \\ &= \big((f^*_{\mathcal{S},U} \otimes f^*_{\mathcal{T},U})(\sigma)\big)^{\sim}(y) \\ &= \big(f^*_{\mathcal{S},y} \otimes f^*_{\mathcal{T},y}\big)\big(\widetilde{\sigma}(f(y))\big). \end{aligned} \tag{3.5.4}$$

The same result is extended by linearity to arbitrary (not necessarily decomposable) tensors; hence, we conclude that

$$\boldsymbol{\tau}(y,z) = \big(f^*_{\mathcal{S},y} \otimes f^*_{\mathcal{T},y}\big)(z), \qquad (y,z) \in f^*(\mathcal{S} \otimes_{\mathcal{A}} \mathcal{T}). \tag{3.5.5}$$

Since we arrived at the last equality having chosen arbitrary $U \in \mathfrak{T}_X$ and σ satisfying (3.5.1), this shows that $\boldsymbol{\tau}$ is in fact *well defined*, thus answering the question raised in the first steps of the proof.

Finally, by (3.5.5) and the fact that (see also (1.4.4))

$$\big(f^*_{\mathcal{S},y} \otimes f^*_{\mathcal{T},y}\big)(z) \in f^*(\mathcal{S})_y \otimes_{f^*(\mathcal{A})_y} f^*(\mathcal{T})_y \cong \big(f^*(\mathcal{S}) \otimes_{f^*(\mathcal{A})} f^*(\mathcal{T})\big)_y,$$

the restriction of $\boldsymbol{\tau}$ to the stalk $\{y\} \times (\mathcal{S} \otimes_{\mathcal{A}} \mathcal{T})_{f(y)} = (f^*(\mathcal{S} \otimes_{\mathcal{A}} \mathcal{T}))_y$ is an $f^*(\mathcal{A})_y$-*isomorphism* of the form

$$\big(f^*(\mathcal{S} \otimes_{\mathcal{A}} \mathcal{T})\big)_y \xrightarrow{\ \simeq\ } \big(f^*(\mathcal{S}) \otimes_{f^*(\mathcal{A})} f^*(\mathcal{T})\big)_y,$$

for every $y \in Y$. Therefore, $\boldsymbol{\tau}$ is an isomorphism as in the statement and the proof is now complete. □

A byproduct of equality (3.5.5) is the following useful formula:

$$\boldsymbol{\tau}(y, w \otimes u) = (y,w) \otimes (y,u), \quad (y, w \otimes u) \in \{y\} \times \Omega_{f(y)} \otimes_{\mathcal{A}_{f(y)}} \mathcal{L}_{f(y)}. \tag{3.5.5$'$}$$

Note. For another proof of Lemma 3.5.1, without explicit construction of the isomorphism $\boldsymbol{\tau}$, we refer to Mallios [62, Chap. VI, p. 27].

By means of Lemma 3.5.1 we define the morphism

$$\partial^* := \boldsymbol{\tau} \circ f^*(\partial). \tag{3.5.6}$$

It will be shown that ∂^* is the Maurer-Cartan differential of $f^*(\mathcal{G})$.

For every $(y,z) \in f^*(\mathcal{G}) = Y \times_X \mathcal{G}$, (3.5.6) yields

$$\partial^*(y,z) = \boldsymbol{\tau}\left(f^*(\partial)(y,z)\right) = \boldsymbol{\tau}(y, \partial(z)). \tag{3.5.7}$$

Omitting $\boldsymbol{\tau}$ in the last equalities, one can equivalently write

$$\partial^*(y,z) \equiv (y, \partial(z)); \qquad (y,z) \in f^*(\mathcal{G}) = Y \times_X \mathcal{G}, \tag{3.5.7$'$}$$

which facilitates, in practice, the calculations.

The definition of ∂^* is depicted in the following diagram, where (3.3.4) has been applied.

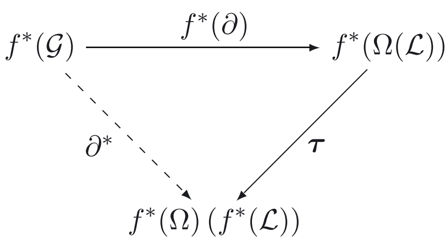


Diagram 3.5

We also need the following auxiliary result:

3.5.2 Lemma. *There exists a monomorphism of sheaves of groups*

$$i^* : f^*(\mathcal{A}ut(\mathcal{L})) \longrightarrow \mathcal{A}ut(f^*(\mathcal{L})).$$

Proof. For an arbitrary $(y,z) \in \{y\} \times \mathcal{A}ut(\mathcal{L})_{f(y)} = f^*(\mathcal{A}ut(\mathcal{L}))_y$, there exists a $g \in \mathrm{Aut}(\mathcal{L}|_U)$, where $U \in \mathfrak{T}_X$ with $f(y) \in U$, such that

$$z = [g]_{f(y)} = \widetilde{g}(f(y)).$$

Since $\mathrm{Aut}(\mathcal{L}|_U) \cong \mathcal{A}ut(\mathcal{L})(U)$, we have that $g = \widetilde{g}$ within the previous isomorphism. We denote by

$$id_{f^{-1}(U)} \times_U g : f^{-1}(U) \times_U (\mathcal{L}|_U) \longrightarrow f^{-1}(U) \times_U (\mathcal{L}|_U)$$

the automorphism obtained by restricting $id_{f^{-1}(U)} \times g$ to the fiber product $f^{-1}(U) \times_U (\mathcal{L}|_U)$; that is,

$$id_{f^{-1}(U)} \times_U g := \big(id_{f^{-1}(U)} \times g\big)\big|_{f^{-1}(U)\times_U(\mathcal{L}|_U)} \in \mathrm{Aut}(f^*(\mathcal{L})|_U). \tag{3.5.8}$$

As a result,

$$(id_{f^{-1}(U)} \times_U g)^{\sim} \in \mathcal{A}ut\big(f^*(\mathcal{L})\big)(f^{-1}(U)).$$

Now, we let

$$i^*(y,z) := (id_{f^{-1}(U)} \times_U g)^{\sim}(y), \tag{3.5.9}$$

which determines an element of the stalk $\mathcal{A}ut(f^*(\mathcal{L}))_y$.

The map i^*, given by (3.5.9), is *well defined*. Indeed, assume that

$$z = [g']_{f(y)} = \widetilde{g'}(f(y)),$$

where $g' \in \mathrm{Aut}(\mathcal{L}|_{U'})$ and $U' \in \mathfrak{T}_Y$ with $f(y) \in U'$. Then the analog of equality (3.5.9) is

$$i^*(y,z) = (id_{f^{-1}(U')} \times_{U'} g')^{\sim}(y). \tag{3.5.9'}$$

In order to show that the right-hand sides of (3.5.9) and (3.5.9′) coincide, we denote by

$$r_V^U : \mathrm{Aut}(\mathcal{L}|_U) \longrightarrow \mathrm{Aut}(\mathcal{L}|_V),$$
$$R_{W'}^W : \mathrm{Aut}(f^*(\mathcal{L})|_W) \longrightarrow \mathrm{Aut}(f^*\mathcal{L}|_{W'}),$$

the restriction maps (for open $V \subseteq U$ and $W' \subseteq W$) of the presheaves

$$U \longmapsto \mathrm{Aut}(\mathcal{L}|_U); \qquad U \in \mathfrak{T}_X,$$
$$W \longmapsto \mathrm{Aut}(f^*(\mathcal{L})|_W); \qquad W \in \mathfrak{T}_Y,$$

generating the sheaves $\mathcal{A}ut(\mathcal{L})$ and $\mathcal{A}ut(f^*(\mathcal{L}))$, respectively. In particular, for every open $V \subseteq U$, we have that

$$r_V^U(g) = g\big|_{\mathcal{L}_V} =: g|_V, \tag{3.5.10}$$

$$R_{f^{-1}(V)}^{f^{-1}(U)}(h) = h\big|_{f^{-1}(V)\times_V(\mathcal{L}|_V)} =: h\big|_{f^{-1}(V)}, \tag{3.5.11}$$

if $g \in \mathrm{Aut}(\mathcal{L}|_U)$ and $h \in \mathrm{Aut}\big(f^{-1}(U) \times_U (\mathcal{L}|_U)\big) = \mathrm{Aut}\big(f^*(\mathcal{L})|_{f^{-1}(U)}\big)$.

Under the above notations, our assumption

$$[g]_{f(x)} = z = [g']_{f(x)}$$

implies the existence of an open $V \subseteq U \cap U'$ such that $r_V^U(g) = r_V^{U'}(g')$, i.e., $g|_V = g'|_V$. Hence, if $R_{W,y} : \mathrm{Aut}(f^*(\mathcal{L})|_W) \to \mathcal{A}ut(f^*(\mathcal{L}))_y$ ($W \in \mathfrak{T}_Y$) is the canonical map into germs (see (1.2.6)), we obtain, in virtue of Diagram 1.5 and equality (3.5.8),

$$\begin{aligned}
(id_{f^{-1}(U')} \times_{U'} g')^{\sim}(y) &= R_{f^{-1}(U'),y}(id_{f^{-1}(U')} \times_{U'} g') \\
&= \big(R_{f^{-1}(V),y} \circ R_{f^{-1}(V)}^{f^{-1}(U')}\big)(id_{f^{-1}(U')} \times_{U'} g') \\
&= R_{f^{-1}(V),y}\big((id_{f^{-1}(U')} \times_{U'} g')\big|_{f^{-1}(V)\times_V(g'|_V)}\big) \\
&= R_{f^{-1}(V),y}(id_{f^{-1}(V)} \times_V g|_V)
\end{aligned}$$

$$\begin{aligned} &= \big(R_{f^{-1}(V),y} \circ R^{f^{-1}(U)}_{f^{-1}(V)}\big)(id_{f^{-1}(U)} \times_U g) \\ &= R_{f^{-1}(U),y}(id_{f^{-1}(U)} \times_U g) \\ &= (id_{f^{-1}(U)} \times_U g)^{\widetilde{}}(y), \end{aligned}$$

which proves that i^* is indeed well defined.

We show that i^* is *continuous* in a way similar to that of $\boldsymbol{\tau}$ in Lemma 3.5.1. More precisely, we fix an arbitrary $(y_\mathrm{o}, z_\mathrm{o}) \in f^*(\mathcal{A}ut(\mathcal{L}))$. As before, we can find a $g_\mathrm{o} \in \mathrm{Aut}(\mathcal{L}|_{U_\mathrm{o}})$ with $z_\mathrm{o} = [g_\mathrm{o}]_{f(y_\mathrm{o})} = \widetilde{g_\mathrm{o}}(f(y_\mathrm{o}))$, for an open $U_\mathrm{o} \subseteq X$ with $f(y_\mathrm{o}) \in U_\mathrm{o}$. Denoting by

$$f^*_{U_\mathrm{o}} : \mathcal{A}ut(\mathcal{L})(U_\mathrm{o}) \longrightarrow f^*(\mathcal{A}ut(\mathcal{L}))(f^{-1}(U_\mathrm{o}))$$

the corresponding canonical (adjunction) map with

$$\big(f^*_{U_\mathrm{o}}(\sigma)\big)(y) = \big(y, \sigma(f(y))\big),$$

for every $\sigma \in \mathcal{A}ut(\mathcal{L})(U_\mathrm{o}) \cong \mathrm{Aut}(\mathcal{L}|_{U_\mathrm{o}})$ and $y \in f^{-1}(U_\mathrm{o})$, we easily verify that the open set $B := f^*_{U_\mathrm{o}}(\widetilde{g_\mathrm{o}})(f^{-1}(U_\mathrm{o}))$, in the basis for the topology of $f^*(\mathcal{A}ut(\mathcal{L}))$, has the form

$$B = \big\{\big(y, \widetilde{g_\mathrm{o}}(f(y))\big) \,\big|\, y \in f^{-1}(U_\mathrm{o})\big\} = f^{-1}(U_\mathrm{o}) \times_{U_\mathrm{o}} \widetilde{g_\mathrm{o}}(U_\mathrm{o}).$$

Hence, evaluating i^* at any element of B, we see that

$$\begin{gathered} i^*\big(y, \widetilde{g_\mathrm{o}}(f(y))\big) = \big(id_{f^{-1}(U_\mathrm{o})} \times_{U_\mathrm{o}} g_\mathrm{o}\big)^{\widetilde{}}(y) = \\ \big((id_{f^{-1}(U_\mathrm{o})} \times_{U_\mathrm{o}} g_\mathrm{o})^{\widetilde{}} \circ p_1 |_B\big)\, \big(y, \widetilde{g_\mathrm{o}}(f(y))\big); \end{gathered}$$

i.e., $i^*|_B = (id_{f^{-1}(V_\mathrm{o})} \times_{V_\mathrm{o}} g_\mathrm{o})^{\widetilde{}} \circ p_1 |_B$, which proves the continuity at $(y_\mathrm{o}, z_\mathrm{o})$, and analogously everywhere. Thus i^* is a sheaf morphism, since it commutes with the projections.

Also, i^* is a *monomorphism*, for if

$$i^*(y,z) = (id_{f^{-1}(U)} \times_U g)^{\widetilde{}}(y) = (id_{f^{-1}(U')} \times_{U'} g')^{\widetilde{}}(y) = i^*(y,z'),$$

then there is an open $V \subseteq U \cap U'$ such that $f(y) \in V$ and (see also (3.5.11))

$$\begin{gathered} R^{f^{-1}(U)}_{f^{-1}(V)}\big(id_{f^{-1}(U)} \times_U g\big) = \big(id_{f^{-1}(U)} \times_U g\big)\big|_{f^{-1}(V) \times_V (\mathcal{L}|_V)} = \\ R^{f^{-1}(U')}_{f^{-1}(V)}\big(id_{f^{-1}(U')} \times_{U'} g'\big) = \big(id_{f^{-1}(U')} \times_{U'} g'\big)\big|_{f^{-1}(V) \times_V (\mathcal{L}|_V)}. \end{gathered}$$

The previous equalities, combined with (3.5.8) and (3.5.10), imply that

$$id_{f^{-1}(V)} \times_V (g|_V) = id_{f^{-1}(V)} \times_V (g'|_V),$$

from which it follows that

$$g|_{\mathcal{L}_V} =: g|_V = g'|_V := g'|_{\mathcal{L}_V} \in \mathrm{Aut}(\mathcal{L}|_V).$$

Consequently, $z = [g]_{f(y)} = [g']_{f(y)} = z'$, as required.

Finally, we show that i^* is a *morphism of sheaves of groups*. To this end let us first look at the *product* $z \cdot z' \in \mathcal{A}ut(\mathcal{L})_{f(y)}$ of two elements

$$z = [g]_{f(y)} = \widetilde{g}(f(y)) \qquad \text{and} \qquad z' = [g']_{f(y)} = \widetilde{g'}(f(y)),$$

where $g \in \mathrm{Aut}(\mathcal{L}|_U)$, $g' \in \mathrm{Aut}(\mathcal{L}|_{U'})$, and $U, U' \in \mathfrak{T}_X$ with $f(y) \in U \cap U'$. By the definition of an operation in a sheaf with an algebraic structure (see the concluding part of Subsection 1.2.2), in conjunction with (3.5.10), we obtain over $V = U \cap U'$:

$$\begin{aligned} z \cdot z' = [g]_{f(y)} \cdot [g']_{f(y)} = r_{V,f(y)}\big(r_V^U(g) \circ r_V^{U'}(g')\big) = \\ r_{V,f(y)}(g|_V \circ g'|_V) = (g|_V \circ g'|_V)^{\widetilde{\ }}(f(y)) = [g|_V \circ g'|_V]_{f(y)}. \end{aligned} \tag{3.5.12}$$

Hence, we immediately check that (3.5.9) and (3.5.12) give

$$\begin{aligned} i^*\big((y,z),(y,z')\big) &= i^*(y, z \cdot z') \\ &= \big(id_{f^{-1}(V)} \times_V (g|_V \circ g'|_V)\big)^{\widetilde{\ }}(y) \\ &= \big((id_{f^{-1}(V)} \times_V (g|_V)) \circ (id_{f^{-1}(V)} \times_V (g'|_V))\big)^{\widetilde{\ }}(y). \end{aligned} \tag{3.5.13}$$

Using similar arguments for the product structure of $\mathcal{A}ut(f^*(\mathcal{L}))_y$, we transform (3.5.13) into

$$\begin{aligned} i^*\big((y,z),(y,z')\big) &= \big((id_{f^{-1}(V)} \times_V g|_V) \circ (id_{f^{-1}(V)} \times_V g'|_V)\big)^{\widetilde{\ }}(y) \\ &= R_{f^{-1}(V),y}\big(R^{f^{-1}(U)}_{f^{-1}(V)}(id_{f^{-1}(U)} \times_U g) \circ \\ &\qquad\qquad \circ R^{f^{-1}(U')}_{f^{-1}(V)}(id_{f^{-1}(U')} \times_{U'} g')\big) \\ &= R_{f^{-1}(U),y}(id_{f^{-1}(U)} \times_U g) \cdot R_{f^{-1}(U'),y}(id_{f^{-1}(U')} \times_{U'} g') \\ &= \big(id_{f^{-1}(U)} \times_U g\big)^{\widetilde{\ }}(y) \cdot \big(id_{f^{-1}(U')} \times_{U'} g'\big)^{\widetilde{\ }}(y) \\ &= i^*(y,z) \cdot i^*(y,z'); \end{aligned}$$

that is, i^* is a morphism of sheaves of groups. □

From the initial representation $\rho : \mathcal{G} \to \mathcal{A}ut(\mathcal{L})$ we now obtain the representation

$$\rho^* := i^* \circ f^*(\rho), \tag{3.5.14}$$

shown also in the next diagram.

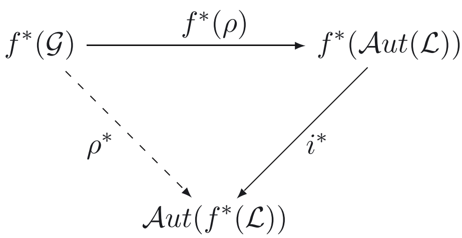


Diagram 3.6

Hence, for every $(y, z) \in f^*(\mathcal{G}) = Y \times_X \mathcal{G}$, equality (1.4.5) implies that

$$\rho^*(y, z) = i^*(f^*(\rho)(y, z)) = i^*(y, \rho(z)). \tag{3.5.15}$$

Omitting i^* in the last expression, we equivalently write

$$\rho^*(x, z) \equiv (x, \rho(z)), \tag{3.5.15$'$}$$

which, like (3.5.7′), is quite convenient in practice.

In Section 3.3 we have seen that $\rho : \mathcal{G} \to \mathcal{A}ut(\mathcal{L})$ induces an action $\Delta : \mathcal{G} \times_X \Omega(\mathcal{L}) \to \Omega(\mathcal{L})$. Similarly, $\rho^* : f^*(\mathcal{G}) \to \mathcal{A}ut(f^*(\mathcal{L}))$ induces an action

$$\Delta^* : f^*(\mathcal{G}) \times_Y f^*(\Omega)\left(f^*(\mathcal{L})\right) \longrightarrow f^*(\Omega)\left(f^*(\mathcal{L})\right),$$

where, as in (3.3.4), $f^*(\Omega)\left(f^*(\mathcal{L})\right) := f^*(\Omega) \otimes_{f^*(\mathcal{A})} f^*(\mathcal{L})$.

Using the isomorphism $\boldsymbol{\tau}$ of Lemma 3.5.1, we obtain the last auxiliary result, needed in the proof of the main theorem of this section.

3.5.3 Lemma. *With the previous notations, equality*

$$\Delta^*\big((y, a), \boldsymbol{\tau}(y, w)\big) = \boldsymbol{\tau}\big(y, \Delta(a, \omega)\big)$$

holds true, for every $(y, a) \in f^*(\mathcal{G})_y$ *and* $(y, w) \in f^*(\Omega(\mathcal{L}))_y$.

The statement is depicted in the following diagram, where the map ψ is given by $\psi((y,a),(y,w)) := (y,(a,w))$ and $id_{f^*(\mathcal{G})} \times_Y \boldsymbol{\tau}$ actually denotes the restriction of $id_{f^*(\mathcal{G})} \times \boldsymbol{\tau}$ to the indicated fiber product.

$$\begin{array}{ccc}
f^*(\mathcal{G}) \times_Y f^*(\Omega(\mathcal{L})) & \xrightarrow{\;id_{f^*(\mathcal{G})} \times_Y \boldsymbol{\tau}\;} & f^*(\mathcal{G}) \times_Y f^*(\Omega)\,(f^*(\mathcal{L})) \\
\psi \big\downarrow & & \\
f^*(\mathcal{G} \times_X \Omega(\mathcal{L})) & & \big\downarrow \Delta^* \\
f^*(\Delta) \big\downarrow & & \\
f^*(\Omega(\mathcal{L})) & \xrightarrow{\;\boldsymbol{\tau}\;} & f^*(\Omega)\,(f^*(\mathcal{L}))
\end{array}$$

Diagram 3.7

Proof. Since $(y,a) \in f^*(\mathcal{G})_y = \{y\} \times \mathcal{G}_{f(y)}$, there exists a $g \in \mathcal{G}(U)$ such that $a = g(f(y))$, for some $U \in \mathfrak{T}_X$ with $f(y) \in U$. Similarly, for $(y,\omega) \in f^*(\Omega)_y = \{y\} \times \Omega_{f(y)}$, there exists a $\sigma \in \Omega(U) \otimes_{\mathcal{A}(U)} \mathcal{L}(U)$ such that $w = \widetilde{\sigma}(f(y)) = [\sigma]_{f(y)}$, with the same U as before. Because $\rho(g) \in \mathcal{A}ut(\mathcal{L})(U) \cong \mathrm{Aut}(\mathcal{L}|_U)$, we obtain the corresponding induced automorphism of sections (in a slightly simplified form of (3.3.2)) $\overline{\rho(g)}_U : \mathcal{L}(U) \xrightarrow{\simeq} \mathcal{L}(U)$. Therefore, by (3.3.7) and (3.3.7′),

$$\Delta(a,w) = \rho(a).w = \big((1 \otimes \overline{\rho(g)}_U)(\sigma)\big)^{\sim}(f(y)),$$

where $1 = 1_{\Omega(U)}$. As a consequence, also taking into account (3.5.2) with the appropriate modifications,

$$\begin{aligned}
\boldsymbol{\tau}\big(y, \Delta(a,w)\big) &= \boldsymbol{\tau}\big(y, \big((1 \otimes \overline{\rho(g)}_U)(\sigma)\big)^{\sim}(f(y))\big) \\
&= \big((f^*_{\Omega,U} \otimes f^*_{\mathcal{L},U})\big((1 \otimes \overline{\rho(g)}_U)(\sigma)\big)\big)^{\sim}(y) \\
&= \big(\big(f^*_{\Omega,U} \otimes (f^*_{\mathcal{L},U} \circ \overline{\rho(g)}_U)\big)(\sigma)\big)^{\sim}(y).
\end{aligned} \tag{3.5.16}$$

On the other hand, since $f^*_{\mathcal{G},U}(g) \in f^*(\mathcal{G})(f^{-1}(U))$, the automorphism

$$\rho^*(f^*_{\mathcal{G},U}(g)) \in \mathcal{A}ut(f^*(\mathcal{L}))(f^{-1}(U)) \cong \mathrm{Aut}\big(f^*(\mathcal{L})|_{f^{-1}(U)}\big) \tag{3.5.17}$$

induces the automorphism of sections

$$\overline{\rho^*(f^*_{\mathcal{G},U}(g))}_{f^{-1}(U)} : f^*(\mathcal{L})(f^{-1}(U)) \xrightarrow{\simeq} f^*(\mathcal{L})(f^{-1}(U)).$$

Hence, working as in (3.5.16), we obtain

$$\begin{aligned}
\Delta^*\big((y,a),\boldsymbol{\tau}(y,w)\big) &= \Delta^*\big((y,g(f(x))),\boldsymbol{\tau}(y,\widetilde{\sigma}(f(y)))\big)\\
&= \Delta^*\big(f^*_{\mathcal{G},U}(g)(y),\big((f^*_{\Omega,U}\otimes f^*_{\mathcal{L},U})(\sigma)\big)^{\sim}(y)\big)\\
&= \Big(\big(1\otimes\overline{\rho^*(f^*_{\mathcal{G},U}(g))}_{f^{-1}(U)}\big)\big((f^*_{\Omega,U}\otimes f^*_{\mathcal{L},U})(\sigma)\big)\Big)^{\sim}(y)\\
&= \Big(\big(f^*_{\Omega,U}\otimes\big(\overline{\rho^*(f^*_{\mathcal{G},U}(g))}_{f^{-1}(U)}\circ f^*_{\mathcal{L},U}\big)\big)(\sigma)\Big)^{\sim}(y),
\end{aligned} \tag{3.5.18}$$

where $1 = 1_{f^*(\Omega)(f^{-1}(U))}$.

Comparing (3.5.16) and (3.5.18) we see that, in order to complete the proof, it suffices to show the following equality

$$\overline{\rho^*(f^*_{\mathcal{G},U}(g))}_{f^{-1}(U)}\circ f^*_{\mathcal{L},U} = f^*_{\mathcal{L},U}\circ\overline{\rho(g)}_U, \tag{3.5.19}$$

after the identifications $\mathcal{A}ut(\mathcal{L})(U)\cong \mathrm{Aut}(\mathcal{L}|_U)$ and $\mathcal{A}ut(f^*(\mathcal{L}))(f^{-1}(U))\cong \mathrm{Aut}(f^*(\mathcal{L})|_{f^{-1}(U)})$. We first work out the right-hand side of (3.5.19). Since, for any $\ell\in\mathcal{L}(U)$,

$$(f^*_{\mathcal{L},U}\circ\overline{\rho(g)}_U)(\ell)\in f^*(\mathcal{L})(f^{-1}(U)),$$

it follows from (1.4.3) that, for every $y\in f^{-1}(U)$,

$$\begin{aligned}
&\big((f^*_{\mathcal{L},U}\circ\overline{\rho(g)}_U)(\ell)\big)(y) = \big(f^*_{\mathcal{L},U}\big(\overline{\rho(g)}_U(\ell)\big)\big)(y)\\
&= \big(y,\big(\overline{\rho(g)}_U(\ell)\big)(f(y))\big) = \big(y,\rho(g)\big(\ell(f(y))\big)\big),
\end{aligned} \tag{3.5.20}$$

where $\rho(g)\in\mathrm{Aut}(\mathcal{L}|_U)$. Analogously, in virtue of (3.5.17) and for every $\ell\in\mathcal{L}(U)$ and $y\in f^{-1}(U)$, the left-hand side of (3.5.19) yields

$$\begin{aligned}
\Big(\big(\overline{\rho^*(f^*_{\mathcal{G},U}(g))}_{f^{-1}(U)}\circ f^*_{\mathcal{L},U}\big)(\ell)\Big)(y) &= \Big(\big(\overline{\rho^*(f^*_{\mathcal{G},U}(g))}_{f^{-1}(U)}\big(f^*_{\mathcal{L},U}(\ell)\big)\Big)(y)\\
&= \rho^*(f^*_{\mathcal{G},U}(g))\big(f^*_{\mathcal{L},U}(\ell))(y)\big) = \rho^*(f^*_{\mathcal{G},U}(g))\big(y,\ell(f(y))\big);
\end{aligned}$$

that is,

$$\Big(\big(\overline{\rho^*(f^*_{\mathcal{G},U}(g))}_{f^{-1}(U)}\circ f^*_{\mathcal{L},U}\big)(\ell)\Big)(y) = \rho^*(f^*_{\mathcal{G},U}(g))\big(y,\ell(f(y))\big). \tag{3.5.21}$$

However, thinking of $\rho^*(f^*_{\mathcal{G},U}(g)) \equiv \overline{\rho^*}_{f^{-1}(U)}(f^*_{\mathcal{G},U}(g))$ as a section belonging to $\mathcal{A}ut(f^*(\mathcal{L}))(f^{-1}(U))$ and taking into account (3.5.15) and (3.5.9), we check that, for every $q \in f^{-1}(U)$,

$$\begin{aligned}\rho^*\left(f^*_{\mathcal{G},U}(g)\right)(q) = \rho^*\big(f^*_{\mathcal{G},U}(g)(q)\big) = \rho^*\big(q, g(f(y))\big) = \\ i^*\big(q, \rho\big(g(f(q))\big)\big) = i^*\big(q, \rho(g)(f(q))\big) = \big(id_{f^{-1}(U)} \times_U \rho(g)\big)^{\sim}(q).\end{aligned}$$

Therefore, the identification $\mathcal{A}ut(f^*(\mathcal{L}))(f^{-1}(U)) \cong \mathrm{Aut}(f^*(\mathcal{L})|_{f^{-1}(U)})$ and the last equality imply that the section $\rho^*(f^*_{\mathcal{G},U}(g))$, now interpreted as an element of $\mathrm{Aut}(f^*(\mathcal{L})|_{f^{-1}(U)})$, has the form

$$\rho^*(f^*_{\mathcal{G},U}(g)) = id_{f^{-1}(U)} \times_U \rho(g). \tag{3.5.22}$$

As a consequence of (3.5.22), equality (3.5.21) is transformed into

$$\begin{aligned}\Big(\big(\overline{\rho^*(f^*_{\mathcal{G},U}(g))}_{f^{-1}(U)} \circ f^*_{\mathcal{L},U}\big)(\ell)\Big)(y) = \rho^*\big(f^*_{\mathcal{G},U}(g)\big)\big(y, \ell(f(y))\big) \\ = \big(id_{f^{-1}(U)} \times_U \rho(g)\big)\big(y, \ell(f(y))\big) = \big(y, \rho(g)(\ell(f(y)))\big).\end{aligned}$$

In other words, for every $\ell \in \mathcal{L}(U)$ and $y \in f^{-1}(U)$,

$$\Big(\big(\overline{\rho^*(f^*_{\mathcal{G},U}(g))}_{f^{-1}(U)} \circ f^*_{\mathcal{L},U}\big)(\ell)\Big)(y) = \big(y, \rho(g)(\ell(f(y)))\big).$$

Comparing the preceding equality with (3.5.20), we obtain (3.5.19) which completes the proof. □

The previous lemmata now make straightforward the proof of the main result of this section, namely:

3.5.4 Theorem. *The quadruple $f^*(\mathcal{G}) \equiv \big(f^*(\mathcal{G}), \rho^*, f^*(\mathcal{L}), \partial^*\big)$, with ρ^* and ∂^* defined by (3.5.14) and (3.5.6), respectively, is a Lie sheaf of groups.*

Proof. The only remaining matter to be proven is that ∂^* is a Maurer-Cartan differential; that is, condition (3.3.8) is satisfied. Indeed, for every $(y,a),(y,b) \in \{y\} \times \mathcal{G}_{f(y)} = f^*(\mathcal{G})_y$, we have that

$$\begin{aligned}\partial^*\big((y,a)\cdot(y,b)\big) &= \partial^*\big((y, a\cdot b)\big) \\ \text{(by (3.5.7))} \qquad &= \boldsymbol{\tau}(y, \partial(a\cdot b))\end{aligned}$$

$$
\begin{aligned}
& &&= \boldsymbol{\tau}\big(y, \rho(b^{-1}).\partial(a) + \partial(b)\big)\\
&\text{(by Lemma 3.5.1)} &&= \boldsymbol{\tau}\big((y, \rho(b^{-1}).\partial(a)\big) + \boldsymbol{\tau}\big(y, \partial(b)\big)\\
&\text{(by (3.3.7), (3.5.7))} &&= \boldsymbol{\tau}\big(y, \Delta(b^{-1}, \partial(a))\big) + \partial^*(y,b)\\
&\text{(by Lemma 3.5.3)} &&= \Delta^*\big((y, b^{-1}), \boldsymbol{\tau}(y, \partial(b)\big) + \partial^*(y,b)\\
&\text{(by the analog of (3.3.7) for } \rho^*) &&= \rho^*(y, b^{-1}).\partial^*(y,a) + \partial^*(y,b)\\
& &&= \rho^*((y,b)^{-1}).\partial^*(y,a) + \partial^*(y,b),
\end{aligned}
$$

which completes the proof. □

The Lie sheaf of groups $f^*(\mathcal{G}) \equiv \big(f^*(\mathcal{G}), \rho^*, f^*(\mathcal{L}), \partial^*\big)$ is called the ***pull-back Lie sheaf of groups of*** $\mathcal{G} \equiv (\mathcal{G}, \rho, \mathcal{L}, \partial)$.

We close the present chapter with a result supplementing the previous theorem and motivating a generalization of the notion of the morphism of Lie sheaves of groups.

With the notations of the previous discussion, we define the morphism

$$f^*_{\mathcal{G}} : f^*(\mathcal{G}) \to \mathcal{G} \quad \text{with} \quad f^*_{\mathcal{G}} := \mathrm{pr}_2|_{f^*(\mathcal{G})},$$

where $\mathrm{pr}_2 : Y \times \mathcal{G} \to \mathcal{G}$ is the projection to the second factor. Then we obtain the following commutative diagram

$$
\begin{array}{ccc}
f^*(\mathcal{G}) & \xrightarrow{\;f^*_{\mathcal{G}}\;} & \mathcal{G}\\
{\scriptstyle \pi^*_{\mathcal{G}}}\big\downarrow & & \big\downarrow{\scriptstyle \pi_{\mathcal{G}}}\\
Y & \xrightarrow{\;f\;} & X
\end{array}
$$

Diagram 3.8

where the vertical maps are the projections of the corresponding sheaves. The same diagram means that $f^*_{\mathcal{G}}$ is an ***f-morphism***, i.e., $f^*_{\mathcal{G}}$ *covers* (or *projects* to) f. Similarly, we define the f-morphisms $f^*_{\mathcal{L}} : f^*(\mathcal{L}) \to \mathcal{L}$ and $f^*_{\Omega} : f^*(\Omega) \to \Omega$. Thus we prove:

3.5.5 Corollary. *The pair* $(f^*_{\mathcal{G}}, f^*_{\mathcal{L}})$ *is an* f*-morphism of Lie sheaves of groups between* $f^*(\mathcal{G}) \equiv (f^*(\mathcal{G}), \rho^*, f^*(\mathcal{L}), \partial^*)$ *and* $\mathcal{G} \equiv (\mathcal{G}, \rho, \mathcal{L}, \partial)$*; that* *is, analogously to Definition 3.4.1, we obtain the commutative diagrams*

$$\begin{array}{ccc} f^*(\mathcal{G}) \times_Y f^*(\mathcal{L}) & \xrightarrow{\;f^*_{\mathcal{G}} \times f^*_{\mathcal{L}}\;} & \mathcal{G} \times_X \mathcal{L} \\ {\scriptstyle \delta^*}\big\downarrow & & \big\downarrow{\scriptstyle \delta} \\ f^*(\mathcal{L}) & \xrightarrow{\;f^*_{\mathcal{L}}\;} & \mathcal{L} \end{array}$$

Diagram 3.9

and

$$\begin{array}{ccc} f^*(\mathcal{G}) & \xrightarrow{\;f^*_{\mathcal{G}}\;} & \mathcal{G} \\ {\scriptstyle \partial^*}\big\downarrow & & \big\downarrow{\scriptstyle \partial} \\ f^*(\Omega) \otimes_{f^*(\mathcal{A})} f^*(\mathcal{L}) & \xrightarrow{\;f^*_{\Omega} \otimes f^*_{\mathcal{L}}\;} & \Omega \otimes_{\mathcal{A}} \mathcal{L} \end{array}$$

Diagram 3.10

In Diagram 3.9, the action δ^* *is given by*

$$\delta^*\big((y,a),(y,\ell)\big) := \big(y, \delta(a,\ell)\big); \qquad (y,a) \in f^*(\mathcal{G})_y,\ (y,\ell) \in f^*(\mathcal{L})_y,$$

while the top morphism on the same diagram is appropriately restricted to the fiber product of the domain.

Proof. The commutativity of Diagram 3.9 is an immediate consequence of the definition of the morphisms involved.

On the other hand, for any

$$(y,z) \in f^*(\Omega \otimes_{\mathcal{A}} \mathcal{L})_{f(y)} \cong \{y\} \times \big(\Omega_{f(y)} \otimes_{\mathcal{A}_{f(y)}} \mathcal{L}_{f(y)}\big),$$

the analogs of (3.5.5) and (1.4.4), applied to the present data, imply that

$$\begin{aligned} (f^*_{\Omega} \otimes f^*_{\mathcal{L}})\,(\boldsymbol{\tau}(y,z)) &= \big((f^*_{\Omega} \otimes f^*_{\mathcal{L}}) \circ (f^*_{\Omega,y} \otimes f^*_{\mathcal{L},y})\big)\,(z) \\ &= \big((f^*_{\Omega} \circ f^*_{\Omega,y}) \otimes (f^*_{\mathcal{L}} \circ f^*_{\mathcal{L},y})\big)\,(z) \\ &= \big(id_{\Omega_{f(y)}} \otimes id_{\mathcal{L}_{f(y)}}\big)(z) = z. \end{aligned} \tag{3.5.23}$$

Hence, for every $(y,a) \in f^*(\mathcal{G})$, equalities (3.5.7) and (3.5.23) yield

$$\big((f^*_\Omega \otimes f^*_\mathcal{L}) \circ \partial^*\big)(y,a) = (f^*_\Omega \otimes f^*_\mathcal{L})\,(\boldsymbol{\tau}(y,\partial(a))) = \partial(a) = (\partial \circ f^*_\mathcal{G})(y,a),$$

by which we verify the commutativity of Diagram 3.10 and complete the proof. □

Note. Given the Lie sheaf of groups $\mathcal{G} \equiv (\mathcal{G}, \rho, \mathcal{L}, \partial)$, over the topological space X, and a continuous map $f : X \to Y$, one can form the push-out quadruple $\big(f_*(\mathcal{G}), f_*(\rho), f_*(\mathcal{L}), f_*(\partial)\big)$, where $f_*(\mathcal{G})$ is a sheaf of groups, $f_*(\mathcal{L})$ is an $f_*(\mathcal{A})$-module of Lie algebras, and $f_*(\rho) : f_*(\mathcal{G}) \to f_*(\mathcal{A}ut(\mathcal{L}))$, $f_*(\partial) : f_*(\mathcal{G}) \to f_*(\Omega \otimes_\mathcal{A} \mathcal{L})$ are appropriate morphisms.

In analogy to the case of the pull-back, we would like to have a quadruple of the form $(f_*(\mathcal{G}), \rho_*, f_*(\mathcal{L}), \partial_*)$, where $\rho_* : f_*(\mathcal{G}) \to \mathcal{A}ut(f_*(\mathcal{L}))$ and $\partial_* : f_*(\mathcal{G}) \to f_*(\Omega) \otimes_{f_*(\mathcal{A})} f_*(\mathcal{L})$. However, in general, we cannot find a reasonable way to go from $f_*(\Omega \otimes_\mathcal{A} \mathcal{L})$ to $f_*(\Omega) \otimes_{f_*(\mathcal{A})} f_*(\mathcal{L})$, which, combined with $f_*(\partial)$, would give a differential ∂_*. The same remark applies to ρ_*. Of course, we go from $f_*(\Omega) \otimes_{f_*(\mathcal{A})} f_*(\mathcal{L})$ to $f_*(\Omega \otimes_\mathcal{A} \mathcal{L})$, but this is not of interest in our case.

Therefore, the push-out functor applied to a Lie sheaf of groups does not, in general, lead to a Lie sheaf of groups.

Chapter 4

Principal sheaves

When one tries to state in a general algebraic formalism the various notions of fiber space: general fiber space (without structure group, and maybe not even locally trivial); or fiber bundles with topological structure group as expounded in the book of Steenrod (The Topology of Fiber Bundles, Princeton University Press); or the "differentiable" and "analytic" (real or complex) variants of these notions; or the notions of algebraic fiber spaces (over an abstract field k) – one is led in a natural way to the notion of fiber space with a structure sheaf $\underline{G}$.

A. Grothendieck [36, p. 1]

Principal sheaves, one of the fundamental concepts of this work, were originally defined by A. Grothendieck (see [36]). They represent the abstract analog of principal bundles and provide the natural space in which abstract connections live (cf. Chapter 6). For this reason, principal sheaves are defined here in a slightly different way from Grothendieck's original one; namely, they are fiber spaces whose structure sheaf and structure type is a Lie sheaf of groups (instead of a simple sheaf of groups).

Principal sheaves also constitute the non commutative, so to speak, counterpart of vector sheaves. As a matter of fact, vector sheaves and their geometry can be reduced to that of principal sheaves by means of the principal sheaf of frames (cf. Chapter 5). The main part of this chapter is devoted to the study of morphisms and isomorphisms of principal sheaves, as well as to their cohomological classification.

4.1. Basic definitions and properties

Let $\mathcal{G} \equiv (\mathcal{G}, \rho, \mathcal{L}, \partial)$ be a Lie sheaf of groups over X, as in Definition 3.3.3. We denote by $\pi_{\mathcal{G}} : \mathcal{G} \to X$ the projection of $\mathcal{G}$ on X.

4.1.1 Definition. A ***principal sheaf with structure sheaf*** $\mathcal{G}$, denoted by $\mathcal{P} \equiv (\mathcal{P}, \mathcal{G}, X, \pi)$, is a sheaf of sets $(\mathcal{P}, \pi, X)$ such that:

i) There exists an *action* $\delta : \mathcal{P} \times_X \mathcal{G} \to \mathcal{P}$ of $\mathcal{G}$ on (the right of) $\mathcal{P}$.

ii) For every $x \in X$, there exists an *open neighborhood* U of x in X and an *isomorphism of sheaves of sets*

$$\phi_U : \mathcal{P}|_U \xrightarrow{\simeq} \mathcal{G}|_U, \tag{4.1.1}$$

satisfying

$$\phi_U(p \cdot g) = \phi_U(p) \cdot g, \qquad (p, g) \in \mathcal{P}|_U \times_U \mathcal{G}|_U. \tag{4.1.2}$$

For the sake of brevity, a principal sheaf, as above, is called a ***$\mathcal{G}$-principal sheaf***.

The (center) dots in (4.1.2) denote the action of $\mathcal{G}$ on the right of $\mathcal{P}$ and $\mathcal{G}$, respectively. The same equality means that each ϕ_U is a ***$\mathcal{G}|_U$-equivariant (iso)morphism*** or it has the ***equivariance property***, with respect to the said actions. Clearly, ϕ_U^{-1} is also $\mathcal{G}|_U$-equivariant, i.e.,

$$\phi_U^{-1}(g \cdot g') = \phi_U^{-1}(g) \cdot g', \qquad (g, g') \in \mathcal{G}|_U \times_U \mathcal{G}|_U. \tag{4.1.2'}$$

Condition ii) of the previous definition is equivalent to the existence of an open covering $\mathcal{U} = \{U_\alpha \subseteq X \mid \alpha \in I\}$ of X and a family of $\mathcal{G}|_{U_\alpha}$-equivariant isomorphisms, called ***coordinate mappings*** or, simply, ***coordinates,***

$$\phi_\alpha : \mathcal{P}|_{U_\alpha} \xrightarrow{\simeq} \mathcal{G}|_{U_\alpha}; \qquad \alpha \in I, \tag{4.1.3}$$

also satisfying the *equivariance property*

(4.1.4) $$\phi_\alpha(p \cdot g) = \phi_\alpha(p) \cdot g, \qquad (p,g) \in \mathcal{P}|_{U_\alpha} \times_{U_\alpha} \mathcal{G}|_{U_\alpha}.$$

The section-wise variant of (4.1.4), written (according to convention (1.1.3)) in the form $\phi_\alpha(s \cdot g) = \phi_\alpha(s) \cdot g$, for every $(s,g) \in \mathcal{P}(U_\alpha) \times \mathcal{G}(U_\alpha)$, shows that the induced morphism $\phi_\alpha \equiv (\overline{\phi_\alpha})_{U_\alpha}$ is $\mathcal{G}(U_\alpha)$-equivariant.

An open covering $\mathcal{U} = (U_\alpha)$, over which a family of coordinates (ϕ_α) is defined, will be called a ***local frame*** or a ***coordinatizing covering***. We shall mainly use the first term because of its simplicity. If we want to specify both the covering and the associated coordinates, we refer to the local frame $\mathcal{U} \equiv (\mathcal{U}, (\phi_\alpha))$ or $\mathcal{U} \equiv ((U_\alpha), (\phi_\alpha))$ of $\mathcal{P}$. The open sets U_α of $\mathcal{U}$ are called ***local gauges***.

The covering $\mathcal{U}$ is called a local frame because it determines a family of sections which, in certain cases, are related with ordinary frames (viz. local bases) of vector bundles or vector sheaves.

Quite often it will be convenient to assume that the local frame $\mathcal{U}$ is a *basis for the topology of* X. This can always be done by replacing the original $\mathcal{U}$ with its refinement consisting of all the intersections of the open sets of $\mathcal{U}$ with the open sets of the topology $\mathfrak{T}_X$ of X. The restrictions of the coordinate isomorphisms (4.1.1) to the latter have the same properties as before. Furthermore, as in Mallios [62, Vol. 1, pp. 126–127] (see also the note on page 38), we note that

(4.1.5) *the set of all local frames of $\mathcal{P}$ is a cofinal subset of the set of all* proper *open coverings of its base space X.*

Since, by definition, $\mathcal{P}$ is locally isomorphic to $\mathcal{G}$, we say that $\mathcal{P}$ is of ***structure type*** $\mathcal{G}$.

Some elementary, but useful, properties are given in the sequel.

4.1.2 Proposition. *$\mathcal{G}$ acts freely on $\mathcal{P}$ and freely transitively on its stalks.*

Proof. Assume there exists a $(p,g) \in \mathcal{P} \times_X \mathcal{G}$ such that $p \cdot g = p$. Since $\pi(p) = \pi_{\mathcal{G}}(g) =: x \in U_\alpha$, for some $U_\alpha \in \mathcal{U}$, (4.1.4) yields

$$\phi_\alpha(p) = \phi_\alpha(p \cdot g) = \phi_\alpha(p) \cdot g,$$

from which we get $g = e_x$ (: the neutral/unit element of the stalk $\mathcal{G}_x$), thus proving that the action is free.

On the other hand, if $p, q \in \mathcal{P}_x$ are any elements of the stalk at x, with $x \in U_\alpha$, then there is a $g \in \mathcal{G}_x$ such that $\phi_\alpha(q) = \phi_\alpha(p) \cdot g = \phi_\alpha(p \cdot g)$, thus

$q = p \cdot g$, which proves the transitivity. The above g is uniquely determined by the first property of the action. □

4.1.3 Corollary. *Over each $x \in X$, the respective stalk has the form*

$$\mathcal{P}_x = p \cdot \mathcal{G}_x := \big\{ p \cdot g \mid g \in \mathcal{G}_x \big\},$$

for any $p \in \mathcal{P}_x$.

4.1.4 Proposition. *The map $\boldsymbol{k} : \mathcal{P} \times_X \mathcal{P} \to \mathcal{G}$, defined by*

$$q = p \cdot \boldsymbol{k}(p, q), \tag{4.1.6}$$

is a morphism of sheaves satisfying the following equalities:

$$\begin{aligned} \boldsymbol{k}(p \cdot g, q) &= g^{-1} \cdot \boldsymbol{k}(p, q), \\ \boldsymbol{k}(p, q \cdot g) &= \boldsymbol{k}(p, q) \cdot g, \\ \boldsymbol{k}(q, p) &= \boldsymbol{k}(p, q)^{-1}, \end{aligned}$$

for every $p, q \in \mathcal{P}_x$, $g \in \mathcal{G}_x$, and every $x \in X$.

Proof. The map $\boldsymbol{k}$ is clearly well defined by Proposition 4.1.2. Furthermore, for any $U_\alpha \in \mathcal{U}$,

$$(\mathcal{P} \times_X \mathcal{P})|_{U_\alpha} = \pi^{-1}(U_\alpha) \times_{U_\alpha} \pi^{-1}(U_\alpha)$$

is an open subset of $\mathcal{P} \times_X \mathcal{P}$. Since, for every $p, q \in (\mathcal{P} \times_X \mathcal{P})|_{U_\alpha}$,

$$\phi_\alpha(q) = \phi_\alpha(p \cdot \boldsymbol{k}(p, q)) = \phi_\alpha(p) \cdot \boldsymbol{k}(p, q),$$

we see that $\boldsymbol{k}(p, q) = \phi_\alpha(p)^{-1} \cdot \phi_\alpha(q)$. Hence,

$$\boldsymbol{k}(p, q) = \gamma \circ \big((\alpha \circ \phi_\alpha \circ \mathrm{pr}_1) \times (\phi_\alpha \circ \mathrm{pr}_2) \big) \big|_{\pi^{-1}(U_\alpha) \times_{U_\alpha} \pi^{-1}(U_\alpha)},$$

where γ is the multiplication and α the inversion of $\mathcal{G}$ (cf. Subsection 1.1.2), while $\mathrm{pr}_i : \pi^{-1}(U_\alpha) \times \pi^{-1}(U_\alpha) \to \pi^{-1}(U_\alpha)$ is the projection to the i-th factor ($i = 1, 2$). This proves the continuity of $\boldsymbol{k}$ on the subsheaf over U_α. The same arguments hold for every $\alpha \in I$, thus $\boldsymbol{k}$ is a continuous morphism.

The equalities of the statement are a direct consequence of (4.1.6). □

4.1.5 Remarks. 1) Propositions 4.1.2 and 4.1.4 show that principal sheaves in the sense of Definition 4.1.1 satisfy the conditions of Grothendieck's original definition (see [36, Definition 3.4.2, p. 32]). However, as already explained in the introduction of this chapter, principal sheaves that locally look like $\mathcal{G}$ are better suited for the geometric study we have in mind.

2) In the *present* chapter, the assumption that the structure group $\mathcal{G}$ is a Lie sheaf of groups is *not* necessary. As in [36], what is essentially needed here is that $\mathcal{G}$ is only a *sheaf of groups*. However, the former assumption will be crucial for the theory of connections and related topics, developed from Chapter 6 onwards. Thus, to maintain a certain homogeneity in the exposition, we adhere to Definition 4.1.1 and indicate –when necessary– those cases in which the group structure alone is sufficient.

A local frame $\mathcal{U} \equiv (\mathcal{U}, (\phi_\alpha))$ of $\mathcal{P}$ determines a particular family of sections. More precisely, we give the following:

4.1.6 Definition. The ***natural (local) sections*** of $\mathcal{P}$, with respect to $\mathcal{U}$, are the sections given by

$$s_\alpha := \phi_\alpha^{-1} \circ \mathbf{1}|_{U_\alpha} \in \mathcal{P}(U_\alpha), \quad \alpha \in I. \tag{4.1.7}$$

Equivalently, the natural sections are defined by

$$s_\alpha = \phi_\alpha^{-1}(\mathbf{1}|_{U_\alpha}), \tag{4.1.7$'$}$$

with ϕ_α now denoting the induced morphism of sections.

By the previous definition, a coordinate map gives rise to a natural section. The converse is also true, namely we have:

4.1.7 Proposition. *Let $s \in \mathcal{P}(U)$ be a section of $\mathcal{P}$ over the open $U \subseteq X$. Then there exists a $\mathcal{G}|_U$-equivariant isomorphism $\phi_U : \mathcal{P}|_U \to \mathcal{G}|_U$ (coordinate), whose corresponding natural section is precisely s.*

Proof. We define the map

$$\psi_U : \mathcal{G}|_U \longrightarrow \mathcal{P}|_U \;:\; g \mapsto s(\pi_{\mathcal{G}}(g)) \cdot g.$$

It is a continuous morphism (by the continuity of the action δ), admitting also an inverse ϕ_U, given by

$$\phi_U(p) = \boldsymbol{k}\big(s(\pi(p)), p\big); \qquad p \in \mathcal{P}|_U,$$

and whose continuity is guaranteed by Proposition 4.1.4. If s_U denotes the natural section of $\mathcal{P}$ induced by ϕ_U, then (4.1.7) yields

$$s_U(x) = \phi_U^{-1}(\mathbf{1}(x)) = \psi_U(\mathbf{1}(x)) = \psi_U(e_x) = s(x); \qquad x \in U,$$

which concludes the proof. □

4.1.8 Corollary. *There is a bijection between local frames of $\mathcal{P}$ and families of local sections, whose domains cover X. In particular, if $\mathcal{P}$ admits a global section, then $\mathcal{P}$ is $\mathcal{G}$-equivariantly isomorphic with $\mathcal{G}$.*

We single out some typical examples.

4.1.9 Examples.

(a) Principal sheaves from principal bundles

Let (P, G, X, π_P) be a smooth principal bundle and let $\mathcal{P} \equiv (\mathcal{P}, X, \pi)$ be the sheaf of germs of its *smooth* sections, i.e.,

$$\mathcal{P} := \mathbf{S}\big(U \longmapsto \Gamma(U, P)\big),$$

if $\Gamma(U, P)$ is the set of *smooth sections* of P over U. Thus, $\mathcal{P}(U) \cong \Gamma(U, P)$.

We also consider the Lie sheaf of groups $\mathcal{G} = \mathcal{C}_X^\infty(G)$, obtained from the Lie group G, as in Example 3.3.6(a). Then $\mathcal{G}$ acts on $\mathcal{P}$ (from the right) by means of the local actions

$$\Gamma(U, P) \times C^\infty(U, G) \longrightarrow \Gamma(U, P) : (s, g) \longmapsto s \cdot g,$$

for every $U \in \mathfrak{T}_X$.

By definition, the local structure of P is described by a family of G-equivariant isomorphisms (trivializations) $\Phi_\alpha : P|_{U_\alpha} \xrightarrow{\simeq} U_\alpha \times G$, over an open covering, say $\mathcal{U} = \{U_\alpha \subseteq X \,|\, \alpha \in I\}$. Thus, for every open $V \subseteq U_\alpha$, we can define the bijection

$$\phi_{\alpha,V} : \Gamma(V, P) \longrightarrow C^\infty(V, G) \;:\; \sigma \mapsto \mathrm{pr}_2 \circ \Phi_\alpha \circ \sigma, \tag{4.1.8}$$

whose inverse is

$$\phi_{\alpha,V}^{-1} : C^\infty(V, G) \longrightarrow \Gamma(V, P) \;:\; g \mapsto \Phi_\alpha^{-1} \circ (id_V, g). \tag{4.1.8$'$}$$

Both maps are $C^\infty(V, G)$-equivariant (iso)morphisms, as a result of the equivariance of Φ_α and Φ_α^{-1}. Varying V in U_α, we see that the (presheaf) isomorphism $\{\phi_{\alpha,V} \,|\, V \subseteq U_\alpha \text{ open}\}$ generates a $\mathcal{G}|_{U_\alpha}$-equivariant isomorphism $\phi_\alpha : \mathcal{P}|_{U_\alpha} \to \mathcal{G}|_{U_\alpha}$, with inverse

$$\phi_\alpha^{-1} = \mathbf{S}\big((\phi_{\alpha,V}^{-1})_{V \subseteq U_\alpha}\big).$$

Hence, we conclude that:

The sheaf $(\mathcal{P}, \mathcal{G}, X, \pi)$ of germs of smooth sections of a principal bundle (P, G, X, π_P) is a $\mathcal{G}$-principal sheaf, where $\mathcal{G}$ is the sheaf $\mathcal{C}_X^\infty$ of germs of smooth G-valued maps on X.

Of course, there is a topological analog of the previous result, but in this case the structure sheaf $\mathcal{C}_X^0(G)$ (: sheaf of germs of continuous G-valued maps on X) is not a Lie sheaf of groups.

For future use, let us connect the natural sections (σ_α) of P with the natural sections (s_α) of $\mathcal{P}$, over $\mathcal{U}$. By definition, $\Phi_\alpha(\sigma_\alpha(x)) = (x, e)$, for every $x \in U_\alpha$. On the other hand, if $\widetilde{\sigma}_\alpha \in \mathcal{P}(U_\alpha)$ is the section associated to σ_α by the identification $\Gamma(U_\alpha, P) \cong \mathcal{P}(U_\alpha)$, then (see also (1.2.10) and (1.2.13))

$$\phi_\alpha(\widetilde{\sigma}_\alpha(x)) = \phi_\alpha([\sigma_\alpha]_x) = [\phi_{\alpha,U_\alpha}(\sigma)]_x = [c_e]_x,$$

where $c_e : U_\alpha \to G$ is the (constant) map with $c_e(x) = e$, for all $x \in U_\alpha$. However, $c_e \in C^\infty(U_\alpha, G)$ corresponds bijectively to $\mathbf{1}|_{U_\alpha} \in \mathcal{G}(U_\alpha)$ via the identification $C^\infty(U_\alpha, G) \cong \mathcal{G}(U_\alpha)$; that is, $\mathbf{1}|_{U_\alpha} = \widetilde{c}_e$. Hence,

$$\phi_\alpha(\widetilde{\sigma}_\alpha(x)) = [c_e]_x = \widetilde{c}_e(x) = \mathbf{1}(x) = \phi_\alpha(s_\alpha(x)),$$

for every $x \in U_\alpha$, from which we obtain the bijection

$$\Gamma(U_\alpha, P) \ni \sigma_\alpha \stackrel{\simeq}{\longmapsto} \widetilde{\sigma}_\alpha = s_\alpha \in \mathcal{P}(U_\alpha), \qquad \alpha \in I. \tag{4.1.9}$$

(b) Principal sheaves from projective systems of principal bundles

Let $(P_i, G_i, X, \pi_i)_{i \in J}$ be a *projective system* of smooth principal bundles. The previous term means that we are given two projective systems $\{P_i, p_{ij}\}$ and $\{G_i, \rho_{ij}\}$ such that each triplet $(p_{ij}, \rho_{ij}, id_X)$ is a principal bundle morphism of (P_j, G_j, X, π_j) into (P_i, G_i, X, π_i), if $j \geq i$.

Then, arguing as in Examples 3.3.6(c) and 4.1.9(a), we define the sheaf $\mathcal{P} = \varprojlim \mathcal{P}_i$, where $(\mathcal{P}_i, \mathcal{G}_i, X, \pi_i)$ are the principal sheaves with

$$\mathcal{P}_i := \mathbf{S}\big(U \longmapsto \Gamma(U, P_i)\big).$$

The Lie sheaf of groups $\mathcal{G} = \varprojlim \mathcal{G}_i$ acts naturally on the right of $\mathcal{P}$ by applying the sheafification process to the local actions

$$\Gamma(U, P_i) \times C^\infty(U, G_i) \longrightarrow \Gamma(U, P_i),$$

induced by the actions $P_i \times G_i \to P_i$, and then by taking projective limits.

If we consider projective systems of principal bundles whose trivializations are defined over the same open covering (U_α) of X, then $\mathcal{P}_i|_{U_\alpha} \cong \mathcal{G}_i|_{U_\alpha}$ and $\mathcal{P}|_{U_\alpha} \cong \mathcal{G}|_{U_\alpha}$; that is, $\mathcal{P}$ is a $\mathcal{G}$-principal sheaf.

In the same way one defines the projective limit of a projective system of principal sheaves.

Coverings of the aforementioned type can be constructed under suitable assumptions. For certain analogous situations we refer to Galanis [30, 31, 32]. In [31] it is also shown that, under certain conditions, the (projective) limit of a projective system of smooth principal bundles (over a Banach space) has the structure of a Fréchet principal bundle.

(c) The pull-back of a principal sheaf

Let $\mathcal{P} \equiv (\mathcal{P}, \mathcal{G}, X, \pi)$ be a principal sheaf over X and let $f : Y \to X$ be a continuous map. We consider the pull-backs $f^*(\mathcal{P})$ and $f^*(\mathcal{G})$ of $\mathcal{P}$ and $\mathcal{G}$, respectively, and the morphism

$$\pi^* : f^*(\mathcal{P}) = Y \times_X \mathcal{P} \longrightarrow \mathcal{P},$$

obtained by restricting the first projection $\mathrm{pr}_1 : Y \times \mathcal{P} \to Y$ to $f^*(\mathcal{P})$. We have already seen in Section 3.5 that $f^*(\mathcal{G})$ is a Lie sheaf of groups. We claim that the quadruple

$$f^*(\mathcal{P}) \equiv (f^*(\mathcal{P}), f^*(\mathcal{G}), Y, \pi^*)$$

is a principal sheaf, called ***the pull-back principal sheaf of*** $\mathcal{P}$.

First we define an action

$$\delta^* : f^*(\mathcal{P}) \times_Y f^*(\mathcal{G}) \longrightarrow f^*(\mathcal{G})$$

by setting

$$\delta^*\big((y,p),(y,g)\big) := (y, \delta(p,g)) = (y, p \cdot g),$$

if δ is the action of $\mathcal{G}$ on $\mathcal{P}$. Formally, if h denotes the obvious isomorphism

$$f^*(\mathcal{P}) \times_Y f^*(\mathcal{G}) \xrightarrow{\simeq} f^*(\mathcal{P} \times_X \mathcal{G}).$$

we have that $\delta^* = f^*(\delta) \circ h$ (see (1.4.5)).

Given a local frame $\mathcal{U} \equiv ((U_\alpha), (\phi_\alpha))$ of $\mathcal{P}$, we form the open covering

$$\mathcal{V} := \big\{ V_\alpha := f^{-1}(U_\alpha) \subseteq Y \,|\, U_\alpha \in \mathcal{U} \big\}.$$

of Y. We immediately check that

$$f^*(\mathcal{P})|_{V_\alpha} = V_\alpha \times_{U_\alpha} \mathcal{P}|_{U_\alpha} = f^*(\mathcal{P}|_{U_\alpha}),$$
$$f^*(\mathcal{G})|_{V_\alpha} = V_\alpha \times_{U_\alpha} \mathcal{G}|_{U_\alpha} = f^*(\mathcal{G}|_{U_\alpha}).$$

Hence, the induced (iso)morphisms

$$\phi^*_\alpha := f^*(\phi_\alpha) : f^*(\mathcal{P})|_{V_\alpha} = f^*(\mathcal{P}|_{U_\alpha}) \xrightarrow{\simeq} f^*(\mathcal{G}|_{U_\alpha}) = f^*(\mathcal{G})|_{V_\alpha},$$

given by $\phi^*_\alpha(y,p) = (y, \phi_\alpha(p))$, yield

$$\phi^*_\alpha\left((y,p)\cdot(y,g)\right) = \phi^*_\alpha\left(y, p\cdot g\right) = (y, \phi_\alpha(p\cdot g)) =$$
$$(y, \phi_\alpha(p)\cdot g) = (y,\phi_\alpha(p))\cdot(y,g) = \phi^*_\alpha(y,p)\cdot(y,g),$$

for every $(y,p) \in f^*(\mathcal{P})_y$ and $(y,g) \in f^*(\mathcal{G})_y$. This shows that (ϕ^*_α) are (equivariant) coordinates with respect to $\mathcal{V}$, thus $f^*(\mathcal{P})$ is a principal sheaf.

Let us find the natural sections of $f^*(\mathcal{P})$, over $\mathcal{V}$, needed in Section 6.5. First observe that the unit section $\mathbf{1}^* : Y \to f^*(\mathcal{G})$ of $f^*(\mathcal{G})$ is given by (see also (1.4.2))

$$\mathbf{1}^* = f^*_X(\mathbf{1}), \tag{4.1.10}$$

since, for every $y \in Y$,

$$\mathbf{1}^*(y) := \big(y, e_{f(y)}\big) = \big(y, \mathbf{1}(f(y))\big) = f^*_X(\mathbf{1})(y).$$

Hence, if we denote by $(s^*_\alpha) \in f^*(\mathcal{P})(V_\alpha)$, $\alpha \in I$, the natural sections of $f^*(\mathcal{P})$ with respect to $\mathcal{V}$, equality (4.1.10) implies that

$$\begin{aligned} s^*_\alpha(y) &= \left(f^*(\phi_\alpha)^{-1}(\mathbf{1}^*)\right)(y) = \left(f^*(\phi_\alpha^{-1})(\mathbf{1}^*)\right)(y) \\ &= \big(y, (\phi_\alpha^{-1}\circ \mathbf{1})(f(y))\big)\,\big(y, s_\alpha(f(y))\big) = f^*_{U_\alpha}(s_\alpha)(y), \end{aligned}$$

from which it follows that

$$s^*_\alpha = f^*_{U_\alpha}(s_\alpha), \qquad \alpha \in I. \tag{4.1.11}$$

4.1.10 Remark. Another important, abstract, example is the *principal sheaf of frames of a vector sheaf.* Its detailed study will be deferred to Section 5.2, after laying the necessary groundwork concerning vector sheaves.

4.2. Morphisms of principal sheaves

Taking into account the notations induced in the beginning of Section 3.4, we consider two principal sheaves $\mathcal{P} \equiv (\mathcal{P}, \mathcal{G}, X, \pi_{\mathcal{P}})$ and $\mathcal{Q} \equiv (\mathcal{Q}, \mathcal{H}, X, \pi_{\mathcal{Q}})$, with structure groups $\mathcal{G} \equiv (\mathcal{G}, \mathcal{L}_{\mathcal{G}}, \rho_{\mathcal{G}}, \partial_{\mathcal{G}})$ and $\mathcal{H} \equiv (\mathcal{H}, \mathcal{L}_{\mathcal{H}}, \rho_{\mathcal{H}}, \partial_{\mathcal{H}})$, respectively.

4.2.1 Definition. A ***morphism of principal sheaves*** from $\mathcal{P}$ to $\mathcal{Q}$ is determined by a quadruple $(f, \phi, \overline{\phi}, id_X)$, where (f, id_X) is an ordinary morphism (of sheaves of sets) of $(\mathcal{P}, \pi_{\mathcal{P}}, X)$ into $(\mathcal{Q}, \pi_{\mathcal{Q}}, X)$, and $(\phi, \overline{\phi})$ a morphism of Lie sheaves of groups (as in Definition 3.4.1), so that the ***equivariance property***

$$f(p \cdot g) = f(p) \cdot \phi(g) \tag{4.2.1}$$

be satisfied, for every $(p, g) \in \mathcal{P} \times_X \mathcal{G}$.

4.2.2 Remarks. 1) If, according to Remark 4.1.5(2), we consider structure groups $\mathcal{G}$ as only sheaves of groups, then a morphism of principal sheaves is simply a triplet (f, ϕ, id_X) satisfying (4.2.1).

2) Under a slight modification of Definition 4.2.1, one can define morphisms between principal sheaves over *different bases*. An example of this situation is provided by the principal sheaves $f^*(\mathcal{P})$ and $\mathcal{P}$, linked together by the morphism

$$\left(f^*_{\mathcal{P}}, f^*_{\mathcal{G}}, f^*_{\mathcal{L}}, f\right),$$

where, in general, we set

$$f^*_{\mathcal{S}} := \mathrm{pr}_2|_{f^*(\mathcal{S})} : f^*(\mathcal{S}) = Y \times_X \mathcal{S} \longrightarrow \mathcal{P}, \tag{4.2.2}$$

for every sheaf $\mathcal{S}$ over X. With the notations of Example 4.1.9(c) and the terminology induced after Diagram 3.8, $f^*_{\mathcal{P}}$ is an f-morphism i.e.,

$$\pi \circ f^*_{\mathcal{P}} = f \circ \pi^*,$$

while, for every $((y, p), (y, g)) \in f^*(\mathcal{P}) \times_Y f^*(\mathcal{G})$,

$$f^*_{\mathcal{P}}\big((y, p) \cdot (y, g)\big) = f^*_{\mathcal{P}}(y, p) \cdot f^*_{\mathcal{G}}(y, g).$$

By Corollary 3.5.5, the last equality means that $f^*_{\mathcal{P}}$ is equivariant with respect to the action of $f^*(\mathcal{G})$ on $f^*(\mathcal{P})$ and that of $\mathcal{G}$ on $\mathcal{P}$.

Another example of morphisms of principal sheaves over different bases is given in Proposition 4.2.5 below. However, we do not pursue this matter any further, since it will rarely occur in this work.

We introduce the following convenient terminology.

4.2.3 Definition. A morphism $(f, \phi, \overline{\phi}, id_X)$, as in Definition 4.2.1, is said to be an ***isomorphism*** if f and $(\phi, \overline{\phi})$ are isomorphisms of the corresponding structures. In particular, if $\mathcal{G} = \mathcal{H}$, $\mathcal{L}_{\mathcal{G}} = \mathcal{L}_{\mathcal{H}} = \mathcal{L}$ and $(\phi, \overline{\phi}) = (id_{\mathcal{G}}, id_{\mathcal{L}})$, then a morphism $(f, id_{\mathcal{G}}, id_{\mathcal{L}}, id_X)$ of $(\mathcal{P}, \mathcal{G}, X, \pi_{\mathcal{P}})$ into $(\mathcal{Q}, \mathcal{G}, X, \pi_{\mathcal{Q}})$ is called a ***$\mathcal{G}$-morphism*** and will be simply denoted by f, if there is no danger of confusion.

It is immediate that, for an isomorphism as above, $(f^{-1}, \phi^{-1}, \overline{\phi}^{-1}, id_X)$ is also a morphism of principal sheaves, so we get an isomorphism within the category of principal sheaves.

4.2.4 Theorem. *With the notations of Definition 4.2.3, every $\mathcal{G}$-morphism $f : (\mathcal{P}, \mathcal{G}, X, \pi_{\mathcal{P}}) \to (\mathcal{Q}, \mathcal{G}, X, \pi_{\mathcal{Q}})$ is an isomorphism.*

Proof. In virtue of the conclusion of Subsection 1.1.1, it suffices to show that the restrictions $f_x : \mathcal{P}_x \to \mathcal{Q}_x$ are bijections, for all $x \in X$.

For the injectivity of an arbitrary f_x assume that $f(p) = f(p')$, for some $p, p' \in \mathcal{P}_x$. Since, by Proposition 4.1.2, there is a (unique) $g \in \mathcal{G}_x$ such that $p' = p \cdot g$, the $\mathcal{G}$-equivariance of f implies that

$$f(p) = f(p') = f(p \cdot g) = f(p) \cdot g;$$

hence, $g = e_x$, which yields the desired injectivity.

To show the ontoness, we choose an arbitrary $q \in \mathcal{Q}_x$. If $\mathcal{U}$ is a local frame of $\mathcal{P}$, there exists some $U_\alpha \in \mathcal{U}$ with $x \in U_\alpha$. We consider the natural section $s_\alpha \in \mathcal{P}(U_\alpha)$ and the element

$$p := s_\alpha(x) \cdot \boldsymbol{k'}\big(f(s_\alpha(x)), q\big) \in \mathcal{P}_x,$$

where $\boldsymbol{k'}$ is the analog of $\boldsymbol{k}$ for the principal sheaf $\mathcal{Q}$ (see Proposition 4.1.4). Then (4.2.1) implies that

$$f(p) = f\big(s_\alpha(x) \cdot \boldsymbol{k'}(f(s_\alpha(x)), q)\big) = f(s_\alpha(x)) \cdot \boldsymbol{k'}\big(f(s_\alpha(x)), q\big) = q,$$

which terminates the proof. ∎

A byproduct of the preceding proof is the following useful equality

$$f^{-1}(q) = s_\alpha(\pi_{\mathcal{Q}}(q)) \cdot \boldsymbol{k'}\big(f\big(s_\alpha(\pi_{\mathcal{Q}}(q))\big), q\big),$$

for every $q \in \pi_{\mathcal{Q}}^{-1}(U_\alpha)$. Equivalently, for every $U_\alpha \in \mathcal{U}$, we have that

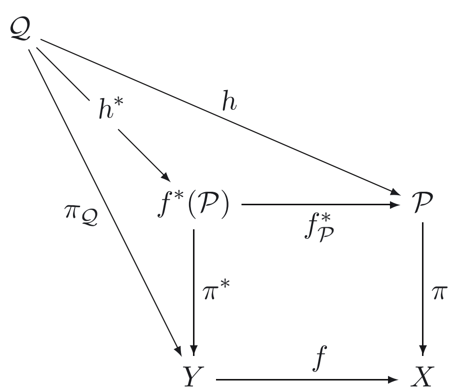
$$f^{-1}\big|_{\pi_{\mathcal{Q}}^{-1}(U_\alpha)} = (s_\alpha \circ \pi_{\mathcal{Q}}) \cdot \big(\boldsymbol{k'} \circ (f \circ s_\alpha \circ \pi_{\mathcal{Q}},\, id)\big)\big|_{\pi_{\mathcal{Q}}^{-1}(U_\alpha)}.$$

We are now in a position to prove a fundamental property of the pull-back of principal sheaves, based on the generalization of morphisms discussed in Remark 4.2.2(2). Namely, with the notations of Example 4.1.9(c) and (4.2.2), we state the following:

4.2.5 Proposition. *Consider a principal sheaf $\mathcal{P} \equiv (\mathcal{P}, \mathcal{G}, X, \pi)$ and its pull-back $f^*(\mathcal{P}) \equiv (f^*(\mathcal{P}), f^*(\mathcal{G}), Y, \pi^*)$ by a continuous map $f : Y \to X$. Then $f^*(\mathcal{P})$ has the following* **universal property***: Given any principal sheaf $\mathcal{Q} \equiv (\mathcal{Q}, f^*(\mathcal{G}), Y, \pi_{\mathcal{Q}})$ and any principal sheaf morphism $(h, f^*_{\mathcal{G}}, f^*_{\mathcal{L}}, f)$ of $\mathcal{Q}$ into $\mathcal{P}$, there exists a unique $f^*(\mathcal{G})$-(iso)morphism h^* of $\mathcal{Q}$ onto $f^*(\mathcal{P})$ such that*

$$f^*_{\mathcal{P}} \circ h^* = h. \tag{4.2.3}$$

The statement is summarized in the following diagram whose three sub-diagrams are commutative.



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Proof. We set

$$h^*(q) := \big(\pi_{\mathcal{Q}}(q), h(q)\big), \qquad q \in \mathcal{Q}.$$

Obviously, h^* is a continuous map such that $\pi^* \circ h^* = \pi_{\mathcal{Q}}$, thus h^* is a morphism of sheaves.

Equality (4.2.3) is an immediate consequence of the definition of h^* and (4.2.2). On the other hand, since h is equivariant with respect to the actions of $f^*(\mathcal{G})$ and $\mathcal{G}$, we check that, for any $q \in \mathcal{Q}_y$ and $(y,g) \in f^*(\mathcal{G})_y$,

$$\begin{aligned} h^*\left(q \cdot (y,g)\right) &= \big(\pi_{\mathcal{Q}}(q \cdot (y,g)), h(q \cdot (y,g))\big) = \big(y, h(q) \cdot f^*_{\mathcal{G}}(y,g)\big) \\ &= (y, h(q) \cdot g) = (y, h(q)) \cdot (y,g) = h^*(q) \cdot (y,g). \end{aligned}$$

Hence, by Theorem 4.2.4 (see also Definition 4.2.3), h^* is a principal sheaf isomorphism.

The uniqueness is proved as follows. Assume there exists another principal sheaf morphism, say $\underline{h}^*$, also satisfying

$$f^*_{\mathcal{P}} \circ \underline{h}^* = h.$$

Of course, $\underline{h}^*$ is $f^*(\mathcal{G})$-equivariant with $\pi^* \circ \underline{h}^* = \pi_{\mathcal{Q}}$. Therefore,

$$\underline{h}^* = (\mathrm{pr}_1 \circ \underline{h}^*, \mathrm{pr}_2 \circ \underline{h}^*) = (\pi^* \circ \underline{h}^*, f^*_{\mathcal{P}} \circ \underline{h}^*) = (\pi_{\mathcal{Q}}, h) = h,$$

where the projections pr_i, $(i = 1, 2)$, are restricted to $f^*(\mathcal{P})$. □

Note. Regarding the previous statement, the assumption that the morphism between $\mathcal{Q}$ and $\mathcal{P}$ has the form $(h, f^*_{\mathcal{G}}, f^*_{\mathcal{L}}, f)$ is essential. Indeed, if we take a morphism of the form $(h, \phi, \overline{\phi}, f)$, where

$$\phi : f^*(\mathcal{G}) \longrightarrow \mathcal{G}, \qquad \overline{\phi} : f^*(\mathcal{L}) \longrightarrow \mathcal{L}$$

are arbitrary morphisms (of Lie sheaves of groups and sheaves of Lie algebras, respectively) over f, then the $f^*(\mathcal{G})$-equivariance of h is not ensured.

4.3. The cocycle of a principal sheaf

Let $\mathcal{P} \equiv (\mathcal{P}, \mathcal{G}, X, \pi)$ be a principal sheaf with a local frame $(\mathcal{U}, (\phi_\alpha))$. Due to the $\mathcal{G}$-equivariance of every ϕ_α and ϕ_α^{-1}, the ***coordinate transformation*** $\phi_\alpha \circ \phi_\beta^{-1} : \mathcal{G}|_{U_{\alpha\beta}} \to \mathcal{G}|_{U_{\alpha\beta}}$ is fully determined by its value at the neutral element of each stalk. More precisely, for every $g \in \mathcal{G}_x$, with $x \in U_{\alpha\beta}$,

$$(\phi_\alpha \circ \phi_\beta^{-1})(g) = (\phi_\alpha \circ \phi_\beta^{-1})(e_x \cdot g) = (\phi_\alpha \circ \phi_\beta^{-1})(e_x) \cdot g.$$

Setting

(4.3.1) $$g_{\alpha\beta}(x) := (\phi_\alpha \circ \phi_\beta^{-1})(e_x); \qquad x \in U_{\alpha\beta},$$

we obtain a family of sections

$$g_{\alpha\beta} \in \mathcal{G}(U_{\alpha\beta}); \qquad \alpha, \beta \in I,$$

satisfying the equalities

$$g_{\alpha\beta} = (\phi_\alpha \circ \phi_\beta^{-1}) \circ \mathbf{1}|_{U_{\alpha\beta}} \tag{4.3.2}$$

$$= (\phi_\alpha \circ \phi_\beta^{-1}) \big(\mathbf{1}|_{U_{\alpha\beta}}\big)\,, \tag{4.3.2$'$}$$

the second of them obviously referring to the induced morphism of sections.

4.3.1 Definition. The family $(g_{\alpha\beta})$ is called the ***coordinate 1-cocycle***, or simply the ***cocycle*** of $\mathcal{P}$, with respect to the local frame $\mathcal{U} \equiv (\mathcal{U}, (\phi_\alpha))$. Each $g_{\alpha\beta} \in \mathcal{G}(U_{\alpha\beta})$ is called a ***transition section***.

4.3.2 Proposition. *Over $U_{\alpha\beta\gamma} := U_\alpha \cap U_\beta \cap U_\gamma$, the* ***cocycle condition***

$$g_{\alpha\gamma} = g_{\alpha\beta} \cdot g_{\beta\gamma}$$

is satisfied. Hence, in the notations of Subsection 1.6.4, $(g_{\alpha\beta}) \in Z^1(\mathcal{U}, \mathcal{G})$.

Proof. For every $x \in U_{\alpha\beta\gamma}$, (4.3.1) and the $\mathcal{G}$-equivariance of the coordinates imply that

$$\begin{aligned} g_{\alpha\gamma}(x) &= \big((\phi_\alpha \circ \phi_\beta^{-1}) \circ (\phi_\beta \circ \phi_\gamma^{-1})\big)(e_x) = (\phi_\alpha \circ \phi_\beta^{-1})(g_{\beta\gamma}(x)) \\ &= (\phi_\alpha \circ \phi_\beta^{-1})(e_x) \cdot g_{\beta\gamma}(x) = g_{\alpha\beta}(x) \cdot g_{\beta\gamma}(x). \end{aligned}$$

□

4.3.3 Corollary. *For every $\alpha, \beta \in I$, the following equalities hold true:*

$$g_{\alpha\alpha} = \mathbf{1}|_{U_\alpha} \quad \textit{and} \quad g_{\beta\alpha} = g_{\alpha\beta}^{-1},$$

where $g_{\alpha\beta}^{-1} \in \mathcal{G}(U_{\alpha\beta})$ is the inverse section of $g_{\alpha\beta}$, defined by (1.1.4).

The relationship of the cocycle $(g_{\alpha\beta})$ with the natural sections and the coordinates of $\mathcal{P}$ is given in the following:

4.3.4 Proposition. *For every $\alpha, \beta \in I$, the compatibility conditions*

$$s_\beta = s_\alpha \cdot g_{\alpha\beta}, \tag{4.3.3}$$

$$\phi_\beta = (g_{\beta\alpha} \circ \pi) \cdot \phi_\alpha, \tag{4.3.4}$$

hold over $U_{\alpha\beta}$ and $\mathcal{P}|_{U_{\alpha\beta}}$, respectively.

Proof. For every $x \in U_{\alpha\beta}$, (4.1.7) yields

$$s_\beta(x) = \phi_\beta^{-1}(e_x) = \phi_\alpha^{-1}\big((\phi_\alpha \circ \phi_\beta^{-1})(e_x)\big) = \\ \phi_\alpha^{-1}(g_{\alpha\beta}(x)) = \phi_\alpha^{-1}(e_x) \cdot g_{\alpha\beta}(x) = s_\alpha(x) \cdot g_{\alpha\beta}(x),$$

which proves (4.3.3).

For the second condition we take any $p \in \mathcal{P}|_{U_{\alpha\beta}}$ with $\pi(p) = x$. Since $s_\alpha(x)$ and p belong to $\mathcal{P}_x$, by Proposition 4.1.2 there is a unique $g \in \mathcal{G}_x$ such that $p = s_\alpha(x) \cdot g$. Hence,

$$\phi_\beta(p) = \phi_\beta(s_\alpha(x) \cdot g) = (\phi_\beta \circ \phi_\alpha^{-1})(e_x) \cdot g = g_{\beta\alpha}(x) \cdot g.$$

On the other hand,

$$\phi_\alpha(p) = \phi_\alpha(s_\alpha(x) \cdot g) = \phi_\alpha(s_\alpha(x)) \cdot g = g.$$

Combining the previous equalities, we obtain (4.3.4). □

4.3.5 Remark. Equality (4.3.3) can be equivalently used to define the cocycle $(g_{\alpha\beta})$; that is, $g_{\alpha\beta} \in \mathcal{G}(U_{\alpha\beta})$ is the (unique) section satisfying (4.3.3). Its continuity is now a result of the equality

$$g_{\alpha\beta} = \boldsymbol{k} \circ (s_\alpha, s_\beta).$$

We shall find a variant of (4.3.4) involving also the restriction maps of the presheaves of sections of $\mathcal{P}$ and $\mathcal{G}$. This will motivate the construction of a principal sheaf from a cocycle.

From the (Lie) sheaf of groups $\mathcal{G}$ we obtain the presheaf $(\mathcal{G}(U_\alpha), \zeta_{\alpha\beta})$ of sections of $\mathcal{G}$ over $\mathcal{U}$, where –for convenience– we have denoted by

$$\zeta_{\alpha\beta} := \zeta_{U_\beta}^{U_\alpha} : \mathcal{G}(U_\alpha) \longrightarrow \mathcal{G}(U_\beta) : \sigma \mapsto \sigma|_{U_\beta}; \qquad U_\beta \subseteq U_\alpha,$$

the ordinary restriction maps. Analogously, we consider the presheaf of sections of $\mathcal{P}$ over $\mathcal{U}$, denoted by $(\mathcal{P}(U_\alpha), \rho_{\alpha\beta})$.

Obviously, each $\mathcal{G}(U_\alpha)$ is a group acting *freely* on the right of $\mathcal{P}(U_\alpha)$. Also, the coordinates of $\mathcal{P}$ induce corresponding equivariant isomorphisms of sections $\phi_\alpha : \mathcal{P}(U_\alpha) \to \mathcal{G}(U_\alpha)$, for all $\alpha \in I$. To have a clearer picture of the situation, we draw the following diagram, for $U_\beta \subseteq U_\alpha$,

$$\begin{array}{ccc} \mathcal{P}(U_\alpha) & \xrightarrow{\rho_{\alpha\beta}} & \mathcal{P}(U_\beta) \\ \Big\downarrow{\scriptstyle \phi_\alpha} & & \Big\downarrow{\scriptstyle \phi_\beta} \\ \mathcal{G}(U_\alpha) & \xrightarrow{\zeta_{\alpha\beta}} & \mathcal{G}(U_\beta) \end{array}$$

DIAGRAM 4.2

which *is not commutative*, as the subsequent calculations show. Indeed, for any $s \in \mathcal{P}(U_\alpha)$ and $x \in U_\beta$, we have that

$$\big((\zeta_{\alpha\beta} \circ \phi_\alpha)(s)\big)(x) = \phi_\alpha(s)|_{U_\beta}(x) = \phi_\alpha(s(x)) =$$
$$(\phi_\alpha \circ \phi_\beta^{-1})\big(\phi_\beta(s)(x)\big) = (\phi_\alpha \circ \phi_\beta^{-1})\Big(\big((\phi_\beta \circ \rho_{\alpha\beta})(s)\big)(x)\Big) =$$
$$(\phi_\alpha \circ \phi_\beta^{-1})(e_x) \cdot \big((\phi_\beta \circ \rho_{\alpha\beta})(s)\big)(x) = g_{\alpha\beta}(x) \cdot \big((\phi_\beta \circ \rho_{\alpha\beta})(s)\big)(x);$$

that is,

$$\zeta_{\alpha\beta} \circ \phi_\alpha = g_{\alpha\beta} \cdot (\phi_\beta \circ \rho_{\alpha\beta}).$$

The preceding equality determines the (restriction) maps

$$\varrho_{\alpha\beta} := g_{\beta\alpha} \cdot \zeta_{\alpha\beta} = \phi_\beta \circ \rho_{\alpha\beta} \circ \phi_\alpha^{-1} \,:\, \mathcal{G}(U_\alpha) \longrightarrow \mathcal{G}(U_\beta). \tag{4.3.5}$$

(Note the difference between the two typefaces $\rho_{\alpha\beta}$ and $\varrho_{\alpha\beta}$.)

Consequently,

(4.3.6) *under the isomorphisms ϕ_α and ϕ_β, each restriction map $\rho_{\alpha\beta}$ is identified with $\varrho_{\alpha\beta} = g_{\beta\alpha} \cdot \zeta_{\alpha\beta}$; thus $\mathcal{P}$ can be essentially recaptured from the presheaf $\big(\mathcal{G}(U_{\alpha\beta}), (\varrho_{\alpha\beta})\big)$ and the cocycle $(g_{\alpha\beta})$.*

This remark will be exploited in the construction of $\mathcal{P}$ from its cocycle, as expounded in Theorem 4.5.1.

4.3.6 Example. Let us continue the study of Example 4.1.9(a) by finding the relationship between the cocycle $(g_{\alpha\beta})$ of the principal bundle P and the cocycle, say $(\gamma_{\alpha\beta})$, of the sheaf of germs of its smooth sections $\mathcal{P}$, both defined over the local frame $\mathcal{U}$. In virtue of (4.1.9), equality (4.3.3) implies that $\widetilde{\sigma}_\beta(x) = \widetilde{\sigma}_\alpha(x) \cdot \gamma_{\alpha\beta}(x)$, for every $x \in U_{\alpha\beta}$, or, by (1.2.10),

$$[\sigma_\beta]_x = [\sigma_\alpha]_x \cdot \gamma_{\alpha\beta}(x).$$

Since $\sigma_\beta = \sigma_\alpha \cdot g_{\alpha\beta}$, it follows that

$$[\sigma_\beta]_x = [\sigma_\alpha \cdot g_{\alpha\beta}]_x = [\sigma_\alpha]_x \cdot [g_{\alpha\beta}]_x.$$

Comparing the last two series of equalities, we have that

$$\gamma_{\alpha\beta}(x) = [g_{\alpha\beta}]_x = \widetilde{g_{\alpha\beta}}(x).$$

Therefore, we obtain the bijection

$$C^\infty(U_{\alpha\beta}, G) \ni g_{\alpha\beta} \longmapsto \widetilde{g_{\alpha\beta}} = \gamma_{\alpha\beta} \in \mathcal{G}(U_{\alpha\beta}). \tag{4.3.7}$$

This is in accordance with the canonical identification of $C^\infty(U_{\alpha\beta}, G)$ with $\mathcal{C}^\infty_X(G)(U_{\alpha\beta})$, owing to the completeness of the presheaf $U \longmapsto C^\infty(U, G)$.

4.4. Morphisms of principal sheaves and cocycles

The results of this section describe the relationship of morphisms and isomorphisms of principal sheaves with cocycles. The case of isomorphisms is important for the classification of principal sheaves.

4.4.1 Theorem. *Let $\mathcal{P} \equiv (\mathcal{P}, \mathcal{G}, X, \pi)$ and $\mathcal{P}' \equiv (\mathcal{P}', \mathcal{G}', X, \pi')$ be two principal sheaves with respective local frames $(\mathcal{U}, (\phi_\alpha))$ and $(\mathcal{U}, (\phi'_\alpha))$, over the same open covering $\mathcal{U} = (U_\alpha)_{\alpha \in I}$ of X. We denote by (s_α), (s'_α) the natural sections of $\mathcal{P}$, $\mathcal{P}'$, respectively, and by $(g_{\alpha\beta})$, $(g'_{\alpha\beta})$ the associated cocycles.*

If $(f, \phi, \overline{\phi}, id_X)$ is a morphism of $\mathcal{P}$ into $\mathcal{P}'$, then there exists a unique 0-cochain $(h_\alpha) \in C^0(\mathcal{U}, \mathcal{G}')$ satisfying the equalities

$$f(s_\alpha) = s'_\alpha \cdot h_\alpha, \tag{4.4.1}$$

$$g'_{\alpha\beta} = h_\alpha \cdot \phi(g_{\alpha\beta}) \cdot h_\beta^{-1}, \tag{4.4.2}$$

over U_α and $U_{\alpha\beta}$ respectively.

Conversely, assume that we are given a morphism $(\phi, \overline{\phi})$ of Lie sheaves of groups of $\mathcal{G}$ into $\mathcal{G}'$, and a 0-cochain (h_α) satisfying (4.4.2). Then there exists a unique morphism of sheaves (of sets) $f : \mathcal{P} \to \mathcal{P}'$ satisfying (4.4.1) and such that $(f, \phi, \overline{\phi}, id_X)$ is a morphism of principal sheaves.

Proof. Let any $x \in U_\alpha$. Since $f(s_\alpha)(x) = f(s_\alpha(x))$ and $s'_\alpha(x)$ belong to the same stalk $\mathcal{P}'_x$, there is a unique $h_\alpha(x) \in \mathcal{G}'_x$ such that

$$f(s_\alpha)(x) = s'_\alpha(x) \cdot h_\alpha(x)$$

(see Proposition 4.1.2). Because we can write $h_\alpha = \boldsymbol{k}' \circ (s'_\alpha, f(s_\alpha))$, where $\boldsymbol{k}'$ is the analog of $\boldsymbol{k}$ for $\mathcal{P}'$, Proposition 4.1.4 implies that h_α is a continuous section of $\mathcal{G}'$ over U_α, satisfying (4.4.1).

Applying f to both sides of (4.3.3), we see that $f(s_\beta) = f(s_\alpha) \cdot \phi(g_{\alpha\beta})$. Substituting $f(s_\alpha)$ and $f(s_\beta)$ with their expressions given by (4.4.1), for α and β respectively, the previous equality is transformed to

$$s'_\beta \cdot h_\beta = s'_\alpha \cdot h_\alpha \cdot \phi(g_{\alpha\beta})$$

or, by the analog of (4.3.3) for $\mathcal{P}'$,

$$s'_\alpha \cdot g'_{\alpha\beta} \cdot h_\beta = s'_\alpha \cdot h_\alpha \cdot \phi(g_{\alpha\beta}).$$

The last equality yields (4.4.2), since $\mathcal{G}(U_{\alpha\beta})$ acts freely on $\mathcal{P}(U_{\alpha\beta})$.

For the converse we proceed as follows. Given an $\alpha \in I$, we define the map $f_\alpha : \mathcal{P}|_{U_\alpha} \to \mathcal{P}'|_{U_\alpha}$ by setting

$$f_\alpha(p) := s'_\alpha(x) \cdot h_\alpha(x) \cdot \phi(g_\alpha(x)), \tag{4.4.3}$$

where $x := \pi(p) \in U_\alpha$ and $g_\alpha(x)$ is the unique element of $\mathcal{G}_x$ with

$$p = s_\alpha(x) \cdot g_\alpha(x). \tag{4.4.4}$$

Using Corollary 4.1.3, we routinely check that $\pi' \circ f_\alpha = \pi$. Moreover, since (4.4.4) is equivalently written as $g_\alpha(x) = \boldsymbol{k}(s_\alpha(x), p)$, we have that

$$f_\alpha = (s'_\alpha \circ \pi) \cdot (h_\alpha \circ \pi) \cdot \big(\phi \circ \boldsymbol{k} \circ (s_\alpha \circ \pi,\, id)\big),$$

with π and id restricted to $\mathcal{P}|_{U_\alpha} = \pi^{-1}(U_\alpha)$. Thus f_α is continuous and determines a morphism of sheaves.

The morphism f_α is equivariant, with respect to the actions of $\mathcal{G}|_{U_\alpha}$ and $\mathcal{G}'|_{U_\alpha}$, for if $p \in \mathcal{P}_x$ and $g \in \mathcal{G}_x$, with $x \in U_\alpha$, then $\pi(p \cdot g) = \pi(p)$ and (see also Proposition 4.1.4)

$$f_\alpha(p \cdot g) = s'_\alpha(x) \cdot h_\alpha(x) \cdot \phi\big(\boldsymbol{k}(s_\alpha(x), p \cdot g)\big) =$$
$$s'_\alpha(x) \cdot h_\alpha(x) \cdot \phi\big(\boldsymbol{k}(s_\alpha(x), p)\big) \cdot \phi(g) = f_\alpha(p) \cdot \phi(g).$$

Therefore, f_α is a morphism of principal sheaves between $\mathcal{P}|_{U_\alpha}$ and $\mathcal{P}'|_{U_\alpha}$. We obtain the desired morphism f by gluing all the f_α's together; that is, by setting $f|_{\pi^{-1}(U_\alpha)} := f_\alpha$.

We show that f is a well defined morphism. Indeed, for any $p \in \mathcal{P}$ with $\pi(p) = x \in U_{\alpha\beta}$, we get the analogs of (4.4.3) and (4.4.4)

$$f_\beta(p) = s'_\beta(x) \cdot h_\beta(x) \cdot \phi(g_\beta(x)), \tag{4.4.3$'$}$$

$$p = s_\beta(x) \cdot g_\beta(x). \tag{4.4.4$'$}$$

However, (4.4.4) and (4.4.4′), along with (4.3.3), yield

$$s_\alpha(x) \cdot g_\alpha(x) = s_\beta(x) \cdot g_\beta(x) = s_\alpha(x) \cdot g_{\alpha\beta}(x) \cdot g_\beta(x),$$

from which it follows that

$$g_\alpha(x) = g_{\alpha\beta}(x) \cdot g_\beta(x). \tag{4.4.5}$$

As a result, the analog of (4.3.3) for $\mathcal{P}'$, Corollary 4.3.3, and equalities (4.4.5), (4.4.2) transform the right-hand side of (4.4.3′) into

$$\begin{gathered} s'_\beta(x) \cdot h_\beta(x) \cdot \phi(g_\beta(x)) = \\ (s'_\alpha(x) \cdot g'_{\alpha\beta}(x)) \cdot h_\beta(x) \cdot \phi(g_{\beta\alpha}(x)) \cdot \phi(g_\alpha(x)) = \\ s'_\alpha(x) \cdot \big(h_\alpha(x) \cdot \phi(g_{\alpha\beta}(x)) \cdot h_\beta^{-1}(x)\big) \cdot h_\beta(x) \cdot \phi(g_{\beta\alpha}(x)) \cdot \phi(g_\alpha(x)) = \\ s'_\alpha(x) \cdot h_\alpha(x) \cdot \phi(g_\alpha(x)), \end{gathered}$$

which shows that (4.4.3) and (4.4.3′) coincide on the overlapping, hence f is well defined.

Property (4.4.1) is a direct consequence of (4.4.3), since $p = s_\alpha(x)$ implies that $g_\alpha(x) = e_x$.

Finally, f is the unique morphism satisfying (4.4.1), for if there is another morphism, say $\bar{f}$, with the same property, then, for any $p \in \mathcal{P}_x$ as before, equality (4.4.4) implies that

$$\begin{gathered} \bar{f}(p) = \bar{f}\,(s_\alpha(x) \cdot g_\alpha(x)) = \bar{f}(s_\alpha(x)) \cdot \phi(g_\alpha(x)) = \\ s'_\alpha(x) \cdot h_\alpha(x) \cdot \phi(g_\alpha(x)) = f(s_\alpha(x)) \cdot \phi(g_\alpha(x)) = \\ f(s_\alpha(x) \cdot g_\alpha(x)) = f(p). \end{gathered}$$

This terminates the proof. □

Note. Regarding the assumptions of the previous statement, we observe that we can always define local frames over the same open covering $\mathcal{U}$. This is obviously done by taking the intersection of the original coverings of the frames and then by restricting the respective coordinates to the new covering.

If both $\mathcal{P}$ and $\mathcal{P}'$ are $\mathcal{G}$-principal sheaves, then, taking into account Theorem 4.2.4, we see that Theorem 4.4.1 reduces to:

4.4.2 Theorem. *Let $\mathcal{P} \equiv (\mathcal{P}, \mathcal{G}, X, \pi)$ and $\mathcal{P}' \equiv (\mathcal{P}', \mathcal{G}, X, \pi')$ be two $\mathcal{G}$-principal sheaves with respective local frames $\big(\mathcal{U}, (\phi_\alpha)\big)$ and $\big(\mathcal{U}, (\phi'_\alpha)\big)$. Also* *let* (s_α), (s'_α) *be the natural sections of* $\mathcal{P}$, $\mathcal{P}'$, *respectively, and* $(g_{\alpha\beta})$, $(g'_{\alpha\beta})$ *their associated cocycles. Then, for every* $\mathcal{G}$*-isomorphism* $f \equiv (f, id_{\mathcal{G}}, id_{\mathcal{L}}, id_X)$ *of* $\mathcal{P}$ *onto* $\mathcal{P}'$, *there exists a unique 0-cochain* $(h_\alpha) \in C^0(\mathcal{U}, \mathcal{G})$ *satisfying*

$$f(s_\alpha) = s'_\alpha \cdot h_\alpha, \tag{4.4.6}$$

$$g'_{\alpha\beta} = h_\alpha \cdot g_{\alpha\beta} \cdot h_\beta^{-1}, \tag{4.4.7}$$

over U_α *and* $U_{\alpha\beta}$ *respectively.*

Conversely, a 0-cochain (h_α) *satisfying equality (4.4.7) determines a unique* $\mathcal{G}$*-isomorphism* f *of* $\mathcal{P}$ *onto* $\mathcal{P}'$, *also satisfying (4.4.6).*

In the familiar terminology of non-abelian cohomology (see Subsection 1.6.4 and equality (1.6.38)), two (1-)cocycles satisfying (4.4.7) are said to be ***cohomologous***. Therefore, Theorem 4.4.2 can be rephrased in the following concise form.

4.4.2 Theorem (restated). *Two* $\mathcal{G}$*-principal sheaves* $\mathcal{P}$ *and* $\mathcal{P}'$ *are* $\mathcal{G}$*-isomorphic if and only if they admit cohomologous cocycles.*

For the sake of completeness we also state the following result, whose proof is obvious. For its principal *bundle* analog, as well as that of Theorem 4.4.2, the reader is referred to Bourbaki [13, n° 6.4.4].

4.4.3 Corollary. *There is a bijection between* $\mathcal{G}$*-isomorphisms of* $\mathcal{P}$ *onto* $\mathcal{P}'$ *and 0-cochains* $(h_\alpha) \in C^0(\mathcal{U}, \mathcal{G})$ *satisfying condition (4.4.7).*

4.4.4 Remark. The results of this section hold true if the structure groups of the principal sheaves involved are simply sheaves of groups and the morphisms $(f, \phi, \overline{\phi}, id_X)$ and $(f, id_{\mathcal{G}}, id_{\mathcal{L}}, id_X)$ are replaced by (f, ϕ, id_X) and $(f, id_{\mathcal{G}}, id_X)$, respectively. See also the Remarks 4.1.5(2) and 4.2.2(1).

4.5. Principal sheaves from cocycles

In Section 4.3 we saw that the local structure of a principal sheaf $\mathcal{P}$ leads to the construction of a cocycle $(g_{\alpha\beta})$. Here, following the opposite direction, we construct a principal sheaf from a given cocycle.

Throughout this section we assume that

(4.5.1) $\mathcal{U} = (U_\alpha)_{\alpha\in I}$ *is an open covering of the topological space* X, *which is a basis for its topology.*

Given a (Lie) sheaf of groups $\mathcal{G}$ over X and an open covering $\mathcal{U}$, as above, we prove the following basic result:

4.5.1 Theorem. *A 1-cocycle* $(g_{\alpha\beta}) \in Z^1(\mathcal{U}, \mathcal{G})$ *determines a unique, up to isomorphism, principal sheaf* $(\mathcal{P}, \mathcal{G}, X, \pi)$ *with corresponding cocycle precisely the given* $(g_{\alpha\beta})$.

Proof. The idea of the proof is motivated by (4.3.5) and the discussion in (4.3.6). From the latter we recall that $(\mathcal{G}(U_\alpha), \zeta_{\alpha\beta})$ denotes the presheaf of sections of $\mathcal{G}$ over the basis $\mathcal{U}$, with restriction maps

$$\zeta_{\alpha\beta} := \zeta^{U_\alpha}_{U_\beta} : \mathcal{G}(U_\alpha) \longrightarrow \mathcal{G}(U_\beta) : \sigma \mapsto \sigma|_{U_\beta},$$

if $U_\beta \subseteq U_\alpha$, while the morphisms

$$\varrho_{\alpha\beta} \equiv \varrho^{U_\alpha}_{U_\beta} := g_{\beta\alpha} \cdot \zeta_{\alpha\beta} : \mathcal{G}(U_\alpha) \longrightarrow \mathcal{G}(U_\beta)$$

are candidates for the restriction maps of a new presheaf structure with sections $\mathcal{G}(U_\alpha)$. In both of the preceding definitions, $\varrho_{\alpha\beta}$ and $\zeta_{\alpha\beta}$ are morphisms of sections, while $g_{\alpha\beta} \in \mathcal{G}(U_{\alpha\beta})$ is a *constant* factor.

For every $U_\alpha, U_\beta, U_\gamma$ with $U_\gamma \subseteq U_\beta \subseteq U_\alpha$, and $\sigma \in \mathcal{G}(U_\alpha)$, we see that

$$\begin{gathered}(\varrho_{\beta\gamma} \circ \varrho_{\alpha\beta})(\sigma) = \varrho_{\beta\gamma}\left(g_{\beta\alpha} \cdot \zeta_{\alpha\beta}(\sigma)\right) = \\ g_{\gamma\beta} \cdot \zeta_{\beta\gamma}\left(g_{\beta\alpha} \cdot \zeta_{\alpha\beta}(\sigma)\right) = g_{\gamma\beta} \cdot g_{\beta\alpha} \cdot \zeta_{\beta\gamma}\left(\sigma|_{U_\beta}\right) = \\ g_{\gamma\alpha} \cdot \sigma|_{U_\gamma} = g_{\gamma\alpha} \cdot \zeta_{\alpha\gamma}(\sigma) = \varrho_{\alpha\gamma}(\sigma),\end{gathered}$$

which means that $\varrho_{\alpha\gamma} = \varrho_{\beta\gamma} \circ \varrho_{\alpha\gamma}$, as pictured below.

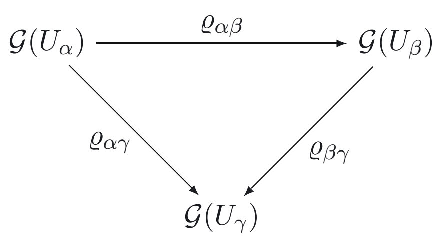

Diagram 4.3

Therefore, in virtue of (4.5.1), the assignment $U_\alpha \mapsto \mathcal{G}(U_\alpha)$, for U_α running in $\mathcal{U}$, together with the maps $\varrho_{\alpha\beta}$, determines a presheaf $P = (\mathcal{G}(U_\alpha), \varrho_{\alpha\beta})$. It should be noted that P *is not a presheaf of groups* since the maps $(\varrho_{\alpha\beta})$ are not group homomorphisms.

If $(\mathcal{P}, \pi, X)$ is the sheaf (of sets) generated by the presheaf P, we shall prove that $\mathcal{P}$ is the principal sheaf we are looking for.

i) There is a *right action* $\delta : \mathcal{P} \times_X \mathcal{G} \longrightarrow \mathcal{P}$ obtained as follows: For each $\alpha \in I$, we define the map

$$\delta_\alpha : \mathcal{G}(U_\alpha) \times \mathcal{G}(U_\alpha) \longrightarrow \mathcal{G}(U_\alpha) \ : \ (\sigma, g) \mapsto \sigma \cdot g.$$

It is routinely checked that δ_α is an action, and the diagram

$$\begin{array}{ccc} \mathcal{G}(U_\alpha) \times \mathcal{G}(U_\alpha) & \xrightarrow{\delta_\alpha} & \mathcal{G}(U_\alpha) \\ {\scriptstyle \varrho_{\alpha\beta} \times \zeta_{\alpha\beta}} \downarrow & & \downarrow {\scriptstyle \varrho_{\alpha\beta}} \\ \mathcal{G}(U_\beta) \times \mathcal{G}(U_\beta) & \xrightarrow{\delta_\beta} & \mathcal{G}(U_\beta) \end{array}$$

Diagram 4.4

is commutative, for every $U_\beta \subseteq U_\alpha$. Then δ is the action generated by the presheaf morphism of local actions (δ_α).

ii) To find the local structure of $\mathcal{P}$, let us fix an open set $U_\alpha \in \mathcal{U}$. Then, all the U_β's, with $U_\beta \subseteq U_\alpha$, form a basis for the topology of U_α. For any such U_β, we define the map

$$\phi_{\alpha, U_\beta} : \mathcal{G}(U_\beta) \longrightarrow \mathcal{G}(U_\beta) : \sigma \mapsto g_{\alpha\beta} \cdot \sigma, \tag{4.5.2}$$

whose domain is the group of sections belonging to the presheaf $P = (\mathcal{G}(U_\alpha), \varrho_{\alpha\beta})$ generating $\mathcal{P}$, whereas its range is the group of sections from the presheaf $(\mathcal{G}(U_\alpha), \zeta_{\alpha\beta})$ generating $\mathcal{G}$. It is easily seen that (4.5.2) is a $\mathcal{G}(U_\beta)$-equivariant bijection, whose inverse is given by $\phi^{-1}_{\alpha, U_\beta}(\tau) = g_{\beta\alpha} \cdot \tau$, if $\tau \in \mathcal{G}(U_\beta)$. Moreover, for every U_γ with $U_\gamma \subseteq U_\beta \subseteq U_\alpha$, we obtain the following commutative diagram.

$$\begin{array}{ccc} \mathcal{G}(U_\beta) & \xrightarrow{\phi_{\alpha, U_\beta}} & \mathcal{G}(U_\beta) \\ {\scriptstyle \varrho_{\beta\gamma}} \downarrow & & \downarrow {\scriptstyle \zeta_{\beta\gamma}} \\ \mathcal{G}(U_\gamma) & \xrightarrow{\phi_{\alpha, U_\gamma}} & \mathcal{G}(U_\gamma) \end{array}$$

Diagram 4.5

As a matter of fact, for every $\sigma \in \mathcal{G}(U_\beta)$,

$$(\zeta_{\beta\gamma} \circ \phi_{\alpha,U_\beta})(\sigma) = (g_{\alpha\beta} \cdot \sigma)|_{U_\gamma} = (g_{\alpha\gamma} \cdot g_{\gamma\beta}) \cdot \sigma|_{U_\gamma} =$$
$$g_{\alpha\gamma} \cdot (g_{\gamma\beta} \cdot \sigma|_{U_\gamma}) = \phi_{\alpha,U_\gamma}(g_{\gamma\beta} \cdot \sigma|_{U_\gamma}) = (\phi_{\alpha,U_\gamma} \circ \varrho_{\beta\gamma})(\sigma).$$

Consequently, the family (ϕ_{α,U_β}), for all $U_\beta \subseteq U_\alpha$, is an $(\mathcal{G}(U_\beta))$-equivariant isomorphism of $(\mathcal{G}(U_\beta), \varrho_{\alpha\beta})$ onto $(\mathcal{G}(U_\beta), \zeta_{\alpha\beta})$, generating thus a $\mathcal{G}|_{U_\alpha}$-equivariant sheaf isomorphism

$$\phi_\alpha : \mathcal{P}|_{U_\alpha} \xrightarrow{\simeq} \mathcal{G}|_{U_\alpha}.$$

In this way, we obtain the family of *coordinates* $(\phi_\alpha)_{\alpha\in I}$ of $\mathcal{P}$, over the open covering $\mathcal{U}$. Hence, $\mathcal{P}$ is a principal sheaf with local frame $(\mathcal{U}, (\phi_\alpha))$.

iii) Let us denote by $(\bar{g}_{\alpha\beta})$ the cocycle of $\mathcal{P}$, with respect to the previous local frame. By (4.3.1), $\bar{g}_{\alpha\beta}(x) = (\phi_\alpha \circ \phi_\beta^{-1})(e_x)$, for every $x \in U_{\alpha\beta}$. Since $\mathcal{G}$ is identified with the sheaf of germs of its (continuous) sections, we may write

$$e_x = [\mathbf{1}|_{U_\gamma}]_x \equiv \mathbf{1}|_{U_\gamma}(x)$$

for some $U_\gamma \subseteq U_{\alpha\beta}$ with $x \in U_\gamma$ (which, of course, exists). Thus, (1.2.17), the map (4.5.2) and its inverse imply

$$\begin{aligned}
\bar{g}_{\alpha\beta}(x) &= (\phi_\alpha \circ \phi_\beta^{-1})(e_x) = \phi_\alpha\big(\phi_\beta^{-1}(\mathbf{1}|_{U_\gamma}(x))\big) \\
&= \phi_\alpha\big(\phi_{\beta,U_\gamma}^{-1}(\mathbf{1}|_{U_\gamma}(x))\big) = \phi_\alpha\left(g_{\gamma\beta}(x)\right) \\
&= \phi_\alpha(e_x) \cdot g_{\gamma\beta}(x) = \phi_\alpha(\mathbf{1}|_{U_\gamma}(x)) \cdot g_{\gamma\beta}(x) \\
&= \phi_{\alpha,U_\gamma}(\mathbf{1}|_{U_\gamma}(x)) \cdot g_{\gamma\beta}(x) = g_{\alpha\gamma}(x) \cdot g_{\gamma\beta}(x) \\
&= g_{\alpha\beta}(x);
\end{aligned}$$

that is, $(\bar{g}_{\alpha\beta}) = (g_{\alpha\beta})$.

Finally, assume there also exists a principal sheaf $(\mathcal{P}', \mathcal{G}, X, \pi')$ with the same cocycle $(g_{\alpha\beta})$ over $\mathcal{U}$. Since $(g_{\alpha\beta})$ is trivially cohomologous to itself, the restated Theorem 4.4.2 implies that $\mathcal{P}$ and $\mathcal{P}'$ are $\mathcal{G}$-isomorphic, by which we complete the proof. □

4.5.2 Corollary. *A principal sheaf $(\mathcal{P}, \mathcal{G}, X, \pi)$ with local frame $\mathcal{U}$ is fully determined, up to isomorphism, by its cocycle $(g_{\alpha\beta}) \in Z^1(\mathcal{U}, \mathcal{G})$.*

Let now $\mathcal{V} = (V_i)_{i\in J}$ be an open refinement of $\mathcal{U} = (U_\alpha)_{\alpha\in I}$, and let $\tau : J \to I$ be a refining map $(: \ V_i \subseteq U_{\tau(i)})$. If $(g_{\alpha\beta}) \in Z^1(\mathcal{U}, \mathcal{G})$, we set

$$\bar{g}_{ij} := g_{\tau(i)\tau(j)}|_{V_{ij}}, \quad \text{with} \quad V_{ij} := V_i \cap V_j \quad \text{and} \quad i, j \in J.$$

It is obvious that $(\bar{g}_{ij})$ is a cocycle of $\mathcal{P}$ over $\mathcal{V}$, i.e., $(\bar{g}_{ij}) \in Z^1(\mathcal{V}, \mathcal{G})$, corresponding to the local coordinates

$$\phi_{\tau(i)}|_{\pi^{-1}(V_i)} : \pi^{-1}(V_i) = \mathcal{P}|_{V_i} \longrightarrow \mathcal{G}|_{V_i}, \qquad i \in J.$$

If we further assume that $\mathcal{V}$ is also a *basis* for the topology of X, then we obtain the following useful result.

4.5.3 Corollary. *Let $\mathcal{P} \equiv (\mathcal{P}, \mathcal{G}, X, \pi)$ be a principal sheaf with local frame $(\mathcal{U}, (\phi_\alpha))$ and associated cocycle $(g_{\alpha\beta}) \in Z^1(\mathcal{U}, \mathcal{G})$. For any refinement $\mathcal{V}$ of $\mathcal{U}$ as above, we denote by $\bar{\mathcal{P}} \equiv (\bar{\mathcal{P}}, \mathcal{G}, X, \bar{\pi})$ the principal sheaf constructed from the restricted cocycle $(\bar{g}_{ij}) \in Z^1(\mathcal{V}, \mathcal{G})$. Then $\mathcal{P}$ and $\bar{\mathcal{P}}$ are $\mathcal{G}$-isomorphic.*

Proof. Since $(\bar{g}_{ij})$ is a cocycle of $\mathcal{P}$, then, by Corollary 4.5.2, $\mathcal{P}$ is isomorphic with the principal sheaf constructed from $(\bar{g}_{ij})$. □

4.5.4 Remark. For the sake of completeness, let us find an explicit expression of the isomorphism between $\bar{\mathcal{P}}$ and $\mathcal{P}$, involving the coverings $\mathcal{V}$ and $\mathcal{U}$: For every $i \in J$, we define the isomorphism $f_i := \phi^{-1}_{\tau(i)} \circ \bar{\phi}_i$ shown in Diagram 4.6 below, where $\phi^{-1}_{\tau(i)}$ is in fact the inverse of $\phi_{\tau(i)}|_{\pi^{-1}(V_i)}$ and $\bar{\phi}_i$ the coordinate of $\bar{\mathcal{P}}$ over V. Every f_i is a $\mathcal{G}|_{V_i}$-isomorphism as the composite of such isomorphisms.

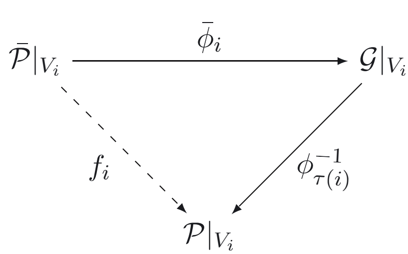

Diagram 4.6

We wish to show that

$$f_i = f_j \quad \text{on} \quad \bar{\mathcal{P}}|_{V_i} \cap \bar{\mathcal{P}}|_{V_j} = \bar{\mathcal{P}}|_{V_{ij}}.$$

For this purpose we take an arbitrary $p \in \bar{\mathcal{P}}|_{V_{ij}}$ with $\bar{\pi}(p) = x$. If $(\bar{s}_i)$ are the natural sections of $\bar{\mathcal{P}}$ with respect to $\mathcal{V}$, then

$$\bar{s}_i(x) \cdot a_i = p = \bar{s}_j(x) \cdot a_j,$$

for unique elements $a_i, a_j \in \mathcal{G}_x$. Since $a_j = \bar{g}_{ji}(x) \cdot a_i$, it follows that

$$\begin{aligned} f_j(p) &= (\phi^{-1}_{\tau(j)} \circ \bar{\phi}_j)(\bar{s}_j(x) \cdot a_j) = \phi^{-1}_{\tau(j)}\big(\bar{\phi}_j(\bar{s}_j(x)) \cdot a_j\big) \\ &= \phi^{-1}_{\tau(j)}(e_x \cdot a_j) = \phi^{-1}_{\tau(j)}(\bar{g}_{ji}(x) \cdot a_i) = \phi^{-1}_{\tau(j)}(\bar{g}_{ji}(x)) \cdot a_i \\ &= \phi^{-1}_{\tau(j)}(g_{\tau(j)\tau(i)}(x)) \cdot a_i = \phi^{-1}_{\tau(j)}\big((\phi_{\tau(j)} \circ \phi^{-1}_{\tau(i)})(e_x)\big) \cdot a_i \\ &= \phi^{-1}_{\tau(i)}(a_i) = \phi^{-1}_{\tau(i)}(\bar{\phi}_i(p)) = f_i(p). \end{aligned}$$

Gluing together all the f_i's we obtain a $\mathcal{G}$-isomorphism as desired.

In particular, the natural sections $(\bar{s}_i)$ of $\bar{\mathcal{P}}$ are related with the natural sections (s_α) of $\mathcal{P}$ by

$$f(\bar{s}_i(x)) = (\phi^{-1}_{\tau(i)} \circ \bar{\phi}_i)(\bar{s}_i(x)) = \phi^{-1}_{\tau(i)}(e_x) = s_{\tau(i)}(x),$$

for every $x \in V_i$. Therefore, the induced morphism of sections yields

$$f(\bar{s}_i) = f_i(\bar{s}_i) = s_{\tau(i)}|_{V_i}, \qquad i \in J.$$

The final result of this section allows us to state an isomorphism criterion, analogous to Theorem 4.4.1, using different local frames of the principal sheaves concerned.

4.5.5 Corollary. *Let $\mathcal{P} \equiv (\mathcal{P}, \mathcal{G}, X, \pi)$ be a principal sheaf with cocycle $(g_{\alpha\beta})$ over a local frame $\mathcal{U} = (U_\alpha), \alpha \in I$, which is a basis for the topology $\mathfrak{T}_X$, and let $\mathcal{Q} \equiv (\mathcal{Q}, \mathcal{G}, X, \pi')$ be another principal sheaf with cocycle $(\gamma_{\alpha'\beta'})$ over a local frame $\mathcal{U}' = (U_{\alpha'}), \alpha' \in I'$, also a basis for $\mathfrak{T}_X$. If $\mathcal{V} = (V_i)$, $i \in J$, is a common refinement of $\mathcal{U}$ and $\mathcal{U}'$, we denote by $\bar{\mathcal{P}}$ and $\bar{\mathcal{Q}}$ the principal sheaves obtained from $\mathcal{P}$ and $\mathcal{Q}$, respectively, by restricting their cocycles to $\mathcal{V}$, as in Corollary 4.5.3. Then the following conditions are equivalent:*

i) $\mathcal{P}$ and $\mathcal{Q}$ are isomorphic.

ii) $\bar{\mathcal{P}}$ and $\bar{\mathcal{Q}}$ are isomorphic.

iii) If $\tau : J \to I$ and $\tau' : J \to I'$ are refining maps for the previous coverings, and $(\bar{g}_{ij}) = (g_{\tau(i)\tau(j)})$, $(\bar{\gamma}_{ij}) = (\gamma_{\tau'(i)\tau'(j)})$ are the cocycles of $\bar{\mathcal{P}}$ and $\bar{\mathcal{Q}}$ respectively, then there exists a 0-cochain $(h_i) \in C^0(\mathcal{V}, \mathcal{G})$ such that

$$\bar{\gamma}_{ij} = h_i \cdot \bar{g}_{ij} \cdot h_j^{-1}, \tag{4.5.3}$$

over V_{ij}, for all indices $i, j \in J$. In other words, $(\bar{\gamma}_{ij})$ and $(\bar{g}_{ij})$ are cohomologous cocycles.

Proof. Direct application of Theorem 4.4.1 and Corollary 4.5.3. □

Taking into account the notations of Corollary 4.5.3, we also write (4.5.3) in the following form

$$\gamma_{\tau'(i)\tau'(j)} = h_i \cdot g_{\tau(i)\tau(j)} \cdot h_j^{-1}. \tag{4.5.4}$$

4.6. Classification of principal sheaves

The unique theorem of this section shows that the equivalence classes of isomorphic $\mathcal{G}$-principal sheaves over X correspond bijectively to the classes of the 1st cohomology set of X with coefficients in $\mathcal{G}$.

4.6.1 Definition. Two principal sheaves $(\mathcal{P}, \mathcal{G}, X, \pi)$ and $(\mathcal{P}', \mathcal{G}, X, \pi')$ are said to be ***equivalent*** if there exists a $\mathcal{G}$-(iso)morphism $f \equiv (f, id_{\mathcal{G}}, id_{\mathcal{L}}, id_X)$ of $\mathcal{P}$ onto $\mathcal{P}'$ (see Definition 4.2.3).

It is clear that $\mathcal{G}$-morphisms induce indeed an equivalence relation. We denote by $[\mathcal{P}]$ the class of $\mathcal{P}$ and by

$$\boldsymbol{P}_{\mathcal{G}}(X) \tag{4.6.1}$$

the set of all equivalence classes obtained in this manner.

From Subsection 1.6.4 (see also the comment following Theorem 4.4.2) we recall that, if $(g_{\alpha\beta}) \in Z^1(\mathcal{U}, \mathcal{G})$, then $[(g_{\alpha\beta})]_{\mathcal{U}} \in H^1(\mathcal{U}, \mathcal{G})$ is the class of all the cocycles (over $\mathcal{U}$), which are cohomologous to $(g_{\alpha\beta})$, and $[(g_{\alpha\beta})] = t_{\mathcal{U}}([(g_{\alpha\beta})]_{\mathcal{U}}) \in H^1(X, \mathcal{G})$, where $t_{\mathcal{U}} : H^1(\mathcal{U}, \mathcal{G}) \to H^1(X, \mathcal{G})$ is the canonical map. We call $[(g_{\alpha\beta})]$ the ***1st cohomology class of*** $\mathcal{P}$.

4.6.2 Theorem (cohomological classification of principal sheaves). *The sets $\boldsymbol{P}_{\mathcal{G}}(X)$ and $H^1(X, \mathcal{G})$ are in bijective correspondence; that is,*

$$\boldsymbol{P}_{\mathcal{G}}(X) \cong H^1(X, \mathcal{G}).$$

Proof. We define a map $\Phi : \boldsymbol{P}_{\mathcal{G}}(X) \to H^1(X, \mathcal{G})$ as follows: Take any class $[\mathcal{P}] \in \boldsymbol{P}_{\mathcal{G}}(X)$. For its representative $\mathcal{P}$ we choose an arbitrary local frame $\mathcal{U} = (U_\alpha)_{\alpha \in I}$ with associated cocycle $(g_{\alpha\beta}) \in Z^1(\mathcal{U}, \mathcal{G})$. Then we let

$$\Phi([\mathcal{P}]) := [(g_{\alpha\beta})].$$

First we show that Φ is *well defined*, i.e., independent of the choice of the representative and its cocycle over $\mathcal{U}$. To this end assume that $\mathcal{Q}$ is any principal sheaf with $[\mathcal{P}] = [\mathcal{Q}]$ and whose cocycle $(\gamma_{\alpha'\beta'}) \in Z^1(\mathcal{U}', \mathcal{G})$ is defined over a local frame $\mathcal{U}' = (U_{\alpha'})_{\alpha' \in I'}$. We take an arbitrary common refinement $\mathcal{V}$ of $\mathcal{U}$ and $\mathcal{U}'$ with $\mathcal{V} = (V_i)_{i \in J}$, and we consider any refining maps $\tau : J \to I$ and $\tau' : J \to I'$. The cocycle $(\bar{g}_{ij}) \in Z^1(\mathcal{V}, \mathcal{G})$, given by

$$\bar{g}_{ij} = g_{\tau(i)\tau(j)}|_{V_{ij}}, \tag{4.6.2}$$

is a cocycle of $\mathcal{P}$. Similarly,

$$\bar{\gamma}_{ij} = \gamma_{\tau'(i)\tau'(j)}|_{V_{ij}} \tag{4.6.3}$$

is a cocycle of $\mathcal{Q}$. Since, by assumption, $\mathcal{P} \cong \mathcal{Q}$, Corollary 4.5.5 implies that

$$[(\bar{g}_{ij})]_{\mathcal{V}} = [(\bar{\gamma}_{ij})]_{\mathcal{V}}. \tag{4.6.4}$$

On the other hand, specializing Diagram 1.11 to the present case, we obtain the commutative diagram

$$\begin{array}{ccc} H^1(\mathcal{U}, \mathcal{G}) & \xrightarrow{\;t^{\mathcal{U}}_{\mathcal{V}}\;} & H^1(\mathcal{V}, \mathcal{G}) \\ & {\scriptstyle t_{\mathcal{U}}} \searrow \quad \swarrow {\scriptstyle t_{\mathcal{V}}} & \\ & H^1(X, \mathcal{G}) & \end{array}$$

Diagram 4.7

and its analog for $\mathcal{U}'$. Therefore, taking into account (1.6.43), (1.6.40), and (4.6.2) – (4.6.4), we see that

$$\begin{aligned} [(g_{\alpha\beta})] &= t_{\mathcal{U}}\left([(g_{\alpha\beta})]_{\mathcal{U}}\right) = (t_{\mathcal{V}} \circ t^{\mathcal{U}}_{\mathcal{V}})\left([(g_{\alpha\beta})]_{\mathcal{U}}\right) \\ &= t_{\mathcal{V}}\left([(g_{\tau(i)\tau(j)}|_{V_{ij}})]_{\mathcal{V}}\right) = t_{\mathcal{V}}\left([(\bar{g}_{ij})]_{\mathcal{V}}\right) \\ &= t_{\mathcal{V}}\left([(\bar{\gamma}_{ij})]_{\mathcal{V}}\right) = t_{\mathcal{V}}\left([(\gamma_{\tau'(i)\tau'(j)}|_{V_{ij}})]_{\mathcal{V}}\right) \\ &= (t_{\mathcal{V}} \circ t^{\mathcal{U}'}_{\mathcal{V}})\left([(\gamma_{\alpha'\beta'})]_{\mathcal{V}}\right) = t_{\mathcal{U}'}\left([(\gamma_{\alpha'\beta'})]_{\mathcal{U}'}\right) \\ &= [(\gamma_{\alpha'\beta'})], \end{aligned}$$

which proves that Φ is well defined.

It is worth noticing here that, since all the cocycles used above are taken over local frames, the inductive limit (1.6.42) has to be taken with respect to all proper local frames $\mathcal{U}$ of X. This is possible because the latter constitute a cofinal subset of the set of all proper (open) coverings of X, as already commented in (4.1.5).

We now show that Φ is *injective*. Indeed, assume that $\Phi([\mathcal{P}]) = \Phi([\mathcal{Q}])$, for arbitrary $[\mathcal{P}]$, $[\mathcal{Q}] \in \boldsymbol{P}_{\mathcal{G}}(X)$. Let $(g_{\alpha\beta})$ and $(\gamma_{\alpha'\beta'})$ be any cocycles of the representatives $\mathcal{P}$ and $\mathcal{Q}$, respectively. The assumption means that $[(g_{\alpha\beta})] = [(\gamma_{\alpha'\beta'})]$; hence, arguing as above,

$$t_{\mathcal{V}}\left([(g_{\tau(i)\tau(j)}|_{V_{ij}})]_{\mathcal{V}}\right) = t_{\mathcal{V}}\left([(\gamma_{\tau'(i)\tau'(j)}|_{V_{ij}})]_{\mathcal{V}}\right),$$

or, equivalently,

$$t_{\mathcal{V}}\left([(\bar{g}_{ij})]_{\mathcal{V}}\right) = t_{\mathcal{V}}\left([(\bar{\gamma}_{ij})]_{\mathcal{V}}\right) \in H^1(X, \mathcal{G}).$$

Therefore, by the injectivity of $t_{\mathcal{V}}$ (see (1.6.44)), we have that $[(\bar{g}_{ij})]_{\mathcal{V}} = [(\bar{\gamma}_{ij})]_{\mathcal{V}}$, and (by Corollary 4.5.5) $\mathcal{P} \cong \mathcal{Q}$; that is, $[\mathcal{P}] = [\mathcal{Q}]$. This proves the injectivity of Φ.

Finally, let $[(g_{\alpha\beta})] \in H^1(X, \mathcal{G})$ be an arbitrarily chosen cohomology class. Let $(g_{\alpha\beta})$ be a representative cocycle, defined over some open covering $\mathcal{U}$ of X. If $\mathcal{U}$ is a basis for the topology of X, Theorem 4.5.1 ensures the existence of a principal sheaf $\mathcal{P}$ with the given cocycle. Clearly, $\Phi([\mathcal{P}]) = [(g_{\alpha\beta})]$. If $\mathcal{U}$ is not a basis, we can take an open refinement $\mathcal{V}$ of $\mathcal{U}$ with this property. We consider the cocycle $(\bar{g}_{ij})$, restriction of $(g_{\alpha\beta})$ to $\mathcal{V}$. Then

$$\begin{gathered}\Phi([\mathcal{P}]) = [(\bar{g}_{ij})] = t_{\mathcal{V}}\left([(g_{\tau(i)\tau(j)}|_{V_{ij}})]_{\mathcal{V}}\right) = \\ (t_{\mathcal{V}} \circ t^{\mathcal{U}}_{\mathcal{V}})\left([(g_{\alpha\beta})]_{\mathcal{U}}\right) = t_{\mathcal{U}}\left([(g_{\alpha\beta})]_{\mathcal{U}}\right) = [(g_{\alpha\beta})].\end{gathered}$$

The previous arguments show that Φ is a surjective map and conclude the proof. □

Note. It is clear that Remark 4.4.4 also applies in this section.

4.7. Reduction of the structure sheaf

The section is dealing with the notion of reduction in the general context of sheaves of groups. Although the results are valid also for Lie sheaves of groups, we restrict ourselves to sheaves and subsheaves of groups in order to prepare the discussion of Section 10.4 about vector sheaves endowed with Riemannian metrics.

Applying the notations of Remark 4.4.4, we first give the following:

4.7.1 Definition. Let $\mathcal{Q} \equiv (\mathcal{Q}, \mathcal{H}, X, \pi')$ and $\mathcal{P} \equiv (\mathcal{P}, \mathcal{G}, X, \pi)$ be two principal sheaves. Given a morphism of sheaves of groups $\phi : \mathcal{H} \to \mathcal{G}$, we say that $\mathcal{P}$ ***reduces to*** $\mathcal{Q}$, ***relative to*** ϕ, if there is a morphism (f, ϕ, id_X) of $\mathcal{Q}$ into $\mathcal{P}$. In this context, we say that $\mathcal{G}$ ***reduces to*** $\mathcal{H}$***, relative to*** ϕ.

The next proposition gives a useful characterization of a reduction.

4.7.2 Proposition. *A principal sheaf* $\mathcal{P} \equiv (\mathcal{P}, \mathcal{G}, X, \pi)$ *reduces to* $\mathcal{Q} \equiv (\mathcal{Q}, \mathcal{H}, X, \pi')$, *relative to a morphism of sheaves of groups* $\phi : \mathcal{H} \to \mathcal{G}$, *if and only if there is a cocycle* $(g_{\alpha\beta}) \in Z^1(\mathcal{U}, \mathcal{G})$ *of* $\mathcal{P}$ *such that* $g_{\alpha\beta} = \phi(h_{\alpha\beta})$, *for a cocycle* $(h_{\alpha\beta}) \in Z^1(\mathcal{U}, \mathcal{H})$.

Proof. The result is essentially a restatement of Theorem 4.4.1, after a rearrangement of the local frame of $\mathcal{P}$ and the necessary modifications in the notations. Hence, the proof is a variation of that of the aforementioned theorem, applied to the simpler case of principal sheaves with structure sheaf a sheaf of groups.

More precisely, assume that (f, ϕ, id_X) is the morphism of $\mathcal{P}$ into $\mathcal{Q}$, realizing the reduction. Let $(\mathcal{U}, (\psi_\alpha))$ and $(\mathcal{U}, (\chi_\alpha))$ be the local frames of $\mathcal{Q}$ and $\mathcal{P}$, respectively, over the same open covering $\mathcal{U}$ of X, with corresponding natural sections (σ_α) and (τ_α). We denote by $(h_{\alpha\beta})$ and $(\chi_{\alpha\beta})$ the respective cocycles of $\mathcal{Q}$ and $\mathcal{P}$.

Applying Theorem 4.4.1 to the morphism (f, ϕ, id_X), we have that

$$f(\sigma_\alpha) = \tau_\alpha \cdot h_\alpha, \tag{4.7.1}$$

$$\chi_{\alpha\beta} = h_\alpha \cdot \phi(h_{\alpha\beta}) \cdot h_\beta^{-1}, \tag{4.7.2}$$

for an appropriate cochain $(h_\alpha) \in C^0(\mathcal{U}, \mathcal{G})$.

Setting $s_\alpha := \tau_\alpha \cdot h_\alpha$, $\alpha \in I$, we obtain a family of local sections of $\mathcal{P}$ over $\mathcal{U}$, which –by Proposition 4.1.7– determines a new local frame $(\mathcal{U}, (\phi_\alpha))$ of $\mathcal{P}$, whose cocycle $(g_{\alpha\beta})$ is related with $(\chi_{\alpha\beta})$ by $g_{\alpha\beta} = h_\alpha^{-1} \cdot \chi_{\alpha\beta} \cdot h_\beta$. Hence, (4.7.1) and (4.7.2) transform into

$$f(\sigma_\alpha) = s_\alpha, \tag{4.7.3}$$

$$g_{\alpha\beta} = \phi(h_{\alpha\beta}), \tag{4.7.4}$$

the second of which proves the direct part of the statement.

Conversely, suppose that $\mathcal{P}$ is a principal sheaf whose cocycle $(g_{\alpha\beta})$ satisfies (4.7.4), for some cocycle $(h_{\alpha\beta}) \in Z^1(\mathcal{U}, \mathcal{H})$. By the general Theorem 4.5.1, $(h_{\alpha\beta})$ determines a principal sheaf, say $\mathcal{Q} \equiv (\mathcal{Q}, \mathcal{H}, X, \pi')$, with cocycle

$(h_{\alpha\beta})$. Moreover, equality (4.7.4) and the converse part of Theorem 4.4.1 (for $h_\alpha = \mathbf{1}|_{U_\alpha}$) result in the existence of a principal sheaf morphism (f, ϕ, id_X) mapping $\mathcal{Q}$ into $\mathcal{P}$, as desired. □

To facilitate the proof of the next result, we recall that the morphism f, mentioned in the converse part of the preceding proof, is constructed by gluing together the local morphisms $f_\alpha : \mathcal{Q}|_{U_\alpha} \to \mathcal{P}|_{U_\alpha}$, given by

$$f_\alpha(q) := s_\alpha(x) \cdot \phi(\eta_\alpha(x)), \tag{4.7.5}$$

for every $q \in \mathcal{Q}|_{U_\alpha}$, with $\pi'(q) = x \in U_\alpha$, and $\eta_\alpha \in \mathcal{H}(U_\alpha)$ determined by $q = \sigma_\alpha(x) \cdot \eta_\alpha(x)$, if (σ_α) are the natural sections of $\mathcal{Q}$ over $\mathcal{U}$.

4.7.3 Corollary. *Within the framework of Proposition 4.7.2, if $\phi : \mathcal{H} \to \mathcal{G}$ is an injective morphism, then so is f.*

Proof. Let q, r be two arbitrary elements of $\mathcal{Q}$ with $f(q) = f(r)$. Clearly $\pi'(q) = \pi'(r) =: x \in U_\alpha$, for some $\alpha \in I$. Then, in virtue of (4.7.5),

$$f(q) = s_\alpha(x) \cdot \phi(\eta_\alpha(x)) \quad \text{and} \quad f(r) = s_\alpha(x) \cdot \phi(\zeta_\alpha(x)),$$

with $\eta_\alpha(x)$ and $\zeta_\alpha(x)$ determined, respectively, by

$$q = s_\alpha(x) \cdot \eta_\alpha(x) \quad \text{and} \quad r = s_\alpha(x) \cdot \zeta_\alpha(x). \tag{4.7.6}$$

The injectivity of ϕ yields $\eta_\alpha(x) = \zeta_\alpha(x)$ and, by (4.7.6), $q = r$. □

Let us now consider the particular case of a *subsheaf of groups* $\mathcal{H} \subseteq \mathcal{G}$ and take ϕ to be the natural inclusion morphism $i : \mathcal{H} \hookrightarrow \mathcal{G}$. Then a reduction of $\mathcal{P} \equiv (\mathcal{P}, \mathcal{G}, X, \pi)$ to some $\mathcal{Q} \equiv (\mathcal{Q}, \mathcal{H}, X, \pi')$ amounts to the existence of a cocycle $(g_{\alpha\beta}) \in Z^1(\mathcal{U}, \mathcal{G})$ such that $(g_{\alpha\beta}) \in Z^1(\mathcal{U}, \mathcal{H})$. Thus we have the following natural definition:

4.7.4 Definition. The structure sheaf $\mathcal{G}$ of a principal sheaf $\mathcal{P}$ ***reduces to a subsheaf of groups*** $\mathcal{H}$ if $\mathcal{G}$ reduces to $\mathcal{H}$ relative to the natural inclusion morphism $i : \mathcal{H} \hookrightarrow \mathcal{G}$.

The reduction of $\mathcal{G}$ to the subsheaf of groups $\mathcal{H}$ means that there is a principal sheaf $\mathcal{Q} \equiv (\mathcal{Q}, \mathcal{H}, X, \pi')$ and a morphism (f, i, id_X) of $\mathcal{Q}$ into $\mathcal{P}$. As a consequence of Corollary 4.7.3, $f : \mathcal{Q} \to \mathcal{P}$ is 1-1, thus $\mathcal{Q}$ can be identified with $f(\mathcal{Q})$. Since f is a morphism of sheaves, $\mathcal{Q} \equiv f(\mathcal{Q})$ may be considered as a subsheaf of $\mathcal{P}$; hence, $(\mathcal{Q}, \mathcal{H}, X, \pi') \cong (f(\mathcal{Q}), \mathcal{H}, X, \pi')$ can be thought of as a ***principal subsheaf of*** $(\mathcal{P}, \mathcal{G}, X, \pi)$; that is, $(\mathcal{Q}, \mathcal{H}, X, \pi')$ is a principal sheaf such that its total/sheaf space $\mathcal{Q}$ is a subsheaf of $\mathcal{P}$ and its structure sheaf $\mathcal{H}$ is a subsheaf of groups of $\mathcal{G}$. As a consequence, one infers the next result.

4.7.5 Corollary. *The reduction of the structure sheaf $\mathcal{G}$ of a principal sheaf $\mathcal{P} \equiv (\mathcal{P}, \mathcal{G}, X, \pi)$ to a subsheaf of groups $\mathcal{H}$ of $\mathcal{G}$ is equivalent to the reduction of $\mathcal{P}$ to a principal subsheaf $\mathcal{Q} \equiv (\mathcal{Q}, \mathcal{H}, X, \pi')$, relative to $i : \mathcal{H} \hookrightarrow \mathcal{G}$.*

Chapter 5

Vector and associated sheaves

More generally, we will find that many locally defined $\mathcal{V}$-valued objects of interest (e.g., wavefunctions) become globally defined when thought of as taking their values in some associated vector bundle.

G. Naber [81, p. 49]

Here we deal with sheaves associated with a principal sheaf $\mathcal{P}$, in particular those arising from representations of the structure sheaf of $\mathcal{P}$. Such representations often lead to structures simpler than the original principal sheaf.

We start with vector sheaves, whose structure is first described independently of the general theory of associated sheaves. In the sequel we show that a vector sheaf is associated with its *principal sheaf of frames*. The latter is an important example of an abstract principal sheaf, already mentioned in Remark 4.1.10. The sheaf of frames of a vector sheaf serves as the link between the two major geometrical categories of sheaves we are interested in, namely, vector and principal sheaves, in complete analogy to ordinary vector and principal fiber bundles. In Chapter 7, we show that the study of connections on vector sheaves is reduced to the study of connections on principal sheaves.

Other types of associated sheaves are also studied in detail.

5.1. Vector sheaves

In this section we present the fundamental notions and properties of the theory of vector sheaves, which will be encountered in this work. The complete study of them, as well as their geometry, is the main content of Mallios [62, Vol. II], where the reader is referred to for topics not treated here.

We start with a fixed algebraized space $(X, \mathcal{A})$. Later on, in the study of connections on vector and associated sheaves (see Chapter 7), $\mathcal{A}$ will be completed to a differential triad.

We recall that an $\mathcal{A}$-module is a sheaf of abelian groups $(\mathcal{E}, \pi, X)$ whose stalks $\mathcal{E}_x$ are $\mathcal{A}_x$-modules and the "scalar" multiplication $\mathcal{A} \times_X \mathcal{E} \to \mathcal{E}$ is continuous (see Subsection 1.1.2).

5.1.1 Definition. An $\mathcal{A}$-module $\mathcal{E} \equiv (\mathcal{E}, \pi, X)$ is called a ***vector sheaf of (finite) rank n*** if, for every $x \in X$, there exists an open neighborhood U of x and an $\mathcal{A}|_U$-isomorphism

$$\psi_U : \mathcal{E}|_U \xrightarrow{\simeq} \mathcal{A}^n|_U,$$

between the $\mathcal{A}|_U$-modules figuring as the domain and range of ψ_U. Hence, $\mathcal{E}$ is a ***locally free $\mathcal{A}$-module***.

Equivalently, we can find an open covering $\mathcal{U} = (U_\alpha)$, $\alpha \in I$, of X and a family of $\mathcal{A}|_{U_\alpha}$-isomorphisms

$$\psi_\alpha : \mathcal{E}|_{U_\alpha} \xrightarrow{\simeq} \mathcal{A}^n|_{U_\alpha}, \qquad \alpha \in I. \tag{5.1.1}$$

From Subsection 1.3.2 we also recall that $\mathcal{A}^n$ is the $\mathcal{A}$-module generated by the complete presheaf of $\mathcal{A}(U)$-modules

$$U \longmapsto \mathcal{A}(U)^n,$$

where U is running the topology $\mathfrak{T}_X$ of X. Equivalently,

$$\mathcal{A}^n = \underbrace{\mathcal{A} \times_X \mathcal{A} \times_X \cdots \times_X \mathcal{A}}_{n-\text{factors}} \cong \bigoplus^n \mathcal{A}.$$

Therefore, $\mathcal{A}^n$ is a (globally) free module such that

$$\mathcal{A}^n(U) \cong \mathcal{A}(U)^n, \qquad U \in \mathfrak{T}_X. \tag{5.1.2}$$

As in the case of principal sheaves, we use the following terminology (see also [62, Vol. I, p. 126]): (ψ_α) are the ***coordinates*** of $\mathcal{E}$ over $\mathcal{U}$, and $\mathcal{U}$ is a ***local frame*** or a ***coordinatizing covering***. The open sets of $\mathcal{U}$ are also called ***local gauges***. We often write $\mathcal{U} \equiv (\mathcal{U}, (\psi_\alpha)) \equiv ((U_\alpha), (\psi_\alpha))$ if we want to specify all the previous elements involved in the local structure of the vector sheaf.

Standard examples of vector sheaves comprise the sheaf of germs of smooth sections of a finite-dimensional smooth vector bundle and the pull-back of a vector sheaf (compare with Examples 4.1.9(a) and (c)).

Also, the sheaf Ω^1_X of germs of differential 1-forms on a smooth n-dimensional manifold X (defined in Example 2.1.4(a)) is a vector sheaf. Over any chart U_α of X, we have that $\Omega^1_X|_{U_\alpha} \cong \mathcal{A}^n|_{U_\alpha}$, where $\mathcal{A} = \mathcal{C}^\infty_X$ is the sheaf of germs of smooth functions on X (defined in the same example).

Analogously to Example 4.1.9(b), a projective system of Banach vector bundles in the sense of Galanis [32] induces a $\mathcal{C}^\infty_X$-module, where X is the common base of the bundles of the system. Note that the projective limit of such bundles is an (infinite-dimensional) Fréchet space, hence the sheaf of its sections cannot be a vector sheaf.

Examples of $\mathcal{A}$-modules and vector sheaves, associated with principal sheaves, will be given in Section 5.4.

The local frame $(\mathcal{U}, (\psi_\alpha))$ determines, for each $\alpha \in I$, a family of ***natural sections*** $e^\alpha_i \in \mathcal{E}(U_\alpha)$, $i = 1, \ldots, n = \mathrm{rank}(\mathcal{E})$, given by

$$e^\alpha_i(x) := \psi^{-1}_\alpha(0_x, \ldots, 1_x, \ldots, 0_x); \qquad x \in U_\alpha, \tag{5.1.3}$$

where 1_x (in the i-th entry) is the unit of the algebra $\mathcal{A}_x$. Moreover, if

$$\epsilon_i : X \ni x \longmapsto (0_x, \ldots, 1_x, \ldots, 0_x) \in \mathcal{A}^n$$

denotes the i-th natural global section of $\mathcal{A}^n$, then (5.1.3) takes the equivalent form

$$e^\alpha_i = \psi^{-1}_\alpha \circ \epsilon_i|_{U_\alpha} \equiv \psi^{-1}_\alpha(\epsilon_i|_{U_\alpha}). \tag{5.1.3$'$}$$

Given a vector sheaf, an immediate consequence of the definitions is the following:

5.1.2 Proposition. *For each $\alpha \in I$, the sections $\{e_i^\alpha \,|\, 1 \leq i \leq n\}$ form a* ***basis*** *of the $\mathcal{A}(U_\alpha)$-module $\mathcal{E}(U_\alpha)$. Consequently, $\{e_i^\alpha(x) \,|\, 1 \leq i \leq n\}$ is a basis of $\mathcal{E}_x$, for every $x \in U_\alpha$.*

Conversely, we have:

5.1.3 Proposition. *Let $(\mathcal{E}, \pi, X)$ be an $\mathcal{A}$-module. Assume that $\mathcal{U} = (U_\alpha)$, $\alpha \in I$, is an open covering of X and*

$$e^\alpha := \{e_i^\alpha \,|\, 1 \leq i \leq n\} \subseteq \mathcal{E}(U_\alpha)^n$$

a family of sections such that, for each $\alpha \in I$, $\{e_i^\alpha(x) \,|\, 1 \leq i \leq n\}$ is a basis of $\mathcal{E}_x$, for every $x \in U_\alpha$. Then $\mathcal{E}$ is a vector sheaf with local frame $\mathcal{U}$, whose corresponding natural sections coincide with (e^α).

Proof. For a fixed $\alpha \in I$ and every open $V \subseteq U_\alpha$, we define the $\mathcal{A}(V)$-isomorphisms

$$\psi_{\alpha,V} : \mathcal{E}(V) \longrightarrow \mathcal{A}(V)^n : s \mapsto (s_1, \ldots, s_n),$$

where the sections $s_i \in \mathcal{A}(V)$ are determined by

$$s(x) = \sum_{i=1}^{n} s_i(x) \cdot e_i^\alpha(x), \qquad x \in V.$$

Varying V in U_α, we obtain a presheaf isomorphism $(\psi_{\alpha,V})$ generating an $\mathcal{A}|_{U_\alpha}$-isomorphism $\psi_\alpha : \mathcal{E}|_{U_\alpha} \to \mathcal{A}^n|_{U_\alpha}$.

Let now $\bar{e} = \{\bar{e}_1, \ldots, \bar{e}_1\}$ be the natural basis of $\mathcal{E}(U_\alpha)$ induced by ψ_α. Then

$$\bar{e}_1(x) = \sum_{j=1}^{n} \lambda_{ij}(x) e_j^\alpha(x).$$

Applying ψ_α to the preceding equality, it follows that $\lambda_{ij}(x) = \delta_{ij}(x)$, for every $x \in U_\alpha$. Therefore, $\bar{e} = e^\alpha$. □

Let $(\mathcal{U}, (\psi_\alpha))$ be a local frame of $\mathcal{E}$. For any $\alpha, \beta \in I$ with $U_{\alpha\beta} = U_\alpha \cap U_\beta \neq \emptyset$, the ***coordinate transformation***

$$\psi_{\alpha\beta} := \psi_\alpha \circ \psi_\beta^{-1} : \mathcal{A}^n|_{U_{\alpha\beta}} \longrightarrow \mathcal{A}^n|_{U_{\alpha\beta}} \tag{5.1.4}$$

is an $\mathcal{A}|_{U_{\alpha\beta}}$-automorphism of $\mathcal{A}^n|_{U_{\alpha\beta}}$, i.e.,

$$\psi_{\alpha\beta} \in \mathrm{Aut}_{\mathcal{A}|_{U_{\alpha\beta}}}(\mathcal{A}^n|_{U_{\alpha\beta}}).$$

Since (using the natural sections –and working as in the case of ordinary vector spaces),

$$\mathrm{Aut}_{\mathcal{A}|_{U_{\alpha\beta}}}(\mathcal{A}^n|_{U_{\alpha\beta}}) \cong \mathrm{GL}(n, \mathcal{A}(U_{\alpha\beta})), \tag{5.1.5}$$

$\psi_{\alpha\beta}$ corresponds bijectively to a matrix, called hereafter the ***transition matrix of*** $\mathcal{E}$***, with respect to*** $U_{\alpha\beta}$, thus

$$\psi_{\alpha\beta} \equiv \left(g_{ij}^{\alpha\beta}\right) \in \mathrm{GL}(n, \mathcal{A}(U_{\alpha\beta})). \tag{5.1.6}$$

The entries of the matrix are determined by

$$e_i^{\beta} = \sum_{j=1}^{n} g_{ji}^{\alpha\beta} \cdot e_j^{\alpha}; \qquad i = 1, \ldots, n, \tag{5.1.6$'$}$$

as shown by elementary calculations.

Applying (3.2.7), we may think of the coordinate transformations as sections of the general linear group sheaf; that is,

$$\psi_{\alpha\beta} \in \mathcal{GL}(n, \mathcal{A})(U_{\alpha\beta}).$$

It is a matter of routine checking to verify the ***cocycle condition***

$$\psi_{\alpha\gamma} = \psi_{\alpha\beta} \circ \psi_{\beta\gamma},$$

over every $U_{\alpha\beta\gamma} \neq \emptyset$. This actually proves:

5.1.4 Proposition. *For a given local frame* $(\mathcal{U}, (\psi_\alpha))$ *of* $\mathcal{E}$*, the family of coordinate transformations* $(\psi_{\alpha\beta})$ *identifies with a 1-cocycle (over* $\mathcal{U}$*) with coefficients in the general linear group sheaf of order* n*; that is,*

$$(\psi_{\alpha\beta}) \in Z^1(\mathcal{U}, \mathcal{GL}(n, \mathcal{A})), \tag{5.1.7}$$

if n *is the rank of* $\mathcal{E}$*.*

As in the case of principal sheaves (see Section 4.6, as well as Subsection 1.6.4) $(\psi_{\alpha\beta})$ determines the (1-dimensional) cohomology class of $\mathcal{U}$

$$[(\psi_{\alpha\beta})]_{\mathcal{U}} \in H^1(\mathcal{U}, \mathcal{GL}(n, \mathcal{A})),$$

and the 1st cohomology class of X with coefficients in $\mathcal{GL}(n,\mathcal{A})$

$$[(\psi_{\alpha\beta})] \in H^1(X, \mathcal{GL}(n,\mathcal{A})).$$

We refer to the latter as the ***1st cohomology class of*** $\mathcal{E}$.

To obtain a cohomological classification of vector sheaves we first need to elaborate on the notion of a morphism of vector sheaves. We repeat formally the relative definition given at the end of Subsection 1.1.2.

5.1.5 Definition. Let $\mathcal{E} \equiv (\mathcal{E}, \pi, X)$ and $\mathcal{E}' \equiv (\mathcal{E}', \pi', X)$ be two vector sheaves over the same base X. An $\mathcal{A}$***-morphism*** $\phi : \mathcal{E} \to \mathcal{E}'$ is a morphism of sheaves of sets such that, for every $x \in X$, the restriction

$$\phi_x := \phi|_{\mathcal{E}_x} : \mathcal{E}_x \longrightarrow \mathcal{E}'_x$$

is an $\mathcal{A}_x$-morphism, i.e., a morphism of $\mathcal{A}_x$-modules. An $\mathcal{A}$-morphism with an inverse is called an $\mathcal{A}$***-isomorphism***.

We now prove the vector sheaf analog of Theorem 4.4.1.

5.1.6 Theorem. *Let $\mathcal{E} \equiv (\mathcal{E}, \pi, X)$ and $\mathcal{E}' \equiv (\mathcal{E}', \pi', X)$ be two vector sheaves of corresponding ranks m and n. If $(\psi_{\alpha\beta})$ and $(\psi'_{\alpha\beta})$ are their respective cocycles over the same open covering $\mathcal{U} = (U_\alpha)_{\alpha\in I}$, then an $\mathcal{A}$-morphism f of $\mathcal{E}$ into $\mathcal{E}'$ determines a unique family of $\mathcal{A}|_{U_\alpha}$-morphisms*

$$h_\alpha : \mathcal{A}^m|_{U_\alpha} \longrightarrow \mathcal{A}^n|_{U_\alpha}; \qquad \alpha \in I,$$

such that the equalities

$$f = (\psi'_\alpha)^{-1} \circ h_\alpha \circ \psi_\alpha, \tag{5.1.8}$$

$$\psi'_{\alpha\beta} \circ h_\beta = h_\alpha \circ \psi_{\alpha\beta} \tag{5.1.9}$$

hold over $\mathcal{E}|_{U_\alpha}$ and $U_{\alpha\beta}$, respectively.

Conversely, a family of $\mathcal{A}|_{U_\alpha}$-morphisms satisfying (5.1.9) determines a unique $\mathcal{A}$-morphism f also verifying (5.1.8).

Proof. Let f be an $\mathcal{A}$-morphism of $\mathcal{E}$ into $\mathcal{E}'$. Restricting f to $\mathcal{E}|_{U_\alpha}$ and setting

$$h_\alpha := \psi'_\alpha \circ f \circ \psi_\alpha^{-1}; \qquad \alpha \in I,$$

we obtain a family of $\mathcal{A}|_{U_\alpha}$-morphisms satisfying (5.1.8). Furthermore, applying (5.1.4) to the equality

$$(\psi'_\alpha)^{-1} \circ h_\alpha \circ \psi_\alpha = (\psi'_\beta)^{-1} \circ h_\beta \circ \psi_\beta \qquad (\text{over } U_{\alpha\beta}),$$

we get (5.1.9). The uniqueness of (h_α) follows immediately from (5.1.8).

Conversely, assume the existence of a family (h_α) as in the second part of the statement. For every $\alpha \in I$, we define the $\mathcal{A}|_{U_\alpha}$-morphism

$$f_\alpha := (\psi'_\alpha)^{-1} \circ h_\alpha \circ \psi_\alpha \;:\; \mathcal{E}|_{U_\alpha} \longrightarrow \mathcal{E}'|_{U_\alpha}.$$

Since, over $\mathcal{E}|_{U_{\alpha\beta}}$,

$$\begin{aligned} f_\beta &= (\psi'_\beta)^{-1} \circ h_\beta \circ \psi_\beta = (\psi'_\alpha)^{-1} \circ \psi'_{\alpha\beta} \circ h_\beta \circ \psi_\beta \\ &= (\psi'_\alpha)^{-1} \circ h_\alpha \circ \psi_{\alpha\beta} \circ \psi_\beta = (\psi'_\alpha)^{-1} \circ h_\alpha \circ \psi_\alpha \\ &= f_\alpha, \end{aligned}$$

the f_α's can be glued together to yield an $\mathcal{A}$-morphism f. Equality (5.1.8) is merely the definition of f_α, for every $\alpha \in I$. The uniqueness of f follows again from (5.1.8). □

Analogously to (5.1.5), we have the identification

(5.1.10) $$\mathrm{Hom}_{\mathcal{A}|_{U_\alpha}}(\mathcal{A}^m|_{U_\alpha}, \mathcal{A}^n|_{U_\alpha}) \cong M_{m\times n}(\mathcal{A}(U_\alpha)),$$

thus each h_α can be identified with an $m \times n$ matrix (h^α_{ij}). Equality (5.1.8) shows that the entries of this matrix satisfy

(5.1.11) $$f(e^\alpha_i(x)) = \sum_{j=1}^{n} h^\alpha_{ji}(x)\,`e^\alpha_j(x); \qquad i = 1, \ldots, m,$$

for every $x \in U_\alpha$. Here $`e^\alpha_i$ $(i = 1, \ldots, n)$ are the natural sections of $\mathcal{E}'$ over $U_\alpha$. The back prime ( `) has been used for obvious typographical reasons. Condition (5.1.9) now takes the matrix form

(5.1.9′) $$\left(`g^{\alpha\beta}_{ij}\right) \cdot \left(h^\beta_{ij}\right) = \left(h^\alpha_{ij}\right) \cdot \left(g^{\alpha\beta}_{ij}\right),$$

where $(`g^{\alpha\beta}_{ij})$ is the transition matrix corresponding to the coordinate transformation $\psi'_{\alpha\beta}$.

In the case of isomorphisms, Theorem 5.1.6 takes the following form:

5.1.7 Theorem. *Let $\mathcal{E} \equiv (\mathcal{E}, \pi, X)$ and $\mathcal{E}' \equiv (\mathcal{E}', \pi', X)$ be two vector sheaves of rank n with cocycles $(\psi_{\alpha\beta})$ and $(\psi'_{\alpha\beta})$, respectively, over the same open covering $\mathcal{U}$ of X. Then an $\mathcal{A}$-isomorphism f of $\mathcal{E}$ onto $\mathcal{E}'$ determines a unique family of $\mathcal{A}|_{U_\alpha}$-automorphisms of $\mathcal{A}^n|_{U_\alpha}$*

$$h_\alpha \in \mathrm{Aut}_{\mathcal{A}|_{U_\alpha}}(\mathcal{A}^n|_{U_\alpha}); \qquad \alpha \in I,$$

satisfying the equalities

(5.1.12) $$f = (\psi'_\alpha)^{-1} \circ h_\alpha \circ \psi_\alpha; \qquad \textit{on} \quad \mathcal{E}|_{U_\alpha},$$

(5.1.13) $$\psi'_{\alpha\beta} = h_\alpha \circ \psi_{\alpha\beta} \circ h_\beta^{-1}; \qquad \textit{on} \quad U_{\alpha\beta}.$$

Conversely, a family (h_α) *satisfying (5.1.13) determines a unique* $\mathcal{A}$*-isomorphism* f *verifying (5.1.12).*

For the sake of completeness, we remark that, in analogy to Theorem 4.4.2, (h_α) can be interpreted as a 0-cochain with coefficients in $\mathcal{GL}(n, \mathcal{A})$; that is,

$$(h_\alpha) \in C^0(\mathcal{U}, \mathcal{GL}(n, \mathcal{A})),$$

after the identifications (cf. (3.2.7) and (5.1.5))

(5.1.14) $$\mathrm{Aut}_{\mathcal{A}|_{U_\alpha}}(\mathcal{A}^n|_{U_\alpha}) \cong \mathrm{GL}(n, \mathcal{A}(U_\alpha)) \cong \mathcal{GL}(n, \mathcal{A})(U_\alpha).$$

Identifying h_α with an invertible matrix $(h^\alpha_{ij}) \in \mathrm{GL}(n, \mathcal{A}(U_\alpha))$, we rewrite (5.1.13) in the matrix form

(5.1.13′) $$\left({}^{\backprime}g^{\alpha\beta}_{ij}\right) = \left(h^\alpha_{ij}\right) \cdot \left(g^{\alpha\beta}_{ij}\right) \cdot \left(h^\beta_{ij}\right)^{-1}.$$

Finally, let us observe that if we deal with isomorphic vector sheaves with 1-cocycles over *different* open coverings of the base, we can prove the vector sheaf analogs of Corollaries 4.5.3 and 4.5.5.

Following the notation of Mallios [62, p. 128], we denote by

(5.1.15) $$\boldsymbol{\Phi}^n_{\mathcal{A}}(X)$$

the *set of equivalence classes of* $\mathcal{A}$*-isomorphic vector sheaves of rank* n *over* X. Hence, we are in a position to prove the following cohomological classification theorem, which is the vector sheaf analog of Theorem 4.6.2.

5.1.8 Theorem. *The sets* $\boldsymbol{\Phi}^n_{\mathcal{A}}(X)$ *and* $H^1(X, \mathcal{GL}(n, \mathcal{A}))$ *are in bijective correspondence, i.e.,*

$$\boldsymbol{\Phi}^n_{\mathcal{A}}(X) \cong H^1(X, \mathcal{GL}(n, \mathcal{A})).$$

Proof. Since we follow the general pattern of the proof of Theorem 4.6.2 (with the necessary modifications), we only give an outline of its main steps.

To a class $[\,\mathcal{E}\,] \in \boldsymbol{\Phi}^n_{\mathcal{A}}(X)$ we assign the cohomology class $[(\psi_{\alpha\beta})]$, if $(\psi_{\alpha\beta})$ is the cocycle determined by the coordinate transformations (over any local frame) of the representative $\mathcal{E}$, after the identifications of Proposition 5.1.4. This is a well defined injection.

The surjectivity of the previous assignment is proved by showing that a cocycle $(\psi_{\alpha\beta}) \in Z^1(\mathcal{U}, \mathcal{GL}(n, \mathcal{A}))$ determines a vector sheaf of rank n, with corresponding coordinate transformations coinciding with the given cocycle. To this end we assume that $\mathcal{U}$ is a *basis for the topology* $\mathfrak{T}_X$ and consider the presheaf of sections of $\mathcal{A}^n$, $\big(\mathcal{A}^n(U_\alpha), \lambda_{\alpha\beta}\big)$, where the restriction maps are the natural restrictions of sections. We define the maps

$$\rho_{\alpha\beta} := \psi_{\alpha\beta}^{-1} \circ \lambda_{\alpha\beta} : \mathcal{A}^n(U_\alpha) \longrightarrow \mathcal{A}^n(U_\beta); \qquad U_\beta \subseteq U_\alpha,$$

where now $\psi_{\alpha\beta}$ is the induced morphism of sections.

For every $s \in \mathcal{A}^n(U_\alpha)$ and $U_\gamma \subseteq U_\beta \subseteq U_\alpha$, in virtue of the cocycle property of $(\psi_{\alpha\beta})$, we have that

$$\begin{aligned}(\rho_{\beta\gamma} \circ \rho_{\alpha\beta})(s) &= \rho_{\beta\gamma}\big(\psi_{\alpha\beta}^{-1}(\lambda_{\alpha\beta}(s)\big) = \rho_{\beta\gamma}\big(\psi_{\alpha\beta}^{-1}(s|_{U_\beta})\big) \\ &= \psi_{\beta\gamma}^{-1}\big(\psi_{\alpha\beta}^{-1}(s|_{U_\beta})|_{U_\gamma}\big) = (\psi_{\beta\gamma}^{-1} \circ \psi_{\alpha\beta}^{-1})(s|_{U_\gamma}) \\ &= \psi_{\alpha\gamma}^{-1}(s|_{U_\gamma}) = (\psi_{\alpha\gamma}^{-1} \circ \lambda_{\alpha\gamma})(s) = \rho_{\alpha\gamma}(s).\end{aligned}$$

Thus, $\big(\mathcal{A}^n(U_\alpha), \rho_{\alpha\beta}\big)$ is a presheaf generating a sheaf $(\mathcal{E}, \pi, X)$. Since each $\mathcal{A}^n(U_\alpha)$ is an $\mathcal{A}(U_\alpha)$-module, it turns out that $\mathcal{E}$ is an $\mathcal{A}$-module.

We check that $\mathcal{E}$ is locally free as follows: Fixing an $\alpha \in I$, for every $U_\beta \subseteq U_\alpha$ we define the $\mathcal{A}(U_\beta)$-isomorphism ((5.1.14) still being in force)

$$\phi_{\alpha, U_\beta} : \mathcal{A}^n(U_\beta) \longrightarrow \mathcal{A}^n(U_\beta) : s \mapsto \psi_{\alpha\beta} \circ s,$$

whose domain consists of the module of sections belonging to the presheaf generating $\mathcal{E}$, whereas the image belongs to the presheaf of sections of $\mathcal{A}^n$. Varying U_β in U_α, the presheaf isomorphism (ϕ_{α, U_β}), for all open $U_\beta \subseteq U_\alpha$, generates an $\mathcal{A}|_{U_\alpha}$-isomorphism $\phi_\alpha : \mathcal{E}|_{U_\alpha} \to \mathcal{A}^n|_{U_\alpha}$.

Let us show that the coordinate transformation $\phi_\alpha \circ \phi_\beta^{-1}$ coincides with $\psi_{\alpha\beta}$, within a natural isomorphism. Indeed, for an arbitrary $a \in \mathcal{A}^n_x$, with $x \in U_{\alpha\beta}$, there is a section $\sigma \in \mathcal{A}^n(U_\gamma)$, for some open $U_\gamma \subseteq U_{\alpha\beta}$, such that $\sigma(x) = a$. Then, by (1.2.13) and the notation of ($\Diamond$) on p. 104, we have:

$$\begin{aligned}(\phi_\alpha \circ \phi_\beta^{-1})(a) &= \phi_\alpha(\phi_\beta^{-1}(\sigma(x))) = \phi_\alpha\big((\phi_{\beta, U_\gamma}^{-1}(\sigma))\tilde{}\,(x)\big) \\ &= \phi_\alpha\big((\psi_{\beta\gamma}^{-1} \circ \sigma)\tilde{}\,(x)\big) = \big(\phi_{\alpha, U_\gamma}(\psi_{\beta\gamma}^{-1} \circ \sigma)\big)\tilde{}\,(x) \\ &= (\psi_{\alpha\gamma} \circ \psi_{\beta\gamma}^{-1} \circ \sigma)\tilde{}\,(x) = (\psi_{\alpha\beta} \circ \sigma)\tilde{}\,(x) \\ &\equiv (\psi_{\alpha\beta} \circ \sigma)(x) = \psi_{\alpha\beta}(\sigma(x)) = \psi_{\alpha\beta}(a).\end{aligned}$$

In the last line of the above series of equalities we have identified $\mathcal{A}^n$ with the sheaf generated by the presheaf of its sections (see also (5.1.2)). □

Note. A slightly different proof is given in Mallios [62, Vol. I, p. 359]. Another proof, based on the classification of principal sheaves and techniques from the sheaf of frames of a vector sheaf, will be given in Corollary 5.2.9.

Before closing this section let us observe that 1-cocycles of the form $(g_{\alpha\beta}) \in Z^1(\mathcal{U}, \mathcal{GL}(n, \mathcal{A}))$ represent cocycles of vector sheaves, as explained in the previous theorem, *and* cocycles of $\mathcal{GL}(n, \mathcal{A})$-principal sheaves. Therefore, applying the notations (4.6.1) (for $\mathcal{G} = \mathcal{GL}(n, \mathcal{A})$) and (5.1.15), the classification Theorems 4.6.2 and 5.1.8 lead to the commutative diagram below, consisting of bijective correspondences.

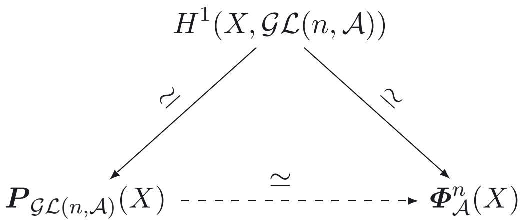

Diagram 5.1

The link between the domain and the range of the dashed horizontal arrow in this diagram can also be realized by the sheaves of frames discussed in the next section (see, in particular, Corollary 5.2.8).

5.2. The sheaf of frames of a vector sheaf

We proceed to the detailed study of the sheaf in the title, one of the most important (abstract) examples of principal sheaves, as mentioned in Remark 4.1.10.

We fix a vector sheaf $(\mathcal{E}, \pi_{\mathcal{E}}, X)$ of rank n, whose local structure is described by the local frame $(\mathcal{U}, (\psi_\alpha))$. We denote by $\mathcal{B}$ the basis for the topology of X, defined in the following way:

$$V \in \mathcal{B} \quad \Longleftrightarrow \quad \exists\ \alpha \in I\ :\ V \subseteq U_\alpha .$$

Of course, one may choose $\mathcal{U}$ to be itself a basis for the topology of X.

For each $V \in \mathcal{B}$, we consider the set

$$\mathrm{Iso}_{\mathcal{A}|_V}(\mathcal{A}^n|_V, \mathcal{E}|_V)$$

consisting of all the $\mathcal{A}|_V$-isomorphisms between the $\mathcal{A}|_V$-modules $\mathcal{A}^n|_V$ and $\mathcal{E}|_V$. It is obvious that

$$V \longmapsto \mathrm{Iso}_{\mathcal{A}|_V}(\mathcal{A}^n|_V, \mathcal{E}|_V), \tag{5.2.1}$$

with V running in $\mathcal{B}$, is a *complete* presheaf with the obvious restriction maps.

5.2.1 Definition. The sheaf generated by the presheaf (5.2.1) is called the ***sheaf of frames of*** $\mathcal{E}$. It is denoted by $\mathcal{P}(\mathcal{E})$.

5.2.2 Proposition. *$\mathcal{P}(\mathcal{E})$ is a $\mathcal{GL}(n, \mathcal{A})$-principal sheaf.*

Proof. For a $V \in \mathcal{B}$, we define the local action

$$\begin{gathered}\delta_V : \mathrm{Iso}_{\mathcal{A}|_V}(\mathcal{A}^n|_V, \mathcal{E}|_V) \times \mathrm{GL}(n, \mathcal{A}(V)) \longrightarrow \mathrm{Iso}_{\mathcal{A}|_V}(\mathcal{A}^n|_V, \mathcal{E}|_V) : \\ (f, g) \longmapsto \delta_V(f, g) \equiv f \cdot g := f \circ g,\end{gathered}$$

under the identifications (5.1.14). Running V in $\mathcal{B}$, we obtain a presheaf morphism (δ_V) generating an action of $\mathcal{GL}(n, \mathcal{A})$ on the right of $\mathcal{P}(\mathcal{E})$.

We now fix an $\alpha \in I$. For every open $V \subseteq U_\alpha$, we define the $\mathrm{GL}(n, \mathcal{A}(V))$-equivariant isomorphism

$$\Phi_{\alpha,V} : \mathrm{Iso}_{\mathcal{A}|_V}(\mathcal{A}^n|_V, \mathcal{E}|_V) \longrightarrow \mathrm{GL}(n, \mathcal{A}(V)) \; : \; f \mapsto \psi_\alpha \circ f, \tag{5.2.2}$$

once more using (5.1.14). The family $(\Phi_{\alpha,V})_V$, with V varying in U_α, defines a presheaf isomorphism. It generates, in turn, a $\mathcal{GL}(n, \mathcal{A})|_{U_\alpha}$-equivariant sheaf isomorphism

$$\Phi_\alpha : \mathcal{P}(\mathcal{E})|_{U_\alpha} \xrightarrow{\cong} \mathcal{GL}(n, \mathcal{A})|_{U_\alpha}. \tag{5.2.3}$$

Thus, $\mathcal{P}(\mathcal{E})$ is a $\mathcal{GL}(n, \mathcal{A})$-principal sheaf with local frame $(\mathcal{U}, (\Phi_\alpha))$. □

The sheaf of frames $\mathcal{P}(\mathcal{E})$ is fully denoted by

$$\mathcal{P}(\mathcal{E}) \equiv (\mathcal{P}(\mathcal{E}), \mathcal{GL}(n, \mathcal{A}), X, \tilde{\pi}). \tag{5.2.4}$$

5.2.3 Corollary. *Let $\mathcal{E}$ be a vector sheaf and $(\psi_{\alpha\beta}) \in Z^1(\mathcal{U}, \mathcal{GL}(n, \mathcal{A}))$ its cocycle, with respect to a local frame $(\mathcal{U}, (\psi_\alpha))$. If $(g_{\alpha\beta}) \in Z^1(\mathcal{U}, \mathcal{GL}(n, \mathcal{A}))$ is the cocycle of $\mathcal{P}(\mathcal{E})$, with respect to the local frame $(\mathcal{U}, (\Phi_\alpha))$ of Proposition 5.2.2, then*

$$(g_{\alpha\beta}) = (\widetilde{\psi_{\alpha\beta}}) = (\psi_{\alpha\beta})$$

within appropriate isomorphisms.

Proof. First observe that the definition of $\mathcal{GL}(n, \mathcal{A})$ and the canonical bijection $\mathrm{GL}(n, \mathcal{A}(U_{\alpha\beta})) \xrightarrow{\simeq} \mathcal{GL}(n, \mathcal{A})(U_{\alpha\beta})$ implies that $\mathrm{I} \xrightarrow{\simeq} \widetilde{\mathrm{I}} = \mathbf{1}$, where, for convenience, I denotes the identity matrix of $\mathrm{GL}(n, \mathcal{A}(U_{\alpha\beta}))$ and $\mathbf{1} = \mathbf{1}|_{U_{\alpha\beta}}$ is the unit section in $\mathcal{GL}(n, \mathcal{A})(U_{\alpha\beta})$.

Then, taking into account the definitions of the cocycles $(g_{\alpha\beta})$ and $(\psi_{\alpha\beta})$ (see (4.3.2), (5.1.4), (5.1.6) and Proposition 5.1.4), we have that

$$\begin{aligned} g_{\alpha\beta} &= (\Phi_\alpha \circ \Phi_\beta^{-1})(\mathbf{1}) = \Phi_\alpha\big(\Phi_\beta^{-1}(\widetilde{\mathrm{I}}\,)\big) \\ &= \Phi_\alpha\big(\big(\Phi^{-1}_{\beta, U_{\alpha\beta}}(\,\mathrm{I}\,)\big)^{\sim}\big) = \big(\Phi_{\alpha, \mathrm{U}_{\alpha\beta}}(\psi_\beta^{-1})\big)^{\sim} \\ &= (\psi_\alpha \circ \psi_\beta^{-1})^{\sim} = \widetilde{\psi_{\alpha\beta}} = \psi_{\alpha\beta}, \end{aligned}$$

after the identifications (5.1.14). □

From the preceding result and (5.1.6) it is clear that each $g_{\alpha\beta}$ can be identified with the matrix of $\psi_{\alpha\beta}$.

Before proceeding, observe that

$$\mathcal{P}(\mathcal{E})(V) \cong \mathrm{Iso}_{\mathcal{A}|_V}(\mathcal{A}^n|_V, \mathcal{E}|_V), \tag{5.2.5}$$

as a result of the completeness of the presheaf (5.2.1) generating $\mathcal{P}(\mathcal{E})$.

If $\sigma_\alpha \in \mathcal{P}(\mathcal{E})(U_\alpha)$, $\alpha \in I$, are the ***natural sections*** of $\mathcal{P}(\mathcal{E})$, over $\mathcal{U}$, then we have:

5.2.4 Corollary. *The natural sections of $\mathcal{P}(\mathcal{E})$ are given by*

$$\sigma_\alpha = \psi_\alpha^{-1}; \qquad \alpha \in I, \tag{5.2.6}$$

after the identification (5.2.5) for $V = U_\alpha$.

Proof. Applying Definition 4.1.6 and working as in the previous proof, we have that

$$\sigma_\alpha := \Phi_\alpha^{-1}(\mathbf{1}) \equiv \Phi^{-1}_{\alpha, U_\alpha}(id) := \psi_\alpha^{-1} \circ id_\alpha = \psi_\alpha^{-1},$$

where now $\mathbf{1} = \mathbf{1}|_{U_\alpha}$ and $id \in \mathrm{Aut}_{\mathcal{A}|_{U_\alpha}}(\mathcal{A}^n|_{U_\alpha})$. □

Note. If we do not apply the identification (5.2.5), then (5.2.6) is replaced by the following equality:

$$\sigma_\alpha = \widetilde{\psi_\alpha^{-1}} = (\widetilde{\psi_\alpha})^{-1}. \tag{5.2.6$'$}$$

Arbitrary $\mathcal{GL}(n, \mathcal{A})$-*principal sheaves* are related with sheaves of frames of vector sheaves as follows:

5.2.5 Proposition. *Let $\mathcal{P} \equiv (\mathcal{P}, \mathcal{GL}(n, \mathcal{A}), X, \pi)$ be an arbitrary principal sheaf. Then there exists a vector sheaf $\mathcal{E}$ of rank n so that $\mathcal{P}(\mathcal{E})$ and $\mathcal{P}$ be $\mathcal{GL}(n, \mathcal{A})$-isomorphic. Therefore, any $\mathcal{GL}(n, \mathcal{A})$-principal sheaf is realized, up to isomorphism, as the sheaf of frames of a vector sheaf of rank n.*

Proof. Let $(g_{\alpha\beta}) \in Z^1(\mathcal{U}, \mathcal{GL}(n, \mathcal{A}))$ be the cocycle of $\mathcal{P}$ with respect to a local frame $\mathcal{U}$. As we have seen in the proof of Theorem 5.1.6, $(g_{\alpha\beta})$ can be considered as the cocycle of a vector sheaf $\mathcal{E}$, and, by Corollary 5.2.3, as the cocycle of the corresponding sheaf of frames $\mathcal{P}(\mathcal{E})$. Therefore, $\mathcal{P} \cong \mathcal{P}(\mathcal{E})$ (by means of a $\mathcal{GL}(n, \mathcal{A})$-isomorphism), as a particular case of Theorem 4.4.2 (restated). □

Concerning isomorphisms of vector sheaves and their sheaves of frames, we obtain:

5.2.6 Proposition. *Two vector sheaves $\mathcal{E} \equiv (\mathcal{E}, \pi, X)$ and $\mathcal{E}' \equiv (\mathcal{E}', \pi', X)$, of rank n, are isomorphic if and only if their corresponding sheaves of frames $\mathcal{P}(\mathcal{E})$ and $\mathcal{P}(\mathcal{E}')$ are $\mathcal{GL}(n, \mathcal{A})$-isomorphic.*

Proof. We have the following sequence of equivalences, for all $\alpha, \beta \in I$:

$$\begin{aligned} \mathcal{E} \cong \mathcal{E}' \quad &\Longleftrightarrow \quad \psi'_{\alpha\beta} = h_\alpha \circ \psi_{\alpha\beta} \circ h_\beta^{-1}, \\ &\Longleftrightarrow \quad g'_{\alpha\beta} = h_\alpha \cdot g_{\alpha\beta} \cdot h_\beta^{-1}, \\ &\Longleftrightarrow \quad \mathcal{P}(\mathcal{E}) \cong \mathcal{P}(\mathcal{E}'). \end{aligned}$$

The first is Theorem 5.1.7, the second is a result of (5.1.14) and Corollary 5.2.3, while the third follows from Theorem 4.4.2. □

5.2.7 Remarks. 1) Assume that we start with an isomorphism of vector sheaves $F : \mathcal{E} \to \mathcal{E}'$ and let $f : \mathcal{P}(\mathcal{E}) \to \mathcal{P}(\mathcal{E}')$ be a $\mathcal{GL}(n, \mathcal{A})$-isomorphism ensured by Proposition 5.2.6. An explicit expression for such an f can be found from (4.4.6), after the calculation of the cochain (h_α) determined by

(5.1.12). Note that in this case, equalities (4.4.7) and (5.1.13) coincide, in virtue of Corollary 5.2.3. The same calculation leads to the following equivalent definition of f: it is the isomorphism generated by the presheaf isomorphism obtained from the family of $\mathrm{GL}(n, \mathcal{A}(V))$-isomorphisms

$$f_V : \mathrm{Iso}_{\mathcal{A}|_V}(\mathcal{A}^n|_V, \mathcal{E}|_V) \longrightarrow \mathrm{Iso}_{\mathcal{A}|_V}(\mathcal{A}^n|_V, \mathcal{E}'|_V) \,:\, g \mapsto F \circ g,$$

where V is running in $\mathcal{B}$.

Another global relationship between f and F will be given in Proposition 5.6.5 in the sequel, where a vector sheaf will be associated with its principal sheaf of frames.

2) Conversely, the construction of a vector sheaf isomorphism F from a given $\mathcal{GL}(n, \mathcal{A})$-isomorphism f will be dealt with in Section 5.6 (see Corollary 5.6.4). We note that in Section 5.6 we discuss the relationship between isomorphisms of vector sheaves and isomorphisms of the corresponding sheaves of frames in the general setting of associated sheaves.

The next result, being a direct consequence of Propositions 5.2.5 and 5.2.6, now explains the comments following Diagram 5.1 regarding its (dashed) horizontal bijection.

5.2.8 Corollary. *The sets $\boldsymbol{\Phi}^n_{\mathcal{A}}(X)$ and $\boldsymbol{P}_{\mathcal{GL}(n,\mathcal{A})}(X)$ are in bijective correspondence.*

Therefore, we are led to a new proof of Theorem 5.1.8, recorded here for the sake of completeness.

5.2.9 Corollary (Cohomological classification of vector sheaves).

$$\boldsymbol{\Phi}^n_{\mathcal{A}}(X) \cong H^1(X, \mathcal{GL}(n, \mathcal{A})).$$

Proof. Corollary 5.2.8 and Theorem 4.6.2 imply, respectively, that

$$\boldsymbol{\Phi}^n_{\mathcal{A}}(X) \;\cong\; \boldsymbol{P}_{\mathcal{GL}(n,\mathcal{A})}(X) \;\cong\; H^1(X, \mathcal{GL}(n, \mathcal{A})). \qquad \square$$

5.3. Associated sheaves: a general construction

We exhibit a general construction by which we obtain sheaves associated with a given principal sheaf. More specific cases, mainly based on various representations of the structure sheaf, will provide some important examples, including the sheaf of frames of a vector sheaf.

Throughout this section we fix a principal sheaf $\mathcal{P} \equiv (\mathcal{P}, \mathcal{G}, X, \pi)$, where, for the sake of simplicity,

> $\mathcal{G}$ is assumed to be only a sheaf of groups. Therefore, morphisms of principal sheaves of this type are defined by triplets of the form (f, ϕ, id_X).

In this respect we refer to Remarks 4.1.5(2) and 4.2.2(1). Later on, in the study of connections on associated sheaves (Chapter 7), $\mathcal{G}$ will necessarily be a Lie sheaf of groups, while the morphisms of principal sheaves will be meant as in Definition 4.2.1.

We further assume that $\mathcal{F} \equiv (\mathcal{F}, \pi_{\mathcal{F}}, X)$ is a given sheaf of sets on which $\mathcal{G}$ *acts from the left*, by an action

$$\delta_F : \mathcal{G} \times_X \mathcal{F} \longrightarrow \mathcal{F} : (g, u) \mapsto \delta_F(g, u) \equiv g \cdot u.$$

For every open $U \subseteq X$, the group $\mathcal{G}(U)$ clearly acts on the right of $\mathcal{P}(U) \times \mathcal{F}(U)$ by setting

$$(s, f) \cdot g := (s \cdot g, g^{-1} \cdot f), \tag{5.3.1}$$

for every $(s, f) \in \mathcal{P}(U) \times \mathcal{F}(U)$ and $g \in \mathcal{G}(U)$. The first component of the right-hand side of (5.3.1) represents the action of $\mathcal{G}(U)$ on the right of $\mathcal{P}(U)$, and the second one is the action of $\mathcal{G}(U)$ on the left of $\mathcal{F}(U)$, induced by δ_F. The action (5.3.1) determines the following equivalence relation on $\mathcal{P}(U) \times \mathcal{F}(U)$:

$$(s, f) \sim_U (t, h) \quad \Longleftrightarrow \quad \exists \; g \in \mathcal{G}(U) \; : \; (t, h) = (s, f) \cdot g, \tag{5.3.2}$$

for every $(s, f), (t, h) \in \mathcal{P}(U) \times \mathcal{F}(U)$.

It is evident that $\sim_U$ is indeed an equivalence relation and the above section $g \in \mathcal{G}(U)$ is unique (by Proposition 4.1.2). Therefore, one obtains the quotient set

$$Q(U) := (\mathcal{P}(U) \times \mathcal{F}(U))/\mathcal{G}(U). \tag{5.3.3}$$

The equivalence class of (s, f) is denoted by

$$[(s, f)]_U \in Q(U). \tag{5.3.4}$$

For any open $U, V \subseteq X$ with $V \subseteq U$, we denote by

$$\rho^U_V : \mathcal{P}(U) \longrightarrow \mathcal{P}(V) \quad \text{and} \quad f^U_V : \mathcal{F}(U) \longrightarrow \mathcal{F}(V)$$

the natural restriction maps of the presheaves of sections corresponding to $\mathcal{P}$ and $\mathcal{F}$, and we define the restriction map

$$q_V^U : Q(U) \longrightarrow Q(V)$$

by setting

$$q_V^U([(s,f)]_U) := [(\rho_V^U(s), f_V^U(f))]_V = [(s|_V, f|_V)]_V, \tag{5.3.5}$$

for every $[(s,f)]_U \in Q(U)$. It is well defined and satisfies the condition

$$q_W^U = q_W^V \circ q_V^U,$$

for all $W, V, U \in \mathfrak{T}_X$ with $W \subseteq V \subseteq U$. As a result, varying U in the topology of X, we obtain the (not necessarily complete) presheaf

$$\big(Q(U),\ q_V^U\big), \tag{5.3.6}$$

generating a sheaf, denoted by

$$\mathcal{Q} \equiv (\mathcal{Q}, \pi_{\mathcal{Q}}, X) := \mathbf{S}\big(U \longmapsto Q(U)\big). \tag{5.3.7}$$

Later on (see Corollary 5.3.6), $\mathcal{Q}$ will be identified with the sheaf (5.3.13), derived by quotienting $\mathcal{P} \times_X \mathcal{F}$ by a global equivalence relation.

5.3.1 Definition. The sheaf $\mathcal{Q}$, defined by (5.3.7), is called the ***sheaf associated with*** $\mathcal{P}$ ***by the action*** δ_F.

We now describe the local structure of $\mathcal{Q}$.

5.3.2 Theorem. *The sheaf* $\mathcal{Q}$ *is of* ***structure type*** $\mathcal{F}$*; that is, there exists an open covering* $\mathcal{U} = (U_\alpha)$ *of* X *and isomorphisms (of sheaves of sets)*

$$\underline{\Phi}_\alpha : \mathcal{Q}|_{U_\alpha} \xrightarrow{\ \simeq\ } \mathcal{F}|_{U_\alpha}, \qquad \alpha \in I.$$

Proof. Let $(\mathcal{U}, (\phi_\alpha))$ be a local frame of $\mathcal{P}$ with natural sections $s_\alpha \in \mathcal{P}(U_\alpha)$. Fixing $\alpha \in I$, for each open $V \subseteq U_\alpha$ we define the map

$$\underline{\Psi}_{\alpha,V} : \mathcal{F}(V) \longrightarrow Q(V) : f \mapsto [(s_\alpha|_V, f)]_V. \tag{5.3.8}$$

It is 1-1, for if $\underline{\Psi}_{\alpha,V}(f) = \underline{\Psi}_{\alpha,V}(f')$, then $(s_\alpha|_V, f') = (s_\alpha|_V, f) \cdot g$, for a unique $g \in \mathcal{G}(V)$. By (5.3.1) and Proposition 4.1.2, $g = \mathbf{1}|_V$, thus $f = f'$.

On the other hand, an arbitrary $[(\sigma,h)]_V \in Q(V)$ can be obtained as the image, via $\underline{\Psi}_{\alpha,V}$, of the element $f := g\cdot h$, where $g \in \mathcal{G}(V)$ is the unique section satisfying $\sigma = s_\alpha|_V \cdot g$. We note that f is a continuous section, since $f = \boldsymbol{k}(s_\alpha,\sigma)\cdot h$ (see Proposition 4.1.4). Hence, $\underline{\Psi}_{\alpha,V}$ is also surjective.

Varying V in U_α, we obtain a presheaf isomorphism $(\underline{\Psi}_{\alpha,V})_{V\subseteq U_\alpha}$ generating an isomorphism of sheaves (of sets) $\underline{\Psi}_\alpha : \mathcal{F}|_{U_\alpha} \to \mathcal{Q}|_{U_\alpha}$ whose inverse, denoted by $\underline{\Phi}_\alpha$, gives the isomorphism of the statement. □

Because of the previous result, $\mathcal{Q}$ is characterized as the ***associated sheaf of type*** $\mathcal{F}$. It is customary to call the isomorphisms $\underline{\Phi}_\alpha$, $\alpha\in I$, the ***coordinates*** of $\mathcal{Q}$ over $\mathcal{U}$.

5.3.3 Proposition. *If $(g_{\alpha\beta})$ is the cocycle of the principal sheaf $\mathcal{P}$, the transformation of coordinates of the associated sheaf $\mathcal{Q}$ (of structure type $\mathcal{F}$) is given by*

$$(\underline{\Phi}_\alpha \circ \underline{\Phi}_\beta^{-1})(u) = g_{\alpha\beta}(x)\cdot u,$$

for every $u\in\mathcal{F}$ with $\pi_{\mathcal{F}}(u) = x \in U_{\alpha\beta}$.

Proof. Let $u \in \mathcal{F}_x$, $x\in U_{\alpha\beta}$. Then $u = h(x)$, for some section $h\in\mathcal{F}(V)$ defined over an open $V\subseteq U_{\alpha\beta}$ with $x\in V$. Then, by the very construction of $\underline{\Psi}_\alpha$,

$$\underline{\Psi}_\alpha(u) = [\underline{\Psi}_{\alpha,V}(h)]_x = \big(\underline{\Psi}_{\alpha,V}(h)\big)^{\sim}(x), \tag{5.3.9}$$

where $[\underline{\Psi}_{\alpha,V}(h)]_x$ denotes the germ of $\underline{\Psi}_{\alpha,V}(h)$ at x, and $\big(\underline{\Psi}_{\alpha,V}(h)\big)^{\sim}$ is the section in $\mathcal{Q}(V)$ obtained from the (presheaf) "section" $\underline{\Psi}_{\alpha,V}(h)\in Q(V)$ (see the notation (◊), p. 104). Similarly,

$$\underline{\Psi}_\beta(u) = (\underline{\Psi}_{\beta,V}(h))^{\sim}(x). \tag{5.3.9$'$}$$

On the other hand, (5.3.8) and (4.3.3) yield

$$\begin{aligned}\underline{\Psi}_{\beta,V}(h) &= [(s_\beta|_V, h)]_V = [(s_\alpha|_V\cdot g_{\alpha\beta}|_V, h)]_V\\ &= [(s_\alpha|_V,\ g_{\alpha\beta}|_V\cdot h)]_V = \underline{\Psi}_{\alpha,V}(g_{\alpha\beta}|_V\cdot h).\end{aligned} \tag{5.3.10}$$

Since $g_{\alpha\beta}|_V\cdot h\in\mathcal{F}(V)$ with $(g_{\alpha\beta}|_V\cdot h)(x) = g_{\alpha\beta}(x)\cdot u$, equalities (5.3.10) and (5.3.9), applied to (5.3.9$'$), lead to

$$\underline{\Psi}_\beta(u) = \big(\underline{\Psi}_{\alpha,V}(g_{\alpha\beta}|_V\cdot h)\big)^{\sim}(x) = \underline{\Psi}_\alpha(g_{\alpha\beta}(x)\cdot u). \tag{5.3.11}$$

This proves the equality of the statement because $\underline{\Phi}_\alpha = \underline{\Psi}_\alpha^{-1}$, for every $\alpha\in I$. □

We rewrite the same change of coordinates in the following convenient form

$$\underline{\Phi}_{\alpha}(v) = g_{\alpha\beta}(x) \cdot \underline{\Phi}_{\beta}(v), \tag{5.3.12}$$

for every $v \in \mathcal{Q}_x$, with $x \in U_{\alpha\beta}$.

As commented earlier, we shall give another, equivalent, interpretation of the associated sheaf of type $\mathcal{F}$. Namely, we shall show that $\mathcal{Q}$ can be identified with the sheaf $(\mathcal{P} \times_X \mathcal{F})/\mathcal{G}$ induced by an analogous global equivalent relation.

As a first step to our goal, we explain the definition and the structure of the aforementioned quotient sheaf: The action $\delta_F : \mathcal{G} \times_X \mathcal{F} \to \mathcal{F}$ determines an action of $\mathcal{G}$ on the right of $\mathcal{P} \times_X \mathcal{F}$ by setting $(p, u) \cdot g := (p \cdot g, g^{-1} \cdot u)$, for every $(p, u) \in \mathcal{P} \times_X \mathcal{F}$ and $g \in \mathcal{G}$ on stalks at the same base point. This induces the following equivalence relation on the fiber product $\mathcal{P} \times_X \mathcal{F}$ (compare with (5.3.1) and (5.3.2)):

$$(p, u) \sim (q, v) \Longleftrightarrow \begin{cases} \pi(p) = \pi(q) = \pi_{\mathcal{F}}(u) = \pi_{\mathcal{F}}(v) = x, \\ \exists \ g \in \mathcal{G}_x \ : \ (p, u) = (q, v) \cdot g := (q \cdot g, g^{-1} \cdot v). \end{cases}$$

Clearly, $g \in \mathcal{G}_x$ is uniquely determined by Proposition 4.1.2.

We denote by $[(p, u)]$ the equivalence class of $(p, u) \in \mathcal{P} \times_X \mathcal{F}$ and by

$$(\mathcal{P} \times_X \mathcal{F})/\mathcal{G} \equiv \mathcal{P} \times_X^{\mathcal{G}} \mathcal{F}, \tag{5.3.13}$$

the resulting quotient space, *topologized with the quotient topology*.

The two notations of (5.3.13) will be used interchangeably. The second notation is reminiscent of an analogous situation for *associated bundles* (see, e.g., Bourbaki [13, n° 6.5.1]).

There is a well defined natural projection

$$\begin{aligned} \bar{\pi} : (\mathcal{P} \times_X \mathcal{F})/\mathcal{G} \equiv \mathcal{P} \times_X^{\mathcal{G}} \mathcal{F} \longrightarrow X \\ [(p, u)] \mapsto \bar{\pi}([(p, u)]) := \pi(p) = \pi_{\mathcal{F}}(u). \end{aligned} \tag{5.3.14}$$

5.3.4 Proposition. *The triplet* $\left(\mathcal{P} \times_X^{\mathcal{G}} \mathcal{F}, \bar{\pi}, X\right)$ *is a sheaf.*

Proof. Let us denote by

$$\kappa : \mathcal{P} \times_X \mathcal{F} \to \mathcal{P} \times_X^{\mathcal{G}} \mathcal{F} \ : \ (p, u) \mapsto [(p, u)]$$

the canonical map, which is continuous and open. Moreover, if π_X is the projection of $\mathcal{P} \times_X \mathcal{F}$ to X (see Subsection 1.1.2), we have the commutative diagram

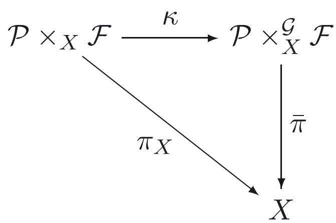

DIAGRAM 5.2

from which (along with the properties of the quotient topology) we see that $\bar{\pi}$ is a continuous map, since $\bar{\pi} \circ \kappa = \pi_X$ is already continuous.

To prove that $\bar{\pi}$ is a local homeomorphism, we proceed as follows: Let $[(p_o, u_o)]$ be an arbitrary element with $\bar{\pi}([(p_o, u_o)]) = x_o$. The sheaf structure of $\mathcal{P}\times_X\mathcal{F}$ guarantees the existence of two open sets $U_o \subseteq X$ and $V_o \subseteq \mathcal{P}\times_X\mathcal{F}$ containing x_o and (p_o, u_o), respectively, so that the map

$$\pi_{X,o} := \pi_X|_{V_o} : V_o \xrightarrow{\simeq} U_o \tag{5.3.15}$$

be a homeomorphism. Then $W_o := \kappa(V_o) \subseteq \mathcal{P}\times_X^{\mathcal{G}}\mathcal{F}$ is an open neighborhood of $[(p_o, u_o)]$. We claim that the continuous map

$$\bar{\pi}_o := \bar{\pi}|_{W_o} \ : \ W_o \longrightarrow U_o$$

is a homeomorphism. To this end we show the following assertions:

i) $\bar{\pi}_o$ is *injective*: If $[(p, u)], [(q, v)] \in W_o$ with $\bar{\pi}_o([(p, u)]) = \bar{\pi}_o([(q, v)])$, then $\pi_{X,o}(p, u) = \pi_{X,o}(q, v)$ and $(p, u) = (q, v)$, as a consequence of the injectivity of (5.3.15).

ii) $\bar{\pi}_o$ is *surjective*: For an arbitrary $x \in U_o$, we see that $\kappa(\pi_{X,o}^{-1}(x)) \in W_o$; hence, Diagram 5.2 implies that $\bar{\pi}_o\big(\kappa(\pi_{X,o}^{-1}(x))\big) = x$.

iii) $\bar{\pi}_o^{-1}$ is *continuous*: This is clear from equality $\bar{\pi}_o^{-1} = \kappa \circ \pi_{X,o}^{-1}$, obtained in the last step of the proof of the surjectivity of $\bar{\pi}_o$.

Therefore, $\bar{\pi}$ is a local homeomorphism at $[(p_o, u_o)]$, by which we close the proof. □

Similarly to Theorem 5.3.2 and Proposition 5.3.3, we have:

5.3.5 Theorem. *The sheaf* $\mathcal{P}\times_X^{\mathcal{G}}\mathcal{F}$ *is of structure type* $\mathcal{F}$*; that is, there exists an open covering* (U_α) *of* X *and coordinates*

$$\widetilde{\Phi}_\alpha : (\mathcal{P}\times_X^{\mathcal{G}}\mathcal{F})\big|_{U_\alpha} \xrightarrow{\simeq} \mathcal{F}|_{U_\alpha}, \qquad \alpha\in I.$$

Moreover, the transformation of coordinates is given by

$$(\widetilde{\Phi}_\alpha\circ\widetilde{\Phi}_\beta^{-1})(u) = g_{\alpha\beta}(x)\cdot u,$$

for every $u\in\mathcal{F}$ *with* $\pi_{\mathcal{F}}(u)=x\in U_{\alpha\beta}$*, if* $(g_{\alpha\beta})$ *is the cocycle of* $\mathcal{P}$*.*

Proof. Let $(\mathcal{U},(\phi_\alpha))$ be a local frame of $\mathcal{P}$ and $s_\alpha\in\mathcal{P}(U_\alpha)$ the associated natural sections. Then, for each $\alpha\in I$, we define the map

$$\widetilde{\Psi}_\alpha : \mathcal{F}|_{U_\alpha} \longrightarrow (\mathcal{P}\times_X^{\mathcal{G}}\mathcal{F})\big|_{U_\alpha} : u\mapsto [(s_\alpha(x),u)], \tag{5.3.16}$$

if $\pi_{\mathcal{F}}(u)=x$. Since we can write

$$\widetilde{\Psi}_\alpha = \kappa\circ\big(s_\alpha\circ\pi_{\mathcal{F}},\, id\big),$$

where id is the identity of $\mathcal{F}|_{U_\alpha}$, we see that $\widetilde{\Psi}_\alpha$ is a morphism of sheaves.

We easily check that $\widetilde{\Psi}_\alpha$ is a bijection whose *inverse* is the map

$$\widetilde{\Phi}_\alpha : (\mathcal{P}\times_X^{\mathcal{G}}\mathcal{F})\big|_{U_\alpha} \longrightarrow \mathcal{F}|_{U_\alpha} : [(p,u)]\mapsto g\cdot u, \tag{5.3.17}$$

where $g=\boldsymbol{k}(s_\alpha(x),p)$, if $x=\pi(p)=\pi_{\mathcal{F}}(u)$. Therefore, $\widetilde{\Psi}_\alpha$ is an isomorphism of sheaves.

Finally, for every $u\in\mathcal{F}_x$ with $x\in U_{\alpha\beta}$, (5.3.16) and (5.3.17) yield

$$\begin{aligned}(\widetilde{\Phi}_\alpha\circ\widetilde{\Phi}_\beta^{-1})(u) = \widetilde{\Phi}_\alpha([(s_\beta(x),u)]) = \widetilde{\Phi}_\alpha([(s_\alpha(x)\cdot g_{\alpha\beta}(x),u)]) =\\ \widetilde{\Phi}_\alpha([(s_\alpha(x),\, g_{\alpha\beta}(x)\cdot u)]) = g_{\alpha\beta}(x)\cdot u,\end{aligned}$$

as stated. □

5.3.6 Corollary. *The sheaves (5.3.7) and (5.3.13) are isomorphic, i.e.,*

$$\mathcal{Q}\cong(\mathcal{P}\times_X\mathcal{F})/\mathcal{G}\equiv\mathcal{P}\times_X^{\mathcal{G}}\mathcal{F}.$$

Proof. For each $\alpha\in I$ we define the isomorphism of sheaves

$$f_\alpha := \widetilde{\Psi}_\alpha\circ\underline{\Phi}_\alpha = \widetilde{\Phi}_\alpha^{-1}\circ\underline{\Phi}_\alpha : \mathcal{Q}|_{U_\alpha}\longrightarrow(\mathcal{P}\times_X^{\mathcal{G}}\mathcal{F})\big|_{U_\alpha},$$

as in the next diagram,

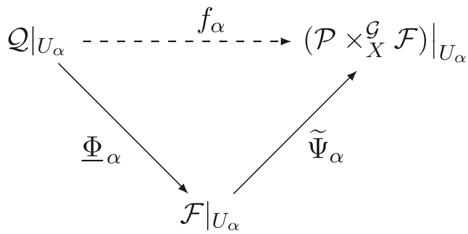

Diagram 5.3

where $(\underline{\Phi}_\alpha)$ and $(\widetilde{\Phi}_\alpha)$ are the coordinates of $\mathcal{Q}$ and $(\mathcal{P} \times_X \mathcal{F})/\mathcal{G}$, respectively.

We claim that $f_\alpha = f_\beta$ on $\mathcal{Q}|_{U_{\alpha\beta}}$, thus the desired isomorphism is obtained by gluing together all the previous isomorphisms. Indeed, for any $q \in \mathcal{Q}|_{U_{\alpha\beta}}$, we set $u := \underline{\Phi}_\beta(q)$, with $\pi_{\mathcal{Q}}(q) = \pi_{\mathcal{F}}(u) = x \in U_{\alpha\beta}$. Then, in virtue of (5.3.16),

$$f_\beta(q) = \widetilde{\Psi}_\beta(\underline{\Phi}_\beta(q)) = \widetilde{\Psi}_\beta(u) = [(s_\beta(x), u)].$$

Hence, applying the transformation of coordinates as in Proposition 5.3.3 and Theorem 5.3.5, we conclude that

$$\begin{aligned} f_\alpha(q) &= (\widetilde{\Psi}_\alpha \circ \underline{\Phi}_\alpha)(q) = \widetilde{\Psi}_\alpha\big((\underline{\Phi}_\alpha \circ \underline{\Phi}_\beta^{-1})(u)\big) \\ &= \widetilde{\Psi}_\alpha(g_{\alpha\beta}(x) \cdot u) = [(s_\alpha(x),\, g_{\alpha\beta}(x) \cdot u)] \\ &= [(s_\alpha(x) \cdot g_{\alpha\beta}(x), u)] = [(s_\beta(x), u)] \\ &= f_\beta(q). \end{aligned}$$

□

As a consequence of the preceding result, the sheaf associated with $\mathcal{P}$ by the action δ_F, will be either $\mathcal{Q}$ or $\mathcal{P} \times^{\mathcal{G}}_X \mathcal{F}$, depending on the particular problem we are dealing with.

For instance, the interpretation of the associated sheaf as the quotient $\mathcal{P} \times^{\mathcal{G}}_X \mathcal{F}$ is convenient in order to prove the following sheaf analog of a classical result, known in the case of associated bundles (cf., e.g., Bourbaki [13, n° 6.5.1], Greub-Halperin-Vanstone [35, p. 198] and Kriegl-Michor [52, p. 381]).

5.3.7 Theorem. *The quadruple* $\big(\mathcal{P} \times_X \mathcal{F}, \bar{\pi}^*(\mathcal{G}), (\mathcal{P} \times_X \mathcal{F})/\mathcal{G} \equiv \mathcal{P} \times^{\mathcal{G}}_X \mathcal{F}, \kappa\big)$ *is a principal sheaf, where* $\bar{\pi}^*(\mathcal{G})$ *is the pull back of* $\mathcal{G}$ *by* $\bar{\pi}$.

Proof. The morphism κ is a local homeomorphism (see the proof of Proposition 5.3.4 and Diagram 5.2); hence, $\mathcal{P} \times_X \mathcal{F}$ is a sheaf with base $\mathcal{Q} = (\mathcal{P} \times_X \mathcal{F})/\mathcal{G} = \mathcal{P} \times_X^{\mathcal{G}} \mathcal{F}$.

There is a natural action

$$* : (\mathcal{P} \times_X \mathcal{F}) \times_{\mathcal{Q}} \bar{\pi}^*(\mathcal{G}) \longrightarrow \mathcal{P} \times_X \mathcal{F}$$

given by

$$(p, u) * (y, g) := (p, u) \cdot g = (p \cdot g, g^{-1} \cdot u),$$

for every $(p, u) \in \mathcal{P} \times_X \mathcal{F}$ and $(y, g) \in \bar{\pi}^*(\mathcal{G})$ projected to the same point of Q, i.e.,

$$\kappa(p, u) = \mathrm{pr}_1|_{\bar{\pi}^*(\mathcal{G})}(y, g) = y \in \mathcal{P} \times_X^{\mathcal{G}} \mathcal{F}.$$

This clearly defines a continuous morphism. The properties of the action are immediately checked, also taking into account that the product of $\bar{\pi}^*(\mathcal{G})$ is given (stalk-wise) by $(y, g) \cdot (y, g') = (y, g \cdot g')$.

Let $\mathcal{U} = (U_\alpha)$ be the open covering of X over which the coordinates $\phi_\alpha : \mathcal{P}|_{U_\alpha} \to \mathcal{G}|_{U_\alpha}$ and $\widetilde{\Phi}_\alpha : (\mathcal{P} \times_X^{\mathcal{G}} \mathcal{F})|_{U_\alpha} \to \mathcal{F}|_{U_\alpha}$, of $\mathcal{P}$ and $\mathcal{P} \times_X^{\mathcal{G}} \mathcal{F}$, respectively, are defined. We form the open covering $\mathcal{V} = (V_\alpha)$ with

$$V_\alpha := \bar{\pi}^{-1}(U_\alpha) = \big(\mathcal{P} \times_X^{\mathcal{G}} \mathcal{F}\big)|_{U_\alpha}.$$

Denoting by $(\mathcal{P} \times_X \mathcal{F})|_{V_\alpha}$ and $\bar{\pi}^*(\mathcal{G})|_{V_\alpha}$ the restrictions of $\mathcal{P} \times_X \mathcal{F}$ and $\bar{\pi}^*(\mathcal{G})$ over V_α, we check that

$$(\mathcal{P} \times_X \mathcal{F})|_{V_\alpha} = \kappa^{-1}(V_\alpha) = \mathcal{P}|_{U_\alpha} \times_{U_\alpha} \mathcal{F}|_{U_\alpha},$$
$$\bar{\pi}^*(\mathcal{G})|_{V_\alpha} = V_\alpha \times_{U_\alpha} \mathcal{G}|_{U_\alpha}.$$

As a result, we can define the maps

$$\chi_\alpha : \bar{\pi}^*(\mathcal{G})|_{V_\alpha} \longrightarrow (\mathcal{P} \times_X \mathcal{F})|_{V_\alpha}; \qquad \alpha \in I,$$

by letting

$$\chi_\alpha(y, g) := \big(\phi_\alpha^{-1}(g),\, g^{-1} \cdot \widetilde{\Phi}_\alpha(y)\big),$$

for every (y, g) in the indicated domain. They are candidates for a system of local coordinates.

First we see that each χ_α is a continuous map satisfying

$$\kappa \circ \chi_\alpha = \mathrm{pr}_1, \tag{5.3.18}$$

where pr_1 is the projection of $\bar{\pi}^*(\mathcal{G})|_{V_\alpha}$ onto V_α. Equality (5.3.18) is proved as follows: If $y = [(p,u)]$, with $\bar{\pi}([(p,u)]) = \pi(p) = x \in U_\alpha$, then (see (5.3.17) with the appropriate modifications) $\widetilde{\Phi}_\alpha(y) = \widetilde{\Phi}_\alpha([(p,u)]) = h \cdot u$, where $h \in \mathcal{G}_x$ is determined by $p = s_\alpha(x) \cdot h$. Therefore, applying (4.1.7) and the equivariance of ϕ_α^{-1},

$$\begin{aligned}(\kappa \circ \chi_\alpha)(y,g) &= [(\phi_\alpha^{-1}(g),\, g^{-1} \cdot (h \cdot u))] = [(s_\alpha(x) \cdot g,\, g^{-1} \cdot (h \cdot u))] \\ &= [(s_\alpha(x), h \cdot u)] = [(s_\alpha(x) \cdot h, u)] = [(p,u)] = y \\ &= \mathrm{pr}_1(y,g),\end{aligned}$$

for every (y,g) in the domain of χ_α. Hence, (5.3.18) is valid.

By its definition, χ_α is a sheaf isomorphism. Moreover, it is $\bar{\pi}^*(\mathcal{G})|_{V_\alpha}$-equivariant. Indeed, for any $(y,g), (y,h) \in \bar{\pi}^*(\mathcal{G})|_{V_\alpha}$, the equivariance of the coordinates of $\mathcal{P}$, the action of $\mathcal{G}$ on $\mathcal{P} \times_X \mathcal{F}$ defined before (5.3.13), the definition of the action $*$, as well as the definition of the pull-back of a sheaf of groups, imply that

$$\begin{aligned}\chi_\alpha\big((y,g)\cdot(y,h)\big) &= \chi_\alpha(y, g \cdot h) \\ &= \big(\phi_\alpha^{-1}(g) \cdot h,\, h^{-1} \cdot (g^{-1} \cdot \widetilde{\Phi}_\alpha(y))\big) \\ &= \big(\phi_\alpha^{-1}(g), g^{-1} \cdot \widetilde{\Phi}_\alpha(y)\big) \cdot h \\ &= \chi_\alpha(y,g) * (y,h).\end{aligned}$$

Therefore, $\big(\mathcal{V}, (\chi_\alpha)\big)$ is a local frame for the principal sheaf structure of the statement. □

We shall connect the sections of the associated sheaf with a particular sort of equivariant morphisms, as specified in the next definition.

5.3.8 Definition. A morphism (of sheaves of sets) $\tau : \mathcal{P} \to \mathcal{F}$ is said to be ***tensorial*** (with respect to the action of $\mathcal{G}$ on the right of $\mathcal{P}$ and on the left of $\mathcal{F}$) if

$$\tau(p \cdot g) = g^{-1} \cdot \tau(p), \qquad (p,g) \in \mathcal{P} \times_X \mathcal{G}.$$

The term "tensorial" is another name for "equivariant" (with respect to the said actions). It coincides with the usual equivariance property (for right actions) defined in the comments after Definition 4.1.1, if we think of $\mathcal{G}$ as acting on the right of $\mathcal{F}$ by setting $u \cdot g := g^{-1} \cdot u$, for every $(u,g) \in \mathcal{F} \times_X \mathcal{G}$. However, we adhere to the first term in conformity with the *tensorial 0-forms* of Kobayashi-Nomizu [49, Vol. I, p. 76].

Analogously to the case of fiber bundles, we have the following relation between tensorial morphisms and sections of the associated sheaf.

5.3.9 Theorem. *The global sections of $\mathcal{P} \times_X^{\mathcal{G}} \mathcal{F}$ are in bijective correspondence with the tensorial morphisms $\tau : \mathcal{P} \to \mathcal{F}$.*

Proof. Let $s \in (\mathcal{P} \times_X^{\mathcal{G}} \mathcal{F})(X)$. We define a map $\tau : \mathcal{P} \to \mathcal{F}$ in the following way: For a $p \in \mathcal{P}$, with $\pi(p) = x \in U_\alpha$, we set $\tau(p) := g_\alpha^{-1} \cdot \widetilde{\Phi}_\alpha(s(x))$, where g_α is determined by $p = s_\alpha(x) \cdot g_\alpha$.

This is a well defined map, for if $x \in U_{\alpha\beta}$, then $\tau(p) := g_\beta^{-1} \cdot \widetilde{\Phi}_\beta(s(x))$, with $p = s_\beta(x) \cdot g_\beta$. But (4.3.3) implies that $g_\alpha = g_{\alpha\beta}(x) \cdot g_\beta$; hence, in virtue of Theorem 5.3.5, we obtain

$$g_\beta^{-1} \cdot \widetilde{\Phi}_\beta(s(x)) = g_\alpha^{-1} \cdot g_{\alpha\beta}(x) \cdot \widetilde{\Phi}_\beta(s(x)) =$$
$$g_\alpha^{-1} \cdot (\widetilde{\Phi}_\alpha \circ \widetilde{\Phi}_\beta^{-1})\big(\widetilde{\Phi}_\beta(s(x))\big) = g_\alpha^{-1} \cdot \widetilde{\Phi}_\alpha(s(x)),$$

which affirms the previous claim.

Since, formally, $\tau(p) = \delta_{\mathcal{F}}\big(\boldsymbol{k}(s_\alpha(x), p),\, \widetilde{\Phi}_\alpha(s(x))\big)$, for every p as above, it follows that τ is a continuous map, thus a morphism of sheaves, as commuting with the projections of the sheaves involved. Its tensoriality is verified by a simple calculation.

We check that the assignment $s \mapsto \tau$, with τ defined as before, is 1–1. Indeed, let s and s' be two sections corresponding to τ and τ' with $\tau = \tau'$. For any $U_\alpha \in \mathcal{U}$, and for every $x \in U_\alpha$, the assumption implies that

$$\widetilde{\Phi}_\alpha(s(x)) = \tau(s_\alpha(x)) = \tau'(s_\alpha(x)) = \widetilde{\Phi}_\alpha(s'(x)),$$

whence $s|_{U_\alpha} = s'|_{U_\alpha}$. Taking all U_α's, we conclude that $s = s'$ on X.

The same assignment is onto. In fact, for a given τ, we define the maps $\sigma_\alpha : U_\alpha \to \mathcal{P} \times_X^{\mathcal{G}} \mathcal{F}$, with $\sigma_\alpha(x) := [(s_\alpha(x), \tau(s_\alpha(x)))]$, for all $\alpha \in I$. They are continuous (local) sections such that, for every $x \in U_{\alpha\beta}$,

$$\sigma_\beta(x) = [(s_\alpha(x) \cdot g_{\alpha\beta}(x),\, g_{\alpha\beta}(x)^{-1} \cdot \tau(s_\alpha(x)))] = \sigma_\alpha(x).$$

We obtain a global section by gluing together all the local sections σ_α, $\alpha \in I$. □

For an open $U \subseteq X$, we denote by $\mathrm{Hom}_{\mathcal{G}|_U}(\mathcal{P}|_U, \mathcal{F}|_U)$ the set of tensorial morphisms of $\mathcal{P}|_U$ into $\mathcal{F}|_U$, with respect to the actions of the sheaf of groups $\mathcal{G}|_U$ on $\mathcal{P}|_U$ and $\mathcal{F}|_U$, respectively. Then the assignment

$$U \longmapsto \mathrm{Hom}_{\mathcal{G}|_U}(\mathcal{P}|_U, \mathcal{F}|_U); \qquad U \in \mathfrak{T}_X,$$

is a complete presheaf, generating the ***sheaf (of germs) of tensorial morphisms of $\mathcal{P}$ into $\mathcal{F}$***

(5.3.19) $$\mathcal{H}om_{\mathcal{G}}(\mathcal{P},\mathcal{F}).$$

Therefore,

(5.3.20) $$\mathcal{H}om_{\mathcal{G}}(\mathcal{P},\mathcal{F})(U) \cong \mathrm{Hom}_{\mathcal{G}|_U}(\mathcal{P}|_U,\mathcal{F}|_U), \qquad U \in \mathfrak{T}_X.$$

5.3.10 Corollary. *The sheaves $\mathcal{H}om_{\mathcal{G}}(\mathcal{P},\mathcal{F})$ and $\mathcal{P} \times_X^{\mathcal{G}} \mathcal{F}$ coincide up to isomorphism.*

Proof. For every open $U \subseteq X$, we consider the map

$$T_U : \mathrm{Hom}_{\mathcal{G}|_U}(\mathcal{P}|_U,\mathcal{F}|_U) \xrightarrow{\simeq} (\mathcal{P} \times_X^{\mathcal{G}} \mathcal{F})(U),$$

obtained by localizing the bijection of Theorem 5.3.9. It is immediately verified that (T_U), with U running in $\mathfrak{T}_X$, is a presheaf isomorphism. Therefore, identifying $\mathcal{P} \times_X^{\mathcal{G}} \mathcal{F}$ with the sheaf of germs of its sections, we prove the assertion. □

Let us now take the particular case $\mathcal{F} = \mathcal{G}$, and consider the ***adjoint action*** of $\mathcal{G}$ on itself (from the left), namely

$$\mathrm{ad} : \mathcal{G} \times_X \mathcal{G} \longrightarrow \mathcal{G} : (a,b) \mapsto \mathrm{ad}(a)(b) := a \cdot b \cdot a^{-1}.$$

The sheaf associated with $\mathcal{P}$ by the adjoint action is denoted by

(5.3.21) $$ad(\mathcal{P}) \equiv \mathcal{P} \times_X^{\mathcal{G}} \mathcal{G}.$$

In this case, the tensorial morphisms $\tau : \mathcal{P} \to \mathcal{G}$, with respect to the action of $\mathcal{G}$ on the right of $\mathcal{P}$ and the adjoint action of $\mathcal{G}$ on itself, satisfy the equality

(5.3.22) $$\tau(p \cdot g) = \mathrm{ad}(g^{-1})(\tau(p)) = g^{-1} \cdot \tau(p) \cdot g.$$

Their set, denoted by $\mathrm{Hom}_{\mathrm{ad}}(\mathcal{P},\mathcal{G})$, is a group under the multiplication defined (point-wise) by

$$(\tau \cdot \tau')(p) := \tau(p) \cdot \tau'(p), \qquad p \in \mathcal{P}.$$

By an obvious localization, we obtain the groups $\mathrm{Hom}_{\mathrm{ad}}(\mathcal{P}|_U, \mathcal{G}|_U)$, $U \in \mathfrak{T}_X$, and the corresponding sheaf of germs $\mathcal{H}om_{\mathrm{ad}}(\mathcal{P}, \mathcal{G})$. Hence, as in Corollary 5.3.10, we have that

$$ad(\mathcal{P}) \cong \mathcal{H}om_{\mathrm{ad}}(\mathcal{P}, \mathcal{G}). \tag{5.3.21$'$}$$

Tensorial morphisms of the previous kind are related with gauge transformations of $\mathcal{P}$. The latter are defined as follows:

5.3.11 Definition. A ***gauge transformation*** of the principal sheaf $\mathcal{P} \equiv (\mathcal{P}, \mathcal{G}, X, \pi)$ is a $\mathcal{G}$-automorphism f of $\mathcal{P}$. The set of such transformations is a group, called the ***group of gauge transformations*** or the ***gauge group of*** $\mathcal{P}$. It is denoted by $GA(\mathcal{P})$.

We clarify that the group operation in $GA(\mathcal{P})$ is given by

$$(f, f') \longmapsto f \circ f'. \tag{5.3.23}$$

Note. The terminology of Definition 5.3.11, adopted here for its simplicity, originates from the gauge theory of physics. However, there is no general agreement regarding the original terminology. In this respect we refer to the comments of Bleecker [10, p. 46] and the definition of Sokolovsky [115, p. 2525]. The notation is taken from [10].

5.3.12 Proposition. *The groups* $GA(\mathcal{P})$ *and* $\mathrm{Hom}_{\mathrm{ad}}(\mathcal{P}, \mathcal{G})$ *are isomorphic.*

Proof. To a gauge transformation f we assign the map $\tau : \mathcal{P} \to \mathcal{G}$, determined by

$$f(p) = p \cdot \tau(p), \qquad p \in \mathcal{P}. \tag{5.3.24}$$

For each $p \in \mathcal{P}$, $\tau(p)$ is uniquely defined by Proposition 4.1.2, thus τ is a well defined map commuting with the projections of $\mathcal{P}$ and $\mathcal{G}$. Its continuity follows from equality $\tau = \boldsymbol{k} \circ (id_{\mathcal{P}}, f)$, which is equivalent to (5.3.24). Hence, τ is a morphism of sheaves. On the other hand, for every $p \in \mathcal{P}$ and $g \in \mathcal{G}$, Proposition 4.1.4 yields

$$\begin{aligned}
\tau(p \cdot g) &= \boldsymbol{k}(p \cdot g, f(p \cdot g)) = \boldsymbol{k}(p \cdot g, f(p) \cdot g) \\
&= g^{-1} \cdot \boldsymbol{k}(p, f(p)) \cdot g = g^{-1} \cdot \tau(p) \cdot g \\
&= \mathrm{ad}(g^{-1})(\tau(p)),
\end{aligned}$$

which proves the tensoriality of τ.

The correspondence $GA(\mathcal{P}) \ni f \mapsto \tau \in \mathrm{Hom}_{\mathrm{ad}}(\mathcal{P}, \mathcal{G})$ is injective, as a consequence of (5.3.24).

Conversely, given a τ as before, we define a morphism $f : \mathcal{P} \to \mathcal{P}$ by equality (5.3.24). Since $f = \gamma \circ (id_{\mathcal{P}}, \tau)$, where γ is the operation of multiplication in $\mathcal{G}$, we obtain a morphism of sheaves. In addition,

$$f(p \cdot g) = p \cdot g \cdot \tau(p \cdot g) = p \cdot \tau(p) \cdot g = f(p) \cdot g,$$

i.e., f is a gauge transformation (see also Theorem 4.2.4).

Finally, if we assume that $f_1 \mapsto \tau_1$, $f_2 \mapsto \tau_2$ and $f_1 \circ f_2 \mapsto \tau$, then

$$p \cdot \tau(p) = (f_1 \circ f_2)(p) = f_1(p \cdot \tau_2(p)) = f_1(p) \cdot \tau_2(p) = p \cdot \tau_1(p) \cdot \tau_2(p),$$

for every $p \in \mathcal{P}$; that is, $\tau = \tau_1 \cdot \tau_2$. Hence, $f_1 \circ f_2 \mapsto \tau_1 \cdot \tau_2$, which completes the proof. □

5.3.13 Remark. The isomorphism of the preceding proposition may be thought of as an *anti-isomorphism*, if instead of (5.3.23) we consider the multiplication $(f, f') \mapsto f' \circ f$ (see Bleecker [10, Theorem 3.3.2]).

Localizing the isomorphism of Proposition 5.3.12, we get a family of isomorphisms

$$G_U : GA(\mathcal{P}|_U) \xrightarrow{\simeq} \mathrm{Hom}_{\mathrm{ad}}(\mathcal{P}|_U, \mathcal{G}|_U), \tag{5.3.25}$$

with U running the topology of X. This yields a presheaf isomorphism. Thus, if we define the ***sheaf (of germs) of gauge transformations of*** $\mathcal{P}$ to be the sheafification of (5.3.25); that is,

$$\mathcal{GA}(\mathcal{P}) := \mathbf{S}(U \longrightarrow GA(\mathcal{P}|_U)); \qquad U \in \mathfrak{T}_X,$$

we have, in conjunction with (5.3.21′), the following result:

5.3.14 Corollary. *There exists an isomorphism of sheaves of groups*

$$G : \mathcal{GA}(\mathcal{P}) \xrightarrow{\simeq} \mathcal{H}om_{\mathrm{ad}}(\mathcal{P}, \mathcal{G}) \cong ad(\mathcal{P}).$$

Proof. The isomorphism G is generated by the presheaf isomorphism (G_U) defined by (5.3.25). □

5.4. Associated sheaves: particular cases

Here, the general construction of the previous section is specialized to actions defined by morphisms of $\mathcal{G}$ into (Lie) sheaves of groups, or, in particular, by representations of $\mathcal{G}$ into certain (convenient) sheaves. In this way, a vector sheaf is associated with its sheaf of frames by the trivial representation of the general linear sheaf group.

> With the exception of the final case (d), as in Section 5.3, we assume throughout that the structure sheaf $\mathcal{G}$ of $\mathcal{P}$ is only a sheaf of groups.

(a) Associated sheaves from morphisms of sheaves of groups

We consider a morphism of sheaves of groups

$$\varphi : \mathcal{G} \longrightarrow \mathcal{H}. \tag{5.4.1}$$

Following the preliminary notations of Section 5.3, the morphism φ determines the action of $\mathcal{G}$ on the left of $\mathcal{H}$

$$\delta_H : \mathcal{G} \times_X \mathcal{H} \longrightarrow \mathcal{H} : (g,h) \mapsto \delta(g,h) := \varphi(g) \cdot h,$$

and the action on the right of $\mathcal{P} \times_X \mathcal{H}$

$$(\mathcal{P} \times_X \mathcal{H}) \times_X \mathcal{G} \longrightarrow \mathcal{P} \times_X \mathcal{H},$$

given by

$$(p,h) \cdot g := \big(p \cdot g, \varphi(g^{-1}) \cdot h\big). \tag{5.4.2}$$

The presheaf

$$U \longmapsto Q(U) := (\mathcal{P}(U) \times \mathcal{H}(U))/\mathcal{G}(U); \qquad U \in \mathfrak{T}_X,$$

where the quotients are defined with respect to (the localization of) (5.4.2) (see the analog of (5.3.2)), determines a sheaf. This is, by definition, the ***sheaf associated with $\mathcal{P}$ by the morphism of sheaves of groups*** φ, denoted by $\varphi(\mathcal{P})$. In virtue of Corollary 5.3.6 we have that

$$\varphi(\mathcal{P}) \cong (\mathcal{P} \times_X \mathcal{H})/\mathcal{G} \equiv \mathcal{P} \times_X^{\mathcal{G}} \mathcal{H}. \tag{5.4.3}$$

The fact that the structure type of $\varphi(\mathcal{P})$ is the sheaf of groups $\mathcal{H}$ leads to the following variation of Theorem 5.3.2.

5.4.1 Proposition. *$\varphi(\mathcal{P})$ is an $\mathcal{H}$-principal sheaf; in other words, $\varphi(\mathcal{P}) \equiv (\varphi(\mathcal{P}), \mathcal{H}, X, \bar{\pi})$. Moreover, there exists a canonical morphism (of principal sheaves)*

$$(\varepsilon, \varphi, id_X) : (\mathcal{P}, \mathcal{G}, X, \pi) \longrightarrow (\varphi(\mathcal{P}), \mathcal{H}, X, \bar{\pi}).$$

Proof. First we obtain a (right) action $\delta_{\varphi(\mathcal{P})} : \varphi(\mathcal{P}) \times_X \mathcal{H} \longrightarrow \phi(\mathcal{P})$: For an open $U \subseteq X$, using the presheaf generating $\varphi(\mathcal{P})$, we define the map

$$\delta^U_{\varphi(\mathcal{P})} : Q(U) \times \mathcal{H}(U) \longrightarrow Q(U),$$

by setting (see also (5.3.4))

$$\delta^U_{\varphi(\mathcal{P})}\big([(s,h)]_U, h'\big) \equiv [(s,h)]_U \cdot h' := [(s, h \cdot h')]_U. \tag{5.4.4}$$

In virtue of (5.4.2), this is a well defined local action; hence, varying U in $\mathfrak{T}_X$, we obtain a presheaf morphism generating $\delta_{\varphi(\mathcal{P})}$.

In our case, the isomorphisms (5.3.8), describing the local structure of $\varphi(\mathcal{P})$, have the form

$$\underline{\Psi}_{\alpha,V} : \mathcal{H}(V) \longrightarrow Q(V), \quad \text{with} \quad \underline{\Psi}_{\alpha,V}(h) := [(s_\alpha|_V, h)]_V,$$

and are $\mathcal{H}(V)$-equivariant with respect to the action (5.4.4). Therefore, the corresponding coordinates $\underline{\Psi}_\alpha : \mathcal{H}|_{U_\alpha} \to \varphi(\mathcal{P})|_{U_\alpha}$ and $\underline{\Phi}_\alpha = \underline{\Psi}_\alpha^{-1}$ are $\mathcal{H}|_{U_\alpha}$-equivariant. This proves that $\varphi(\mathcal{P})$ is an $\mathcal{H}$-principal sheaf.

For the second result we define the map

$$\varepsilon_U : \mathcal{P}(U) \longrightarrow Q(U) : s \mapsto \varepsilon_U(s) := [(s, \mathbf{1}|_U)]_U,$$

$\mathbf{1}$ denoting the unit section of $\mathcal{H}$. Then, for every $g \in \mathcal{G}(U)$, (5.4.4) yields

$$\begin{aligned} \varepsilon_U(s \cdot g) &= [(s \cdot g, \mathbf{1}|_U)]_U = [(s, \varphi(g))]_U \\ &= [(s, \mathbf{1}|_U)]_U \cdot \varphi(g) = \varepsilon_U(s) \cdot \varphi_U(g). \end{aligned}$$

The desired morphism ε is generated by the presheaf morphism (ε_U). □

5.4.2 Remarks. 1) If we assume that $\mathcal{G} \equiv (\mathcal{G}, \mathcal{L}_\mathcal{G}, \rho_\mathcal{G}, \partial_\mathcal{G})$, $\mathcal{H} \equiv (\mathcal{H}, \mathcal{L}_\mathcal{H}, \rho_\mathcal{H}, \partial_\mathcal{H})$ are Lie sheaves of groups, and $(\varphi, \overline{\varphi})$ is a morphism of $\mathcal{G}$ into $\mathcal{H}$ (see Definition 3.4.1), then Proposition 5.4.1 is a fortiori valid, and the canonical morphism of the statement has the form $(\varepsilon, \varphi, \overline{\varphi}, id_X)$.

2) If we consider the quotient sheaf $\mathcal{P} \times^{\mathcal{G}}_X \mathcal{H}$ (see the discussion preceding Proposition 5.3.4), we can define an action of $\mathcal{H}$ on this quotient by setting, stalk-wise,

$$[(p,h)] \cdot h' := [(p, h \cdot h')],$$

for every $(p, h, h') \in \mathcal{P}_x \times \mathcal{H}_x \times \mathcal{H}_x$ and $x \in X$. In conjunction with the same Proposition 5.3.4, it can be shown that the aforementioned quotient sheaf is a principal sheaf of the described type. This approach, however, is a bit more complicated than the one given in the proof of Proposition 5.4.1.

3) In the context of Remark 2, we define a morphism

$$\bar{\varepsilon} : \mathcal{P} \longrightarrow \mathcal{P} \times_X^{\mathcal{G}} \mathcal{H} : p \mapsto \bar{\varepsilon}(p) := [(p, e_x)],$$

if $p \in \mathcal{P}_x$ and e_x is the neutral element of $\mathcal{H}_x$. As expected, the morphisms $\bar{\varepsilon}$ and ε are related by

$$\bar{\varepsilon} = f \circ \varepsilon \tag{5.4.5}$$

(see also Diagram 5.4), where f is the isomorphism (5.4.3), explicitly constructed in Corollary 5.3.6.

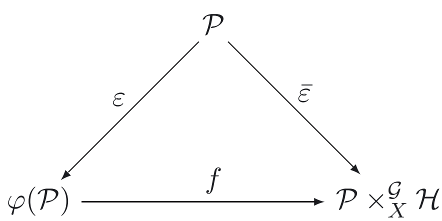

DIAGRAM 5.4

The proof goes as follows: For any $p \in \mathcal{P}$ with $\pi(p) = x \in U_\alpha$, we can find an open $V \subseteq U_\alpha$ and a section $s \in \mathcal{P}(V)$ such that $s(x) = p$. Then

$$\varepsilon(p) = \widetilde{\varepsilon_V(s)}(x) = \big([(s, \mathbf{1}|_V)]_V\big)^{\sim}(x) \in \varphi(\mathcal{P})_x.$$

Thus, as in the proof of Corollary 5.3.6,

$$f(\varepsilon(p)) = f_\alpha(\varepsilon(p)) = \big(\widetilde{\Psi}_\alpha \circ \underline{\Phi}_\alpha\big)(\varepsilon(p)),$$

where now $\underline{\Phi}_\alpha : \varphi(\mathcal{P})|_{U_\alpha} \to \mathcal{G}|_{U_\alpha}$ and $\widetilde{\Psi}_\alpha = \widetilde{\Phi}_\alpha^{-1} : \mathcal{G}|_{U_\alpha} \to (\mathcal{P} \times_X^{\mathcal{G}} \mathcal{H})\big|_{U_\alpha}$ (see Theorems 5.3.2 and 5.3.5, respectively).

On the other hand, the definition of the inverse of (5.3.8), adapted to the present data, yields

$$\underline{\Phi}_\alpha(\varepsilon(p)) = \big(\underline{\Phi}_{\alpha,V}([(s, \mathbf{1}|_V)])\big)^{\sim}(x) = \widetilde{g}(x) \equiv g(x),$$

(after the identification of $\mathcal{G}$ with the sheaf of germs of its sections), where $g \in \mathcal{G}(V)$ is defined by $s = s_\alpha|_V \cdot g$. Therefore, the previous equalities, along with (5.3.16), imply that

$$\begin{aligned} f(\varepsilon(p)) &= \widetilde{\Psi}_\alpha\big(\underline{\Phi}_\alpha(\varepsilon(p))\big) = \widetilde{\Psi}_\alpha(g(x)) \\ &= [(s_\alpha(x), g(x))] = [(s(x) \cdot g(x)^{-1},\, g(x))] \\ &= [(s(x), e_x) \cdot g(x)^{-1}] = [(p, e_x)] \\ &= \bar{\varepsilon}(p), \end{aligned}$$

which gives (5.4.5).

For later use, we prove the following useful result:

5.4.3 Corollary. *Let $\mathcal{U} = (U_\alpha)$ be an open covering of X carrying the local frames of $\mathcal{P}$ and $\varphi(\mathcal{P})$. If $(s_\alpha^{\varphi(\mathcal{P})}) \in C^0(\mathcal{U}, \mathcal{H})$ is the 0-cochain of the natural local sections of $\varphi(\mathcal{P})$, and $(g_{\alpha\beta}^{\varphi(\mathcal{P})}) \in Z^1(\mathcal{U}, \mathcal{H})$ is the corresponding cocycle, then*

$$s_\alpha^{\varphi(\mathcal{P})} = \varepsilon(s_\alpha), \tag{5.4.6}$$

$$g_{\alpha\beta}^{\varphi(\mathcal{P})} = \varphi(g_{\alpha\beta}), \tag{5.4.7}$$

for every $\alpha, \beta \in I$.

Proof. We handle $\varphi(\mathcal{P})$ as in Proposition 5.4.1. The first equality is a consequence of Definition 4.1.6 and (5.3.8). Indeed, based on Diagram 1.7, equality (1.2.9), and convention 1.1.3, we have that

$$\begin{aligned} s_\alpha^{\varphi(\mathcal{P})} &:= \underline{\Psi}_\alpha^{-1}(\mathbf{1}|_{U_\alpha}) = \big(\underline{\Psi}_{\alpha,U_\alpha}^{-1}(\mathbf{1}|_{U_\alpha})\big)^{\sim} \\ &= \big([(s_\alpha, \mathbf{1}|_{U_\alpha})]_{U_\alpha}\big)^{\sim} = \big(\varepsilon_{U_\alpha}(s_\alpha)\big)^{\sim} \\ &= \varepsilon(s_\alpha), \end{aligned}$$

where, in virtue of (1.2.15), we have applied the identifications $\mathbf{1}|_{U_\alpha} \equiv \widetilde{\mathbf{1}|_{U_\alpha}}$ and $s_\alpha \equiv \widetilde{s_\alpha}$.

On the other hand, each transition section $g_{\alpha\beta}^{\varphi(\mathcal{P})}$ is determined by the equality $s_\beta^{\varphi(\mathcal{P})} = s_\alpha^{\varphi(\mathcal{P})} \cdot g_{\alpha\beta}^{\varphi(\mathcal{P})}$ or, equivalently, by $\varepsilon(s_\beta) = \varepsilon(s_\alpha) \cdot g_{\alpha\beta}^{\varphi(\mathcal{P})}$. Since ε is a morphism of principal sheaves, (4.2.1) and (5.4.6) imply that

$$\varepsilon(s_\alpha) \cdot \varphi(g_{\alpha\beta}) = \varepsilon(s_\alpha \cdot g_{\alpha\beta}) = \varepsilon(s_\beta) = \varepsilon(s_\alpha) \cdot g_{\alpha\beta}^{\varphi(\mathcal{P})},$$

from which (5.4.7) follows. $\square$

5.4.4 Remark. For the sake of completeness, we note that (5.4.7) can be obtained directly from Proposition 5.3.3, for $\mathcal{F} = \mathcal{H}$, the action of $\mathcal{G}$ on $\mathcal{H}$ now being given by δ_H as in the beginning of the present section. As a matter of fact, denoting by e_x the neutral element of $\mathcal{H}_x$, we have that

$$g_{\alpha\beta}^{\varphi(\mathcal{P})}(x) = (\underline{\Phi}_\alpha \circ \underline{\Phi}_\beta^{-1})(e_x) = \varphi(g_{\alpha\beta}(x)) \cdot e_x = \varphi(g_{\alpha\beta}(x)),$$

for every $x \in U_{\alpha\beta}$.

(b) Associated sheaves from representations on $\mathcal{A}$-modules

Here we consider *a representation of $\mathcal{G}$ on an $\mathcal{A}$-module $\mathcal{S}$*; that is, a morphism of sheaves of groups

$$\eta : \mathcal{G} \longrightarrow \mathcal{A}ut(\mathcal{S}) \equiv \mathcal{A}ut_{\mathcal{A}}(\mathcal{S}). \tag{5.4.8}$$

From Subsection 1.3.5, we recall that the range of η is the sheaf of germs of $\mathcal{A}$-automorphisms of $\mathcal{S}$. A representation η is equivalent to a left action $\delta_{\mathcal{S}} : \mathcal{G} \times_X \mathcal{S} \longrightarrow \mathcal{S}$ of $\mathcal{G}$ on $\mathcal{S}$ (see the beginning of Section 3.3 and Proposition 3.3.1), generated by the local actions

$$\delta_{\mathcal{S},U}(g, \sigma) \equiv g \cdot \sigma := \eta(g) \circ \sigma,$$

for every $g \in \mathcal{G}(U)$ and $\sigma \in \mathcal{S}(U)$, with U running in $\mathfrak{T}_X$. We also recall that $\eta(g) \in \mathcal{A}ut(\mathcal{S})(U) \cong \mathrm{Aut}_{\mathcal{A}|_U}(\mathcal{S}|_U)$ (see (1.3.13)). Analogously to (5.4.2), $\delta_{\mathcal{S}}$ defines a right action of $\mathcal{G}$ on $\mathcal{P} \times_X \mathcal{S}$. Thus, as in the case (a), we obtain the corresponding associated sheaf

$$\mathcal{M} := \eta(\mathcal{P}) \cong (\mathcal{P} \times_X \mathcal{S})/\mathcal{G} \equiv \mathcal{P} \times_X^{\mathcal{G}} \mathcal{S}, \tag{5.4.9}$$

with projection denoted by π_M. Thus, $\mathcal{M} \equiv (\mathcal{M}, \pi_M, X)$.

Anticipating the next result, we call $\mathcal{M}$ ***the $\mathcal{A}$-module associated with $\mathcal{P}$ by the representation*** η.

5.4.5 Proposition. *The sheaf $\mathcal{M} = \eta(\mathcal{P})$, associated with $\mathcal{P}$ by a representation of $\mathcal{G}$ on an $\mathcal{A}$-module $\mathcal{S}$, is also an $\mathcal{A}$-module, locally $\mathcal{A}$-isomorphic with $\mathcal{S}$. In particular, $\mathcal{M}$ is of structure type $\mathcal{S}$.*

Proof. Working as in the proofs of Theorem 5.3.2 and Proposition 5.4.1, we define the coordinates

$$\underline{\Phi}_\alpha : \mathcal{M}|_{U_\alpha} \xrightarrow{\ \simeq\ } \mathcal{S}|_{U_\alpha}; \qquad \alpha \in I. \tag{5.4.10}$$

In virtue of (5.4.7), they determine the cocycle $(g^{\mathcal{M}}_{\alpha\beta})$ of $\mathcal{M}$, with

$$g^{\mathcal{M}}_{\alpha\beta} = \eta(g_{\alpha\beta}) = \underline{\Phi}_\alpha \circ \underline{\Phi}_\beta^{-1}, \tag{5.4.11}$$

after the identification $\mathcal{A}ut(\mathcal{S})(U_{\alpha\beta}) \cong \mathrm{Aut}_{\mathcal{A}|_{U_{\alpha\beta}}}(\mathcal{S}|_{U_{\alpha\beta}})$.

We define the structure of an $\mathcal{A}$-module on $\mathcal{M}$ by setting

$$u + v := \underline{\Phi}_\alpha^{-1}(\underline{\Phi}_\alpha(u) + \underline{\Phi}_\alpha(v)), \tag{5.4.12}$$

$$\lambda \cdot u := \underline{\Phi}_\alpha^{-1}\big(\lambda \cdot \underline{\Phi}_\alpha(u)\big), \tag{5.4.13}$$

for every $u, v \in \mathcal{M}_x$, $\lambda \in \mathcal{A}_x$, with $x \in U_\alpha$. The previous operations are independent of the choice of the particular U_α containing x. Indeed, this amounts to showing that $(\underline{\Phi}_\alpha \circ \underline{\Phi}_\beta^{-1}) \in \mathrm{Aut}_{\mathcal{A}|_{U_{\alpha\beta}}}(\mathcal{S}|_{U_{\alpha\beta}})$, a fact being true according to (5.4.11).

The two operations are continuous. This is proved by showing that their restrictions over each U_α, namely the maps

$$\begin{aligned} + \ &: \ \mathcal{M}|_{U_\alpha} \times_{U_\alpha} \mathcal{M}|_{U_\alpha} \longrightarrow \mathcal{M}|_{U_\alpha}, \\ \cdot \ &: \ \mathcal{A}|_{U_\alpha} \times_{U_\alpha} \mathcal{M}|_{U_\alpha} \longrightarrow \mathcal{M}|_{U_\alpha} \end{aligned}$$

are continuous. This is the case, since the expressions (5.4.12) and (5.4.13) involve the homeomorphism $\underline{\Phi}_\alpha$ and the respective operations of $\mathcal{S}|_{U_\alpha}$, which are already continuous. From (5.4.10) – (5.4.13) it follows that the coordinates are $\mathcal{A}|_{U_\alpha}$-isomorphisms and $\mathcal{M}$ is of structure type $\mathcal{S}$. □

(c) Associated sheaves from representations on free modules

We apply the previous case (b) to a free module of rank n, i.e., we assume that $\mathcal{S} = \mathcal{A}^n$. By (5.1.14), we have the identification

$$\mathcal{A}ut(\mathcal{A}^n) \equiv \mathcal{A}ut_{\mathcal{A}}(\mathcal{A}^n) \cong \mathcal{GL}(n, \mathcal{A}),$$

therefore, a representation of $\mathcal{G}$ on $\mathcal{A}^n$ can be thought of as a morphism of sheaves of groups

$$\zeta : \mathcal{G} \longrightarrow \mathcal{GL}(n, \mathcal{A}), \tag{5.4.14}$$

which determines the associated sheaf $(\mathcal{E}, \pi_{\mathcal{E}}, X)$, where

$$\mathcal{E} := \zeta(\mathcal{P}) \cong (\mathcal{P} \times_X \mathcal{A}^n)/\mathcal{G} \equiv \mathcal{P} \times^{\mathcal{G}}_X \mathcal{A}^n. \tag{5.4.15}$$

As a consequence of Proposition 5.4.5, we have:

5.4.6 Corollary. *The sheaf $\mathcal{E} \equiv \zeta(\mathcal{P})$, associated with $\mathcal{P}$ by a representation of $\mathcal{G}$ on $\mathcal{A}^n$, is a vector sheaf of rank n.*

Proof. In virtue of Proposition 5.4.5, it suffices to observe that the local coordinates are now maps of the form

$$\underline{\Phi}_\alpha : \mathcal{E}|_{U_\alpha} \xrightarrow{\simeq} \mathcal{A}^n|_{U_\alpha}; \qquad \alpha \in I,$$

which are $\mathcal{A}|_{U_\alpha}$-isomorphisms. □

We call $\mathcal{E} \equiv \zeta(\mathcal{P})$ the ***vector sheaf associated with $\mathcal{P}$ by the representation*** $\zeta : \mathcal{G} \to \mathcal{GL}(n, \mathcal{A})$.

For later use, we note that the analog of (5.4.11) is given by

$$g^{\mathcal{E}}_{\alpha\beta} = \zeta(g_{\alpha\beta}) = \underline{\Phi}_\alpha \circ \underline{\Phi}_\beta^{-1}. \tag{5.4.16}$$

5.4.7 Remark (on the physics jargon). If (P, G, X, π) is a principal bundle, V a (finite-dimensional) vector space and $G \to \mathrm{GL}(V)$ a representation of G on V, then a tensorial map $P \to V$ is called ***particle field*** (see Bleecker [10, p. 43]) or ***matter field*** (see Naber [81, p. 51]).

The same terminology can be applied to the case of tensorial morphisms $\tau : \mathcal{P} \to \mathcal{S}$ (see Definition 5.3.8), if $\mathcal{S}$ is an $\mathcal{A}$-module, or $\mathcal{S} = \mathcal{A}^n$. By the general Theorem 5.3.9, such tensorial morphisms correspond bijectively to the sections of the $\mathcal{A}$-module $\mathcal{M}$ (defined in case 5.4(b)), if $\mathcal{S}$ is an $\mathcal{A}$-module, or to the sections of the vector sheaf $\mathcal{E}$ (obtained in case 5.4(c)), if $\mathcal{S} = \mathcal{A}^n$.

(d) The adjoint sheaf $\rho(\mathcal{P})$

Let $\mathcal{P} \equiv (\mathcal{P}, \mathcal{G}, X, \pi)$ be a principal sheaf where $\mathcal{G}$ is now a *Lie sheaf of groups*, i.e, $\mathcal{G} \equiv (\mathcal{G}, \rho, \mathcal{L}, \partial)$. By definition, $\mathcal{G}$ is provided with a representation $\rho : \mathcal{G} \to \mathcal{A}ut(\mathcal{L})$. Therefore, ρ induces an associated sheaf

$$\rho(\mathcal{P}) \cong (\mathcal{P} \times_X \mathcal{L})/\mathcal{G} \equiv \mathcal{P} \times^{\mathcal{G}}_X \mathcal{L},$$

called the ***ρ-adjoint sheaf*** of $\mathcal{P}$. If ρ is fixed and there is no danger of confusion, we simply say that $\rho(\mathcal{P})$ is the ***adjoint sheaf*** of $\mathcal{P}$. This terminology is influenced by the classical *adjoint bundle* associated with a principal bundle by the adjoint representation of the structure group.

The sheaf $\rho(\mathcal{P})$ is of structure type $\mathcal{L}$, thus there exist an open covering (U_α) of X and $\mathcal{A}|_{U_\alpha}$-isomorphisms (coordinates)

$$\underline{\Phi}_\alpha : \rho(\mathcal{P})|_{U_\alpha} \xrightarrow{\simeq} \mathcal{L}|_{U_\alpha}, \qquad \alpha \in I. \tag{5.4.17}$$

By Proposition 5.3.3, the change of coordinates $\underline{\Phi}_\alpha \circ \underline{\Phi}_\beta^{-1} : \mathcal{L}|_{U_{\alpha\beta}} \to \mathcal{L}|_{U_{\alpha\beta}}$ takes the form

$$(\underline{\Phi}_\alpha \circ \underline{\Phi}_\beta^{-1})(u) = g_{\alpha\beta}(x).u, \tag{5.4.18}$$

for every $u \in \mathcal{L}_x$ and $x \in U_{\alpha\beta}$. The action on the right-hand side of (5.4.18) is that induced by ρ (see the notations and conventions in the beginning of Section 3.3). In particular, the induced morphism of sections $\underline{\Phi}_\alpha \circ \underline{\Phi}_\beta^{-1} : \mathcal{L}(U_{\alpha\beta}) \to \mathcal{L}(U_{\alpha\beta})$ is calculated as follows: For every $\ell \in \mathcal{L}(U_{\alpha\beta})$ and every $x \in U_{\alpha\beta}$, (3.3.1′) implies that

$$\big((\underline{\Phi}_\alpha \circ \underline{\Phi}_\beta^{-1})(\ell)\big)(x) = (\underline{\Phi}_\alpha \circ \underline{\Phi}_\beta^{-1})(\ell(x)) = g_{\alpha\beta}(x).\ell(x) = \big(\rho(g_{\alpha\beta})(\ell)\big)(x);$$

in other words,

$$(\underline{\Phi}_\alpha \circ \underline{\Phi}_\beta^{-1})(\ell) = \rho(g_{\alpha\beta})(\ell), \tag{5.4.19}$$

for every $\ell \in \mathcal{L}(U_{\alpha\beta})$.

The same equalities hold for the change of coordinates $\widetilde{\Phi}_\alpha \circ \widetilde{\Phi}_\beta^{-1}$, if we think of $\rho(\mathcal{P})$ as the associated sheaf $\mathcal{P} \times_X^{\mathcal{G}} \mathcal{L}$ (see Theorem 5.3.5).

5.5. Interrelations with the sheaf of frames

We relate certain associated vector sheaves, obtained earlier, with the corresponding principal sheaves of frames.

We start with an *arbitrary vector sheaf* $\mathcal{E} \equiv (\mathcal{E}, \pi, X)$ of rank n. If $\mathcal{P}(\mathcal{E}) \equiv (\mathcal{P}(\mathcal{E}), \mathcal{GL}(n, \mathcal{A}), X, \tilde{\pi})$ is the principal sheaf of frames of $\mathcal{E}$, then, by what has been said in Section 5.4(c), the *trivial representation* of $\mathcal{GL}(n, \mathcal{A})$ on $\mathcal{A}^n$,

$$id_{\mathcal{GL}(n,\mathcal{A})} : \mathcal{GL}(n, \mathcal{A}) \longrightarrow \mathcal{GL}(n, \mathcal{A}), \tag{5.5.1}$$

determines the associated vector sheaf

$$\widetilde{\mathcal{E}} = (\mathcal{P}(\mathcal{E}) \times_X \mathcal{A}^n)/\mathcal{GL}(n, \mathcal{A}).$$

5.5.1 Proposition. *The sheaves $\widetilde{\mathcal{E}}$ and $\mathcal{E}$ are $\mathcal{A}$-isomorphic.*

Proof. Let $(\psi_{\alpha\beta}) \in Z^1(\mathcal{U}, \mathcal{GL}(n, \mathcal{A}))$ be the cocycle of $\mathcal{E}$, with respect to a local frame $(\mathcal{U}, (\psi_\alpha))$. Then, by Corollary 5.2.3, $(\psi_{\alpha\beta})$ is the cocycle of $\mathcal{P}(\mathcal{E})$. But, (5.4.7), applied to $\varphi = id_{\mathcal{GL}(n,\mathcal{A})}$ and the previous cocycle, shows that the cocycle of $\widetilde{\mathcal{E}}$ is the same $(\psi_{\alpha\beta})$. Therefore, Theorem 5.1.8 leads to the result. □

We rephrase the previous statement in the following useful form:

5.5.2 Corollary. *Every vector sheaf is associated with its principal sheaf of frames by the trivial representation of $\mathcal{GL}(n,\mathcal{A})$; hence,*

$$\mathcal{E} \cong (\mathcal{P}(\mathcal{E}) \times_X \mathcal{A}^n)/\mathcal{GL}(n,\mathcal{A}).$$

The next diagram illustrates Corollary 5.5.2.

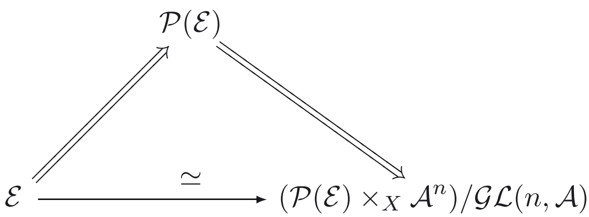

Diagram 5.5

The two double arrows simply indicate the transition from a vector sheaf to its sheaf of frames and back to the vector sheaf associated by the trivial representation, whereas the horizontal arrow represents the isomorphism of Proposition 5.5.1.

On many occasions it is desirable to define a concrete isomorphism of $\widetilde{\mathcal{E}} = (\mathcal{P}(\mathcal{E}) \times_X \mathcal{A}^n)/\mathcal{GL}(n,\mathcal{A})$ onto $\mathcal{E}$, whose existence is theoretically ensured by Corollary 5.5.2. To this end we apply Theorem 5.1.7.

More precisely, since $\widetilde{\mathcal{E}}$ and $\mathcal{E}$ have the same cocycle (see the proof of Proposition 5.5.1), we can take $h_\alpha = id : \mathcal{A}^n|_{U_\alpha} \to \mathcal{A}^n|_{U_\alpha}$, for all $\alpha \in I$; thus a vector sheaf isomorphism $R : \widetilde{\mathcal{E}} \to \mathcal{E}$ can be defined by the family of $\mathcal{A}|_{U_\alpha}$-isomorphisms $R_\alpha : \widetilde{\mathcal{E}}\,|_{U_\alpha} \to \mathcal{E}|_{U_\alpha}$, given by

$$R_\alpha = \psi_\alpha^{-1} \circ \underline{\Phi}_\alpha,$$

and which coincide on the overlappings. We recall that $\psi_\alpha : \mathcal{E}|_{U_\alpha} \to \mathcal{A}^n|_{U_\alpha}$ and $\underline{\Phi}_\alpha : \widetilde{\mathcal{E}}|_{U_\alpha} \to \mathcal{A}^n|_{U_\alpha}$ are the coordinates of $\mathcal{E}$ and $\widetilde{\mathcal{E}}$ (over U_α), respectively. The latter is obtained by the sheafification (see the proof of Theorem 5.3.2 adapted to the present setting) of the isomorphisms, for all V varying in U_α,

$$\underline{\Phi}_{\alpha,V} : \big(\mathrm{Iso}_{\mathcal{A}|_V}(\mathcal{A}^n|_V, \mathcal{E}|_V) \times \mathcal{A}^n(V)\big)/\mathcal{GL}(n,\mathcal{A})(V) \longrightarrow \mathcal{A}^n(V) :$$
$$[(\sigma,h)]_V \longmapsto \underline{\Phi}_{\alpha,V}([(\sigma,h)]_V) = g \cdot h \equiv g \circ h,$$

where $g \in \mathcal{GL}(n, \mathcal{A})(V) \cong \mathrm{Aut}_{\mathcal{A}|_V}(\mathcal{A}^n|_V)$ is given by $\sigma = \sigma_\alpha|_V \circ g \equiv \psi_\alpha^{-1} \circ g$, if σ_α is the natural section of $\mathcal{P}(\mathcal{E})$ over U_α (see also (5.2.5), and Proposition 5.2.2 for the definition of the action of $\mathcal{GL}(n, \mathcal{A})$ on $\mathcal{P}(\mathcal{E})$).

On the other hand, ψ_α^{-1} can be thought of as being generated by the induced morphisms of sections $\overline{(\psi_\alpha^{-1})}_V$, for all open $V \subseteq U_\alpha$; hence, R_α is generated by the sheafification of $\{R_{\alpha,V} := \overline{(\psi_\alpha^{-1})}_V \circ \underline{\Phi}_{\alpha,V} \mid V \subseteq U_\alpha \text{ open}\}$. Therefore, using the above equality determining g, we see that

$$R_{\alpha,V}([(\sigma, h)]_V) = \overline{(\psi_\alpha^{-1})}_V (g \circ h) := \psi_\alpha^{-1} \circ g \circ h = \sigma \circ h.$$

Summarizing, we have shown that

the vector sheaves $(\mathcal{P}(\mathcal{E}) \times_X \mathcal{A}^n)/\mathcal{GL}(n, \mathcal{A})$ and $\mathcal{E}$ become isomorphic by gluing together the isomorphisms (R_α) generated, in turn, by the isomorphisms

$$R_{\alpha,V} : \big(\mathrm{Iso}_{\mathcal{A}|_V}(\mathcal{A}^n|_V, \mathcal{E}|_V) \times \mathcal{A}^n(V)\big)/\mathcal{GL}(n, \mathcal{A})(V) \longrightarrow \mathcal{A}^n(V) : \quad [(\sigma, h)]_V \longmapsto \sigma \circ h, \tag{5.5.2}$$

for all open $V \subseteq U_\alpha$.

Now, starting with a principal sheaf $\mathcal{P} \equiv (\mathcal{P}, \mathcal{G}, X, \pi)$, we consider a *morphism of sheaves of groups* of the form

$$\varphi : \mathcal{G} \longrightarrow \mathcal{GL}(n, \mathcal{A}).$$

(In the case of connections on associated sheaves we shall need a morphism of Lie sheaves of groups $(\varphi, \bar{\varphi})$, with $\bar{\varphi} : \mathcal{L} \to \mathcal{M}_n(\mathcal{A})$. However, this is not necessary in the present considerations.)

According to the results of Section 5.4.(a) (see, in particular, (5.4.3) and Proposition 5.4.1), we obtain an associated principal sheaf, now denoted by

$$\mathcal{P}_\varphi \equiv (\mathcal{P}_\varphi, \mathcal{GL}(n, \mathcal{A}), X, \pi_\varphi), \tag{5.5.3}$$

(instead of $\varphi(\mathcal{P})$), where $\mathcal{P}_\varphi = (\mathcal{P} \times_X \mathcal{GL}(n, \mathcal{A}))/\mathcal{G} \equiv \mathcal{P} \times_X^{\mathcal{G}} \mathcal{GL}(n, \mathcal{A})$.

The same morphism, viewed as a *representation* of $\mathcal{G}$ on $\mathcal{A}^n$, determines the vector sheaf (see also (5.4.15))

$$\mathcal{E}_\varphi \cong (\mathcal{P} \times_X \mathcal{A}^n)/\mathcal{G}, \tag{5.5.4}$$

with corresponding sheaf of frames $\mathcal{P}(\mathcal{E}_\varphi) \equiv (\mathcal{P}(\mathcal{E}_\varphi), \mathcal{GL}(n, \mathcal{A}), X, \tilde{\pi})$. Obviously, the index φ is set to remind ourselves that the respective sheaves are constructed by means of the given morphism φ.

Our aim is to find the relationship between $\mathcal{P}_\varphi$ and $\mathcal{P}(\mathcal{E}_\varphi)$. To help the reader, we picture our quest in Diagram 5.6, where the double arrows indicate the transition from $\mathcal{P}$ to the indicated associated sheaves, while the question mark stands for the relationship we are looking for.

$$\begin{array}{ccc} \mathcal{P} & \Longrightarrow & \mathcal{P}_\varphi \\ \Big\Downarrow & & \vdots\ ? \\ \mathcal{E}_\varphi & \Longrightarrow & \mathcal{P}(\mathcal{E}_\varphi) \end{array}$$

DIAGRAM 5.6

Taking into account Remarks 4.1.5(2) and 4.2.2(1), we first prove the following:

5.5.3 Lemma. *Under the above notations, there exists a morphism of principal sheaves* $(F_{\mathcal{P}}, \varphi, id_X) : (\mathcal{P}, \mathcal{G}, X, \pi) \to (\mathcal{P}(\mathcal{E}_\varphi), \mathcal{GL}(n, \mathcal{A}), X, \tilde{\pi})$.

Proof. Let $(\mathcal{U}, (\phi_\alpha))$ be a local frame of the initial principal sheaf $\mathcal{P}$. For the sake of simplicity, and without loss of generality, we may assume that $\mathcal{U} = (U_\alpha)$ is a basis for the topology of X. Thus $\mathcal{P}$ can be identified with the sheaf generated by the presheaf of sections $\mathcal{P}(U_\alpha)$. Similarly, $\mathcal{P}(\mathcal{E}_\varphi)$ can be constructed as in Section 5.2, over the same basis of topology $\mathcal{U}$.

We define $F_{\mathcal{P}}$ to be the morphism generated by the presheaf morphism

$$\big\{F_{\mathcal{P},\alpha} : \mathcal{P}(U_\alpha) \longrightarrow \mathrm{Iso}_{\mathcal{A}|_{U_\alpha}}(\mathcal{A}^n|_{U_\alpha}, \mathcal{E}_\varphi|_{U_\alpha}) \,|\, U_\alpha \in \mathfrak{T}_X\big\},$$

obtained as follows: For an arbitrary $s \in \mathcal{P}(U_\alpha)$, the $\mathcal{A}|_{U_\alpha}$-isomorphism $F_{\mathcal{P},\alpha}(s) : \mathcal{A}^n|_{U_\alpha} \to \mathcal{E}_\varphi|_{U_\alpha}$ is generated by the presheaf isomorphism

$$\big\{F_{\mathcal{P},\alpha}(s)_V : \mathcal{A}^n(V) \xrightarrow{\ \simeq\ } (\mathcal{P}(V) \times \mathcal{A}^n(V))/\mathcal{G}(V) \,|\, V \subseteq U_\alpha \text{ open}\big\},$$

whose individual isomorphisms are determined by

$$F_{\mathcal{P},\alpha}(s)_V(\sigma) := [(s|_V, \sigma)]_V, \quad \sigma \in \mathcal{A}^n(V). \tag{5.5.5}$$

The equivalence class $[\ \ ]_V$ refers to the construction of $\mathcal{E}_\varphi$ as a sheaf associated with $\mathcal{P}$ (see (5.3.4)).

It is not difficult to see that every $F_{\mathcal{P},\alpha}(s)$ is an $\mathcal{A}^n|_{U_\alpha}$-isomorphism, thus $F_{\mathcal{P}}$ is a well defined morphism of sheaves of sets with domain and range as in the statement. Furthermore, we show that $F_{\mathcal{P}}$ is a morphism of principal sheaves by proving that

$$F_{\mathcal{P},\alpha}(s \cdot g) = F_{\mathcal{P},\alpha}(s) \cdot \varphi(g), \tag{5.5.6}$$

for every $s \in \mathcal{P}(U_\alpha)$ and $g \in \mathcal{G}(U_\alpha)$, where φ now is the induced morphism of sections over U_α.

Indeed, for a fixed open $V \subseteq U_\alpha$, and for every $\sigma \in \mathcal{A}^n(V)$, we first see that

$$\begin{gathered} F_{\mathcal{P},\alpha}(s \cdot g)_V(\sigma) = [((s \cdot g)|_V, \sigma)]_V = \\ [(s|_V,\, \varphi(g|_V) \circ \sigma)]_V = F_{\mathcal{P},\alpha}(s)_V\big(\varphi(g|_V) \circ \sigma\big). \end{gathered} \tag{5.5.7}$$

Since each automorphism $\varphi(g|_V) \in \mathcal{GL}(n,\mathcal{A})(V) \cong \mathrm{Aut}_{\mathcal{A}|_V}(\mathcal{A}^n|_V)$ is also generated by the corresponding family of automorphisms of sections

$$\big\{\, \overline{\varphi(g|_V)}_W : \mathcal{A}^n(W) \xrightarrow{\ \simeq\ } \mathcal{A}^n(W) \mid W \subseteq V \text{ open}\big\}$$

(see (1.2.17)), the definition of the induced morphism of sections implies that

$$\varphi(g|_V)(\sigma) \equiv \overline{\varphi(g|_V)}_V(\sigma) = \varphi(g|_V) \circ \sigma;$$

hence, (5.5.7) is transformed into

$$F_{\mathcal{P},\alpha}(s \cdot g)_V(\sigma) = F_{\mathcal{P},\alpha}(s)_V\big(\varphi(g|_V)(\sigma)\big) = \big(F_{\mathcal{P},\alpha}(s)_V \circ \varphi(g|_V)\big)(\sigma),$$

for every $\sigma \in \mathcal{A}^n(V)$, from which we obtain

$$F_{\mathcal{P},\alpha}(s \cdot g)_V = F_{\mathcal{P},\alpha}(s)_V \circ \varphi(g|_V). \tag{5.5.8}$$

Because the morphisms of presheaves of sections

$$\{F_\alpha(s \cdot g)_V\}_{V \subseteq U_\alpha}, \quad \{F_\alpha(s)_V\}_{V \subseteq U_\alpha}, \quad \{\varphi(g|_V)\}_{V \subseteq U_\alpha}$$

(for all open $V \subseteq U_\alpha$) generate, respectively, the morphisms $F_\alpha(s \cdot g)$, $F_\alpha(s)$ and $\varphi(g)$, it follows that (5.5.8) implies equality (5.5.6).

Finally, we immediately check that $\tilde{\pi} \circ F_{\mathcal{P}} = \pi$, thus $(F_{\mathcal{P}}, \varphi, id_X)$ is a principal sheaf morphism. □

Note. If $\mathcal{G} \equiv (\mathcal{G}, \rho, \mathcal{L}, \partial)$ is a Lie sheaf of groups and we consider a morphism of Lie sheaves of groups $(\varphi, \overline{\varphi}) : (\mathcal{G}, \rho, \mathcal{L}, \partial) \to (\mathcal{GL}(n,\mathcal{A}), \mathcal{A}d, \mathcal{M}_n(\mathcal{A}), \widetilde{\partial})$, with $\overline{\varphi} : \mathcal{L} \to \mathcal{M}_n(\mathcal{A})$, then we obtain an analogous result with the morphism of principal sheaves now given by $(F_{\mathcal{P}}, \varphi, \overline{\varphi}, id_X)$. However, the morphism $\overline{\varphi}$ does not intervene in the construction of $F_{\mathcal{P}}$.

Preparing the study of connections on associated sheaves, we prove the following consequence of the local structure of the sheaves involved in the preceding Lemma.

5.5.4 Corollary. *Let (s_α) be the natural sections of $\mathcal{P}$, over a local frame $\mathcal{U}$, and let $\big(s_\alpha^{\mathcal{P}(\mathcal{E}_\varphi)}\big)$ be the natural sections of $\mathcal{P}(\mathcal{E}_\varphi)$. Then*

$$s_\alpha^{\mathcal{P}(\mathcal{E}_\varphi)} = F_{\mathcal{P}}(s_\alpha), \qquad \alpha \in I. \tag{5.5.9}$$

Proof. First, we recall that, in virtue of (5.2.6′),

$$s_\alpha^{\mathcal{P}(\mathcal{E}_\varphi)} = \widetilde{\psi_\alpha^{-1}} \in \mathcal{P}(\mathcal{E}_\varphi)(U_\alpha), \tag{5.5.10}$$

where $\psi_\alpha^{-1} : \mathcal{A}^n|_{U_\alpha} \to \mathcal{E}|_{U_\alpha}$ is (the inverse of) the coordinate of $\mathcal{E}_\varphi$ over U_α. Since $\mathcal{E}_\varphi$ is now associated with $\mathcal{P}$, the coordinate ψ_α^{-1} is generated by the presheaf isomorphism

$$\big\{\psi_{\alpha,V}^{-1} : \mathcal{A}^n(V) \longrightarrow (\mathcal{P}(V) \times \mathcal{A}^n(V))/\mathcal{G}(V) \,|\, V \subseteq U_\alpha \text{ open}\big\},$$

determined by

$$\psi_{\alpha,V}^{-1}(\sigma) = [(s_\alpha|_V, \sigma)]_V; \qquad \sigma \in \mathcal{A}^n(V), \tag{5.5.11}$$

(see (5.3.8) and the general construction of Theorem 5.3.2). Therefore, equalities (5.5.11), and (5.5.5) for $s = s_\alpha$, imply that

$$\psi_{\alpha,V}^{-1}(\sigma) = [(s_\alpha|_V, \sigma)]_V = F_{\mathcal{P},\alpha}(s_\alpha)_V(\sigma); \qquad \sigma \in \mathcal{A}^n(V),$$

that is, $\psi_{\alpha,V}^{-1} = F_{\mathcal{P},\alpha}(s_\alpha)_V$. Varying V in U_α, we obtain $\psi_\alpha^{-1} = F_{\mathcal{P},\alpha}(s_\alpha)$.

On the other hand, according to (1.1.3) and (1.2.13′), the induced morphisms of sections over U_α, $F_{\mathcal{P}} \equiv \overline{(F_{\mathcal{P}})}_{U_\alpha}$, connected with the presheaf morphism $(F_{\mathcal{P},\alpha})$ yields

$$F_{\mathcal{P}}(s_\alpha)(x) \equiv \overline{(F_{\mathcal{P}})}_{U_\alpha}(s_\alpha)(x) := F_{\mathcal{P}}(s_\alpha(x)) = \widetilde{F_{\mathcal{P},\alpha}(s_\alpha)}(x).$$

Consequently,

$$F_{\mathcal{P}}(s_\alpha) = \widetilde{F_{\mathcal{P},\alpha}(s_\alpha)} = \widetilde{\psi_\alpha^{-1}}$$

This, combined with (5.5.10), leads to the equality of the statement. □

The desired relationship between $\mathcal{P}_\varphi$ and $\mathcal{P}(\mathcal{E}_\varphi)$, represented by the question mark of Diagram 5.6, is now given by the following:

5.5.5 Theorem. *There exists a* $\mathcal{GL}(n,\mathcal{A})$*-isomorphism*

$$\theta \equiv (\theta, id_{\mathcal{GL}(n,\mathcal{A})}, id_X) : (\mathcal{P}_\varphi, \mathcal{GL}(n,\mathcal{A}), X, \pi_\varphi) \longrightarrow (\mathcal{P}(\mathcal{E}_\varphi), \mathcal{GL}(n,\mathcal{A}), X, \tilde{\pi})$$

satisfying the equality

$$\theta \circ \varepsilon = F_{\mathcal{P}}, \tag{5.5.12}$$

where $(F_{\mathcal{P}}, \varphi, id_x)$ *is the morphism of Lemma 5.5.3, and*

$$(\varepsilon, \varphi, id_X) : (\mathcal{P}, \mathcal{G}, X, \pi) \longrightarrow (\mathcal{P}_\varphi, \mathcal{GL}(n,\mathcal{A}), X, \pi')$$

is the canonical morphism defined by Proposition 5.4.1.

Equality (5.5.12) is shown in the commutative diagram below, which in fact completes the upper triangle of Diagram 5.6.

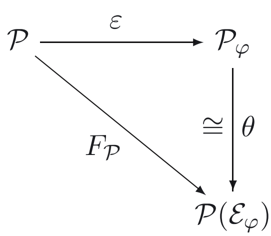

Diagram 5.7

Proof. Working as in the proof of Lemma 5.5.3, we can assume that the local frame $\mathcal{U}$ of $\mathcal{P}$ is a basis for the topology of X, so the sheaves $\mathcal{P}_\varphi$ and $\mathcal{P}(\mathcal{E}_\varphi)$ can be thought of as generated by the respective presheaves

$$U_\alpha \longmapsto (\mathcal{P}(U_\alpha) \times \mathcal{GL}(n,\mathcal{A})(U_\alpha))/\mathcal{G}(U_\alpha),$$
$$U_\alpha \longmapsto \mathrm{Iso}_{\mathcal{A}|_{U_\alpha}}(\mathcal{A}^n|_{U_\alpha}, \mathcal{E}_\varphi|_{U_\alpha}),$$

with U_α running in $\mathcal{U}$.

The isomorphism θ is generated by the presheaf isomorphism

$$\big\{\theta_\alpha : (\mathcal{P}(U_\alpha) \times \mathcal{GL}(n,\mathcal{A})(U_\alpha))/\mathcal{G}(U_\alpha) \longrightarrow \mathrm{Iso}_{\mathcal{A}|_{U_\alpha}}(\mathcal{A}^n|_{U_\alpha}, \mathcal{E}_\varphi|_{U_\alpha}) \mid \alpha \in I\big\},$$

determined in the following manner: Take any class

$$[(s,g)]^{\varphi}_{\alpha} \in (\mathcal{P}(U_\alpha) \times \mathcal{GL}(n,\mathcal{A})(U_\alpha))/\mathcal{G}(U_\alpha)$$

(note the use of the superscript φ and the subscript α in order to avoid confusion with classes of other quotients). Then the isomorphism $\theta_\alpha([(s,g)]^{\varphi}_{\alpha})$ is the one generated by the presheaf morphism (see the construction of $\mathcal{E}_\varphi$ as an associated sheaf)

$$\theta_\alpha\big([(s,g)]^{\varphi}_{\alpha}\big)_V : \mathcal{A}^n(V) \longrightarrow (\mathcal{P}(V) \times \mathcal{A}^n(V))/\mathcal{G}(V),$$

for all open $V \subseteq U_\alpha$, given in turn by

$$\theta_\alpha\big([(s,g)]^{\varphi}_{\alpha}\big)_V(\sigma) := [(s|_V,\, g|_V \circ \sigma)]_V; \quad \sigma \in \mathcal{A}^n(V),$$

the latter equivalence class belonging to the quotient generating $\mathcal{E}_\varphi$. It is easily seen that all the maps involved are well defined and lead to the desired isomorphism (of sheaves of sets).

To check the equivariance of θ, with respect to the given actions, we fix a U_α and an open $V \subseteq U_\alpha$. Then, for any $[(s,g)]^{\varphi}_{\alpha}$ and σ as before, and for any $g' \in \mathcal{GL}(n,\mathcal{A})(U_\alpha) \cong \mathrm{GL}(n,\mathcal{A}(U_\alpha))$, the analog of (5.4.4) for $\mathcal{H} = \mathcal{GL}(n,\mathcal{A})$ implies that

$$\begin{gathered}\theta_\alpha\big([(s,g)]^{\varphi}_{\alpha} \cdot g'\big)_V(\sigma) = \theta_\alpha\big([(s,g\cdot g')]^{\varphi}_{\alpha}\big)_V(\sigma) := \\ \theta_\alpha\big([(s,g\circ g')]^{\varphi}_{\alpha}\big)_V(\sigma) = \big[\big(s|_V,\, g|_V \circ (g|'_V \circ \sigma)\big)\big]_V = \\ \theta_\alpha\big([(s,g)]^{\varphi}_{\alpha}\big)_V(g'|_V \circ \sigma) = \big(\theta_\alpha\big([(s,g)]^{\varphi}_{\alpha}\big)_V \circ g'|_V\big)(\sigma),\end{gathered}$$

from which we get

$$\theta_\alpha\big([(s,g)]^{\varphi}_{\alpha} \cdot g'\big)_V = \theta_\alpha\big([(s,g)]^{\varphi}_{\alpha}\big)_V \circ g'|_V. \tag{5.5.13}$$

Varying V in U_α and taking into account the definition of the action of $\mathcal{GL}(n,\mathcal{A})$ on $\mathcal{P}(\mathcal{E}_\varphi)$ (see Proposition 5.2.2), equality (5.5.13) leads to

$$\theta_\alpha\big([(s,g)]^{\varphi}_{\alpha} \cdot g'\big) = \theta_\alpha\big([(s,g)]^{\varphi}_{\alpha}\big) \circ g' =: \theta_\alpha\big([(s,g)]^{\varphi}_{\alpha}\big) \cdot g',$$

i.e., each isomorphism θ_α is $\mathcal{GL}(n,\mathcal{A})(U_\alpha)$-equivariant. Thus, θ is $\mathcal{GL}(n,\mathcal{A})$-equivariant and determines an isomorphism of principal sheaves.

Finally, for any $s \in \mathcal{P}(U_\alpha)$, the definition of ε (see the second part of the proof of Proposition 5.4.1) implies that

$$(\theta_\alpha \circ \varepsilon_{U_\alpha})(s) = \theta_\alpha\big([(s,\mathbf{1})]^{\varphi}_{\alpha}\big), \quad (\mathbf{1} \equiv \mathbf{1}|_{U_\alpha} \in \mathcal{G}(U_\alpha)). \tag{5.5.14}$$

On the other hand, for each open $V \subseteq U_\alpha$ and $\sigma \in \mathcal{A}^n(V)$, (5.5.5) yields

$$\theta_\alpha\big([(s,\mathbf{1})]^\varphi_\alpha\big)_V(\sigma) = [(s|_V,\sigma)]_V =: F_{\mathcal{P},\alpha}(s)_V(\sigma).$$

Thus, taking all the open $V \subseteq U_\alpha$, we get $\theta_\alpha([(s,\mathbf{1})]^\varphi_\alpha) = F_{\mathcal{P},\alpha}(s)$. The last equality, combined with (5.5.14), implies that $\theta_\alpha \circ \varepsilon_{U_\alpha} = F_{\mathcal{P},\alpha}$, for every $\alpha \in I$. Hence, by sheafification (when U_α is varying in $\mathcal{U}$), we get (5.5.12) which completes the proof. □

We close the present section with the following result concerning natural sections, again anticipating its use in subsequent chapters.

5.5.6 Corollary. *The natural sections* $(s_\alpha^{\mathcal{P}(\mathcal{E}_\varphi)})$ *and* $(s_\alpha^{\mathcal{P}_\varphi})$ *of* $\mathcal{P}(\mathcal{E}_\varphi)$ *and* $\mathcal{P}_\varphi$*, respectively, are linked together by*

$$s_\alpha^{\mathcal{P}(\mathcal{E}_\varphi)} = \theta(s_\alpha^{\mathcal{P}_\varphi}), \qquad \alpha \in I. \tag{5.5.15}$$

Proof. Since the associated principal sheaf $\phi(\mathcal{P})$ of the general case of Proposition 5.4.1 is now denoted by $\mathcal{P}_\varphi$, equality (5.4.6) takes the form $s_\alpha^{\mathcal{P}_\varphi} = \varepsilon(s_\alpha)$. Therefore, (5.5.9) and (5.5.12) yield

$$s_\alpha^{\mathcal{P}(\mathcal{E}_\varphi)} = F_{\mathcal{P}}(s_\alpha) = \theta(\varepsilon(s_\alpha)) = \theta(s_\alpha^{\mathcal{P}_\varphi}).$$ □

5.6. Induced morphisms

The final section of this chapter briefly explains how morphisms of principal sheaves induce morphisms between various sheaves associated with the initial principal sheaves.

We fix two principal sheaves $\mathcal{P} \equiv (\mathcal{P},\mathcal{G},X,\pi)$ and $\mathcal{P}' \equiv (\mathcal{P}',\mathcal{G}',X,\pi')$, as well as two sheaves $\mathcal{F}$ and $\mathcal{F}'$ on which $\mathcal{G}$ and $\mathcal{G}'$ act, respectively, from the left. The corresponding associated sheaves are denoted by $\mathcal{Q} \equiv \mathcal{P} \times_X^{\mathcal{G}} \mathcal{F}$ and $\mathcal{Q}' \equiv \mathcal{P}' \times_X^{\mathcal{G}'} \mathcal{F}'$ (see (5.3.7), (5.3.13), and Corollary 5.3.6).

The general convention of Sections 5.3 and 5.4 is still in force; that is, $\mathcal{G}$ and $\mathcal{G}'$ are (for simplicity) only sheaves of groups, and morphisms of principal sheaves have the form (f,ϕ,id_X).

5.6.1 Definition. Let $\phi : \mathcal{G} \to \mathcal{G}'$ be a morphism of sheaves of groups. A morphism of sheaves $\ell : \mathcal{F} \to \mathcal{F}'$ is said to be ***compatible with*** ϕ if

$$\ell(g\cdot u) = \phi(g)\cdot \ell(u), \qquad (g,u) \in \mathcal{G}\times_X \mathcal{F}.$$

It is clear that the above compatibility condition amounts to the equivariance of ℓ with respect to the aforementioned actions and the morphism ϕ.

5.6.2 Proposition. *Let (f, ϕ, id_X) be a morphism of $\mathcal{P}$ into $\mathcal{P}'$ and let $\ell : \mathcal{F} \to \mathcal{F}'$ be a morphism compatible with ϕ. Then there exists a uniquely determined morphism*

$$F : \mathcal{P} \times_X^{\mathcal{G}} \mathcal{F} \longrightarrow \mathcal{P}' \times_X^{\mathcal{G}'} \mathcal{F}',$$

such that the diagram

$$\begin{array}{ccc} \mathcal{P} \times_X \mathcal{F} & \xrightarrow{\ f \times \ell\ } & \mathcal{P}' \times_X \mathcal{F}' \\ \Big\downarrow{\scriptstyle \kappa} & & \Big\downarrow{\scriptstyle \kappa'} \\ \mathcal{P} \times_X^{\mathcal{G}} \mathcal{F} & \xrightarrow{\quad F \quad} & \mathcal{P}' \times_X^{\mathcal{G}'} \mathcal{F}' \end{array}$$

DIAGRAM 5.8

is commutative, if κ and κ' are the canonical maps.

Proof. We view the associated sheaves as in the discussion before Theorem 5.3.2. Then, for any open $U \subseteq X$, we define the map

$$F_U : (\mathcal{P}(U) \times \mathcal{F}(U))/\mathcal{G}(U) \longrightarrow (\mathcal{P}'(U) \times \mathcal{F}'(U))/\mathcal{G}'(U),$$

by setting

$$F_U([(s,h)]_U) := [(f(s), \ell(h))]_U, \qquad (s,h) \in \mathcal{P}(U) \times \mathcal{F}(U)$$

(for convenience we use the same symbol for the equivalence classes in both the domain and the range of F_U).

The equivariance of f and the compatibility of ℓ with ϕ imply that F_U is well defined. Moreover, by the general considerations of Section 5.3, we see that (F_U) is a presheaf morphism. The desired morphism F is defined to be the morphism of sheaves generated by (F_U).

The uniqueness of F is a simple consequence of Diagram 5.8. $\square$

Note. Another way to define F is to set $F([(p,u)]) := [(f(p), \ell(u))]$, for every $[(p,u)] \in \mathcal{P} \times_X^{\mathcal{G}} \mathcal{F}$, the equivalence classes now being given as in the discussion before Proposition 5.3.4. The continuity of F is proved using the fact that κ is a local homeomorphism.

Thinking of a vector sheaf of rank n as associated with its sheaf of frames by the trivial representation of $\mathcal{GL}(n, \mathcal{A})$, Proposition 5.6.2 leads to:

5.6.3 Corollary. *Let $\mathcal{E}$ and $\mathcal{E}'$ be two vector sheaves over X, of rank m and n respectively, and let $(\mathcal{P}(\mathcal{E}), \mathcal{GL}(m, \mathcal{A}), X, \tilde{\pi})$, $(\mathcal{P}(\mathcal{E}'), \mathcal{GL}(n, \mathcal{A}), X, \tilde{\pi}')$ be the corresponding principal sheaves of frames. If (f, ϕ, id_X) is a morphism of $\mathcal{P}(\mathcal{E})$ into $\mathcal{P}(\mathcal{E}')$ and $\ell : \mathcal{A}^m \to \mathcal{A}^n$ an $\mathcal{A}$-morphism compatible with ϕ, then there exists a unique morphism of vector sheaves $F : \mathcal{E} \to \mathcal{E}'$ such that the diagram*

$$\begin{array}{ccc} \mathcal{P}(\mathcal{E}) \times_X \mathcal{A}^m & \xrightarrow{f \times \ell} & \mathcal{P}(\mathcal{E}') \times_X \mathcal{A}^n \\ \kappa \downarrow & & \downarrow \kappa' \\ \mathcal{E} & \xrightarrow{F} & \mathcal{E}' \end{array}$$

Diagram 5.9

is commutative, where $\mathcal{E}$ and $\mathcal{E}'$ are viewed as associated with their sheaves of frames.

In particular, we obtain the following result, already alluded to in Remark 5.2.7(2).

5.6.4 Corollary. *In addition to the assumptions of the previous statement, we further assume that* $\operatorname{rank}(\mathcal{E}) = \operatorname{rank}(\mathcal{E}') = n$, $\ell := id|_{\mathcal{A}^n}$, *and f is a $\mathcal{GL}(n, \mathcal{A})$-isomorphism of $\mathcal{P}(\mathcal{E})$ onto $\mathcal{P}(\mathcal{E}')$. Then F is an isomorphism of $\mathcal{E}$ onto $\mathcal{E}'$.*

Complementing Remark 5.2.7(1), we prove the following converse of Corollary 5.6.4.

5.6.5 Proposition. *Let $F : \mathcal{E} \to \mathcal{E}'$ be an $\mathcal{A}$-isomorphism of vector sheaves of rank n. Then there is a unique $\mathcal{GL}(n, \mathcal{A})$-isomorphism f of $\mathcal{P}(\mathcal{E})$ onto $\mathcal{P}(\mathcal{E}')$ inducing F.*

Proof. Without loss of generality, we may assume that both $\mathcal{E}$ and $\mathcal{E}'$ have local frames over the same open covering $\mathcal{U} = (U_\alpha)$ of X. For every U_α, we define the $\mathcal{A}|_{U_\alpha}$-isomorphism

$$\mathrm{Iso}_{\mathcal{A}|_{U_\alpha}}(\mathcal{A}^n|_{U_\alpha}, \mathcal{E}|_{U_\alpha}) \longrightarrow \mathrm{Iso}_{\mathcal{A}|_{U_\alpha}}(\mathcal{A}^n|_{U_\alpha}, \mathcal{E}'|_{U_\alpha}) : \sigma \mapsto F \circ \sigma.$$

Localizing this over every open $V \subseteq U_\alpha$, we obtain the $\mathcal{A}|_V$-isomorphisms

$$f_V : \mathrm{Iso}_{\mathcal{A}|_V}(\mathcal{A}^n|_V, \mathcal{E}|_V) \longrightarrow \mathrm{Iso}_{\mathcal{A}|_V}(\mathcal{A}^n|_V, \mathcal{E}'|_V)$$

with $f_V(\sigma) = F \circ \sigma$. It is obvious that $f_V(\sigma \circ g) = f_V(h) \circ g$, for every $\sigma \in \mathrm{Iso}_{\mathcal{A}|_V}(\mathcal{A}^n|_V, \mathcal{E}|_V)$ and $g \in \mathrm{Aut}_{\mathcal{A}|_V}(\mathcal{A}^n|_{U_\alpha}) \cong \mathcal{GL}(n, \mathcal{A})(V)$. Therefore, varying V in the basis for the topology $\mathcal{B}$ (see the beginning of section 5.2), the presheaf isomorphism (f_V) generates a $\mathcal{GL}(n, \mathcal{A})$-isomorphism f of $\mathcal{P}(\mathcal{E})$ onto $\mathcal{P}(\mathcal{E}')$.

In virtue of Corollary 5.6.3, the isomorphism f induces, in its turn, a vector sheaf isomorphism $F' : \mathcal{E} \to \mathcal{E}'$. By the construction of Proposition 5.6.2, adapted to the data of Corollary 5.6.3, F' is generated by the isomorphisms (F'_V), for all open $V \in \mathcal{B}$, given by

$$F'_V([(\sigma, h)]_V) = [(f_V(\sigma), h)]_V := [(F \circ \sigma, h)]_V,$$

for every sections $\sigma \in \mathcal{P}(\mathcal{E})(V) \cong \mathrm{Iso}_{\mathcal{A}|_V}(\mathcal{A}^n|_V, \mathcal{E}|_V)$ and $h \in \mathcal{A}^n(V)$.

On the other hand, F induces the isomorphism $(\bar{F}_V)$ of presheaves of sections. Therefore, if (after (5.5.2)) we identify $[(\sigma, h)]_V$ with $\sigma \circ h$ and $[(F \circ \sigma, h)]$ with $F \circ \sigma \circ h$, the last series of equalities yield

$$F'_V(\sigma \circ h) = F \circ \sigma \circ h = \bar{F}_V(\sigma \circ h),$$

from which we get $F'_V = \bar{F}_V$, for all V. Since F may be thought of as generated by $(\bar{F}_V)$, we have that $F' = F$, thus concluding the proof. □

Based on Proposition 5.4.1 and the notation of (5.4.3), we prove the last result of this section. Before the statement, we would like to draw the reader's attention to the different typefaces ϕ and φ used below, both denoting morphisms of sheaves of groups.

5.6.6 Proposition. *Assume that (f, ϕ, id_X) is a morphism of $(\mathcal{P}, \mathcal{G}, X, \pi)$ into $(\mathcal{P}', \mathcal{G}', X, \pi')$, and $\varphi : \mathcal{G} \to \mathcal{H}$, $\varphi' : \mathcal{G}' \to \mathcal{H}'$ two morphisms of sheaves of groups. We denote by $(\mathcal{P} \times_X^{\mathcal{G}} \mathcal{H}, \mathcal{H}, X, \bar{\pi})$ and $(\mathcal{P}' \times_X^{\mathcal{G}'} \mathcal{H}', \mathcal{H}', X, \bar{\pi}')$ the principal sheaves associated with $\mathcal{P}$ and $\mathcal{P}'$, by φ and φ' respectively. If*

$\ell : \mathcal{H} \to \mathcal{H}'$ is a morphism of sheaves of groups satisfying the compatibility condition $\ell \circ \varphi = \varphi' \circ \phi$, then there is a unique morphism (F, ℓ, id_X) between the previous associated principal sheaves such that the diagrams below are commutative.

$$\begin{array}{ccc}
\mathcal{P} \times_X \mathcal{H} & \xrightarrow{F \times \ell} & \mathcal{P}' \times_X \mathcal{H}' \\
\Big\downarrow{\scriptstyle \kappa} & & \Big\downarrow{\scriptstyle \kappa'} \\
\mathcal{P} \times_X^{\mathcal{G}} \mathcal{H} & \xrightarrow{F} & \mathcal{P}' \times_X^{\mathcal{G}'} \mathcal{H}'
\end{array}$$

Diagram 5.10

$$\begin{array}{ccc}
(\mathcal{P}, \mathcal{G}, X, \pi) & \xrightarrow{(\varepsilon, \varphi, id_X)} & (\mathcal{P} \times_X^{\mathcal{G}} \mathcal{H}, \mathcal{H}, X, \bar{\pi}) \\
{\scriptstyle (f, \phi, id_X)}\Big\downarrow & & \Big\downarrow{\scriptstyle (F, \ell, id_X)} \\
(\mathcal{P}', \mathcal{G}', X, \pi') & \xrightarrow{(\varepsilon', \varphi', id_X)} & (\mathcal{P}' \times_X^{\mathcal{G}'} \mathcal{H}', \mathcal{H}', X, \bar{\pi}')
\end{array}$$

Diagram 5.11

Proof. According to the note following Proposition 5.6.2, we define F by setting $F([(p,h)]) := [(f(p), \ell(h))]$, for every $[(p,h)] \in \mathcal{P} \times_X^{\mathcal{G}} \mathcal{H}$. The compatibility condition of the statement guarantees that F is a well defined morphism of sheaves, making the Diagram 5.10 commutative. Observe that the latter is the principal sheaf analog of Diagram 5.8.

Now, for every $[(p,h)]$, as before, and every $h' \in \mathcal{H}$, Remark 5.4.2(2) implies that

$$F([(p,h)] \cdot h') = F([(p,\, h \cdot h')]) = [(f(p),\, \ell(h \cdot h'))] =$$
$$[(f(p),\, \ell(h) \cdot \ell(h'))] = [(f(p), \ell(h))] \cdot \ell(h') = F([(p,h)]) \cdot \ell(h'),$$

which shows that (F, ℓ, id_X) is a principal sheaf morphism as in the statement.

Finally, by Remark 5.4.2(3), we obtain

$$(F \circ \varepsilon)(p) = F([(p, e_x)]) = [(f(p), \ell(e_x))] = [(f(p), e'_x)] = (\varepsilon' \circ f)(p),$$

for every $p \in \mathcal{P}$ with $\pi(p) = x$; that is, Diagram 5.11 is commutative. □

Chapter 6

Connections on principal sheaves

I personally feel that the next person to propose a new definition of a connection should be summarily executed.

M. Spivak [117, Vol. 5, p. 602]

To the uninitiated, it would seem that the use of fiber bundles and connections to describe the basic forces of nature is a half-baked scheme devised by some clique of mathematicians bent on producing an application for their work. However, physicists themselves found these notions forced upon them by their own perception of nature

D. Bleecker [10, p. xiii]

As the title suggests, the present chapter is devoted to the study of the fundamental geometric notion of *connection* within the context of principal sheaves. It is the abstraction of the classical notion of connection on a principal bundle.

We start with an *operator-like* definition, which turns out to be equivalent to a family of *local sections* of $\Omega(\mathcal{L})$, analogous to the ordinary connection forms, and satisfying (the analog of) the familiar compatibility condition (viz. local gauge equivalence). We explain in detail how the classical case fits into the previous abstract scheme, thus deriving one more definition of (principal bundle) connections, in defiance of M. Spivak's exhortation under the heading of this chapter. Another approach treats connections as sections of the sheaf of connections, an idea originally due (for ordinary connections) to A. Aragnol [1].

The existence of connections is guaranteed by the annihilation of the *Atiyah class* of a principal sheaf. We also study connections linked together by appropriate morphisms of sheaves, and we finish with the *moduli sheaf* of connections. Connections on associated sheaves, in particular connections on vector sheaves, will be treated in Chapter 7.

6.1. Basic definitions and examples

Throughout this chapter, we fix a differential triad $(\mathcal{A}, d, \Omega)$ over a topological space $X \equiv (X, \mathfrak{T}_X)$. Given a Lie sheaf of groups $\mathcal{G} \equiv (\mathcal{G}, \rho, \mathcal{L}, \partial)$, we recall the notations (3.3.4) and (3.3.7), namely, $\Omega(\mathcal{L}) := \Omega \otimes_{\mathcal{A}} \mathcal{L}$ (reminiscent of the Lie algebra valued 1-forms on X), and $\rho(g).\omega$ representing the action of $g \in \mathcal{G}$ on (the right of) $\omega \in \Omega(\mathcal{L})$, induced by the representation $\rho : \mathcal{G} \to \mathcal{A}ut(\mathcal{L})$. In analogy to the classical terminology, sections of $\Omega(\mathcal{L})$ are also called ***1-forms***.

With the previous notations in mind, we come to the main notion of this chapter.

6.1.1 Definition. A ***connection*** on a principal sheaf $\mathcal{P} \equiv (\mathcal{P}, \mathcal{G}, X, \pi)$ is a morphism of sheaves of sets $D : \mathcal{P} \to \Omega(\mathcal{L})$ satisfying the property

$$D(p \cdot g) = \rho(g^{-1}).D(p) + \partial(g), \tag{6.1.1}$$

for every $(p, g) \in \mathcal{P} \times_X \mathcal{G}$. Equivalently, by (3.3.9) and (3.3.10),

$$D(s \cdot g) = \rho(g^{-1}).D(s) + \partial(g), \tag{6.1.1'}$$

for every $s \in \mathcal{P}(U)$, $g \in \mathcal{G}(U)$, and every open $U \subseteq X$.

On the left-hand side of (6.1.1), the center dot denotes the action of $\mathcal{G}$ on (the right) of $\mathcal{P}$. It is clearly distinguished from the action of $\mathcal{G}$ on $\Omega(\mathcal{L})$, denoted by the line dot. Moreover, all the morphisms in (6.1.1′) are the induced morphisms of sections, according to the general convention (1.1.3).

The previous definition depends, obviously, on the representation ρ and the Maurer-Cartan differential ∂ of $\mathcal{G}$. Therefore, it would be more appropriate to call D a (ρ, ∂)-***connection***. However, we systematically apply the simple terminology of Definition 6.1.1, unless we need to make explicit reference to the entities defining the structure of the Lie sheaf of groups $\mathcal{G}$.

Before proceeding to the basic properties of connections, we give two elementary, however useful, examples.

6.1.2 Examples.

(a) The Maurer-Cartan differential

Equality (3.3.8) shows that the Maurer-Cartan differential ∂ is a connection on $\mathcal{G}$, the latter being trivially thought of as a principal sheaf.

The particular case of ∂ derived from the ordinary logarithmic differential (Example 3.3.6(a)) will be examined in Section 6.2 (see Corollary 6.2.2 and the relevant discussion before it).

(b) The canonical local connections of $\mathcal{P}$

Assume that $\mathcal{U} \equiv (\mathcal{U}, (\phi_\alpha))$ is a local frame of a principal sheaf $\mathcal{P}$ with local coordinates $\phi_\alpha : \mathcal{P}|_{U_\alpha} \to \mathcal{G}|_{U_\alpha}$. It is a direct consequence of (4.1.4) and (3.3.8) that each morphism

$$D_\alpha := \partial \circ \phi_\alpha : \mathcal{P}|_{U_\alpha} \longrightarrow \Omega(\mathcal{L})|_{U_\alpha}; \qquad \alpha \in I, \tag{6.1.2}$$

determines a connection on the principal sheaf $\mathcal{P}|_{U_\alpha}$.

The morphisms (D_α) are called the ***canonical local connections*** of $\mathcal{P}$, with respect to the local frame $\mathcal{U}$. Occasionally, (D_α) are also called the ***Maurer-Cartan connections*** of $\mathcal{P}$, with respect to $\mathcal{U}$, because of their relationship with the Maurer-Cartan differential ∂ by means of (6.1.2).

The importance of (D_α) lies in the fact that, under suitable conditions, they can be glued together to determine a global connection. This is explained in Section 6.3 in the sequel.

In particular, if (s_α) are the natural sections of $\mathcal{P}$ over $\mathcal{U}$, we have that

$$D_\alpha(s_\alpha) = 0; \qquad \alpha \in I, \tag{6.1.3}$$

as a result of (4.1.7′) and Proposition 3.3.5.

We now define the analog of the classical local connection forms of a connection.

6.1.3 Definition. Let D be a connection on a principal sheaf $\mathcal{P}$. If $\mathcal{U}$ is a local frame of $\mathcal{P}$, then the ***local connection forms*** of D, with respect to $\mathcal{U}$, are defined to be the sections

$$\omega_\alpha := D(s_\alpha) \in \Omega(\mathcal{L})(U_\alpha), \qquad \alpha \in I. \tag{6.1.4}$$

The previous sections determine a 0-cochain of $\mathcal{U}$ with coefficients in the sheaf $\Omega(\mathcal{L})$, i.e., $(\omega_\alpha) \in C^0(\mathcal{U}, \Omega(\mathcal{L}))$. We apply the classical terminology of local connection forms in order to remind ourselves of the analogy between the above sections and the local connection forms of an ordinary connection. This analogy will be completely clarified by the next two results, as well as by Example 6.2(a) and Theorem 6.2.1, where we explain the relationship of an ordinary connection on a principal bundle with a sheaf morphism D in the sense of Definition 6.1.1.

In the language of physics, the local connection forms are known as the *(local)* ***gauge potentials*** (see, e.g., Bleecker [10, p. 36], Naber [81, p. 36], Nakahara [82, p. 334]). So D could be legitimately called a *(global)* ***gauge potential*** of $\mathcal{P}$.

6.1.4 Proposition. *The following compatibility condition is satisfied over each $U_{\alpha\beta} \neq \emptyset$:*

$$\omega_\beta = \rho(g_{\alpha\beta}^{-1}).\omega_\alpha + \partial(g_{\alpha\beta}). \tag{6.1.5}$$

Proof. In virtue of (4.3.3) and (6.1.1′), we see that

$$\begin{aligned} \omega_\beta = D(s_\beta) &= D(s_\alpha \cdot g_{\alpha\beta}) \\ &= \rho(g_{\alpha\beta}^{-1}).D(s_\alpha) + \partial(g_{\alpha\beta}) \\ &= \rho(g_{\alpha\beta}^{-1}).\omega_\alpha + \partial(g_{\alpha\beta}). \end{aligned}$$

□

Conversely, we have:

6.1.5 Theorem. *A 0-cochain $(\omega_\alpha) \in C^0(\mathcal{U}, \Omega(\mathcal{L}))$, satisfying the compatibility condition (6.1.5), determines a unique connection D on $\mathcal{P}$, whose local connection forms coincide with the given cochain (ω_α).*

Proof. Let $s \in \mathcal{P}(U)$ be any section over an arbitrary $U \subseteq \mathfrak{T}_X$. Since

$$U = \bigcup_{\alpha \in I} (U \cap U_\alpha),$$

for each $\alpha \in I$ there exists a uniquely determined $g_\alpha \in \mathcal{G}(U \cap U_\alpha)$ such that

$$s|_{U\cap U_\alpha} = \big(s_\alpha|_{U\cap U_\alpha}\big) \cdot g_\alpha$$

(see Proposition 4.1.2). We define the map $D_U : \mathcal{P}(U) \to \Omega(\mathcal{L})(U)$ by

$$D_U(s)|_{U\cap U_\alpha} := \rho(g_\alpha^{-1}).(\omega_\alpha|_{U\cap U_\alpha}) + \partial(g_\alpha|_{U\cap U_\alpha}). \tag{6.1.6}$$

We check that D_U is well defined. In fact, over $U \cap U_\beta$ we have the analogous expression

$$D_U(s)|_{U\cap U_\beta} := \rho(g_\beta^{-1}).(\omega_\beta|_{U\cap U_\beta}) + \partial(g_\beta|_{U\cap U_\beta}). \tag{6.1.7}$$

Since $g_\beta = g_{\beta\alpha} \cdot g_\alpha$ holds on $U \cap U_{\alpha\beta}$, omitting (for simplicity) the restrictions figuring in (6.1.6) and (6.1.7), we see that (6.1.5) and Proposition 3.3.5 yield

$$\begin{aligned} \rho(g_\beta^{-1}).\omega_\beta \ + \ \partial(g_\beta) &= \rho(g_\alpha^{-1} \cdot g_{\alpha\beta}).\big(\rho(g_{\alpha\beta}^{-1}).\omega_\alpha + \partial(g_{\alpha\beta})\big) + \partial(g_{\alpha\beta}^{-1} \cdot g_\alpha) \\ &= \rho(g_\alpha^{-1}).\omega_\alpha + \partial(g_\alpha) + \rho(g_\alpha^{-1}).\big(\rho(g_{\alpha\beta}).\partial(g_{\alpha\beta}) + \partial(g_{\alpha\beta}^{-1})\big) \\ &= \rho(g_\alpha^{-1}).\omega_\alpha + \partial(g_\alpha). \end{aligned}$$

This shows that (6.1.6) and (6.1.7) coincide on $U \cap U_{\alpha\beta}$, thus D_U is well defined.

Now it is obvious that (D_U), U running in $\mathfrak{T}_X$, is a presheaf morphism generating a morphism D. To verify that D is a connection, it suffices to show (6.1.1′), for each D_U. Indeed, for any $s \in \mathcal{P}(U)$ and $g \in \mathcal{G}(U)$, we have that (omitting the restrictions in the middle steps)

$$\begin{aligned} D_U(s \cdot g)|_{U\cap U_\alpha} &= D_U\big((s_\alpha \cdot g_\alpha) \cdot g)\big) = D_U\big(s_\alpha \cdot (g_\alpha \cdot g)\big) \\ &= \rho\big((g_\alpha \cdot g)^{-1}\big).\omega_\alpha + \partial(g_\alpha \cdot g) \\ &= \rho(g^{-1}).\big(\rho(g_\alpha^{-1}).\omega_\alpha + \partial(g_\alpha)\big) + \partial(g) \\ &= \rho(g^{-1}).D_U(s)|_{U\cap U_\alpha} + \partial(g|_{U\cap U_\alpha}). \end{aligned}$$

As in the preceding part of the proof, the previous property holds on the entire U.

On the other hand, identifying $\mathcal{P}$ and $\Omega(\mathcal{L})$ with the sheaves of germs of their sections, we see that the induced morphism of sections gives

$$D(s_\alpha) \equiv \bar{D}_U(s_\alpha) = D_U(s_\alpha) = \omega_\alpha,$$

as a particular case of (1.2.17).

Finally, for the uniqueness of D, assume that D' is another connection with $D'(s_\alpha) = \omega_\alpha$. Again D' identifies with the morphism generated by the induced presheaf morphism

$$\left\{\bar{D}'_U : \mathcal{P}(U) \longrightarrow \Omega(\mathcal{L})(U) \,|\, U \subseteq X \text{ open}\right\}.$$

Then, for every $U_\alpha \in \mathcal{U}$, we obtain the commutative diagram

$$\begin{array}{ccc}
\mathcal{P}(U) & \xrightarrow{\;\bar{D}'_U\;} & \Omega(\mathcal{L})(U) \\
\downarrow & & \downarrow \\
\mathcal{P}(U \cap U_\alpha) & \xrightarrow{\;\bar{D}'_{U\cap U_\alpha}\;} & \Omega(\mathcal{L})(U \cap U_\alpha)
\end{array}$$

Diagram 6.1

where the vertical arrows represent the natural restrictions. Thus, applying (6.1.6) and taking into account that D' is also a connection, we have

$$\begin{aligned}
\bar{D}'_U(s)|_{U\cap U_\alpha} &= \bar{D}'_{U\cap U_\alpha}(s|_{U\cap U_\alpha}) = \bar{D}'_{U\cap U_\alpha}(s_\alpha \cdot g_\alpha) \\
&= \rho(g_\alpha^{-1}).(\omega_\alpha|_{U\cap U_\alpha}) + \partial(g_\alpha|_{U\cap U_\alpha}) \\
&= D_U(s)|_{U\cap U_\alpha},
\end{aligned}$$

for all $\alpha \in I$. Hence, $D_U = \bar{D}'_U$, for every open $U \subseteq X$, by which we conclude that $D = D'$. □

6.1.6 Remark. Another way to define D from (ω_α) is by setting

$$D(p) := \rho(g_\alpha^{-1}).\omega_\alpha(x) + \partial(g_\alpha), \tag{6.1.8}$$

for every $p \in \mathcal{P}$ with $\pi(p) = x \in U_\alpha$, where $g_\alpha \in \mathcal{G}$ is (uniquely) determined by $p = s_\alpha(x) \cdot g_\alpha$. Note the difference in the meaning of g_α used in (6.1.6) and (6.1.8).

We prove that D is well defined and satisfies the properties of the statement by a simple modification of the method applied to each D_U above. However, now we should show that D is a continuous map. To this end, we choose an arbitrary $p_o \in \mathcal{P}$ and assume that $\pi(p_o) = x_o \in U_\alpha$. Then there are open neighborhoods V and U of p_o and x_o, respectively, such that $\pi|_V$

is a homeomorphism. Setting $\tau := (\pi|_V)^{-1}$ and $W := \tau(U_\alpha \cap U)$, we find a uniquely determined $g \in \mathcal{G}(U_\alpha \cap U)$ verifying $\tau(x) = s_\alpha(x) \cdot g(x)$, for every $x \in U_\alpha \cap U$. Therefore, (6.1.8) and (3.3.10) yield

$$\begin{aligned} D(\tau(x)) &= \rho(g(x)^{-1}).\omega_\alpha(x) + \partial(g(x)) \\ &= \big(\rho(g^{-1}).\omega_\alpha + \partial(g)\big)(x), \end{aligned}$$

from which it follows that

$$D|_W = \big(\rho(g^{-1}).\omega_\alpha + \partial(g)\big) \circ \pi|_W.$$

This proves the continuity of D at p_{o} and, similarly, on the whole of $\mathcal{P}$.

6.2. Further examples of connections

In addition to the connections ∂ and (D_α), discussed earlier in Examples 6.1.2, we examine the following cases.

(a) Sheaf-theoretic connections from (infinitesimal) connections on principal bundles

Let $P \equiv (P, G, X, \pi_P)$ be a smooth principal bundle. As we have seen in Example 4.1.9(a), the sheaf of germs of its smooth sections determines a principal sheaf $\mathcal{P} \equiv (\mathcal{P}, \mathcal{G}, X, \pi)$, where $\mathcal{G}$ is the Lie sheaf of groups $\mathcal{G} \equiv (\mathcal{G}, \mathcal{A}d, \mathcal{L}, \partial)$ defined in Example 3.3.6(a).

We recall that $\mathcal{G} = \mathcal{C}^\infty_X(G)$ (: the sheaf of germs of smooth G-valued maps on X), $\mathcal{L} = \mathcal{C}^\infty_X(\mathbb{G})$ (: the sheaf of germs of $\mathbb{G}$-valued maps on X), if $\mathbb{G} \cong T_eG$ is the Lie algebra of G, and $\mathcal{A}d$, ∂ are the morphisms obtained by the sheafification of the ordinary adjoint representation and the total (or logarithmic) differential, respectively.

An (open) trivializing covering $\mathcal{U} = (U_\alpha)$ of P also determines a local frame of $\mathcal{P}$. Without loss of generality

we assume that every U_α is the domain of a chart.

Suppose that P admits an (infinitesimal) connection ω in the ordinary sense (see, for instance, Darling [22], Greub-Halperin-Vanstone [35], Kobayashi-Nomizu [49], Naber [81], Sulanke-Wintgen [118]). This amounts to the existence of a family of local connection forms, $\omega_\alpha \in \Lambda^1(U_\alpha, \mathbb{G})$, $\alpha \in I$, satisfying the compatibility condition

$$\omega_\beta = \mathrm{Ad}(g_{\alpha\beta}^{-1}).\omega_\alpha + g_{\alpha\beta}^{-1}.dg_{\alpha\beta}, \tag{6.2.1}$$

where $g_{\alpha\beta} \in C^\infty(U_{\alpha\beta}, G)$ are the transition functions of P. The local connection forms are given by $\omega_\alpha = \sigma_\alpha^* \omega$, if (σ_α) are the natural sections of P over $\mathcal{U}$.

Setting $\mathcal{A} = \mathcal{C}_X^\infty$ (: the sheaf of germs of smooth functions on X, as in Example 2.1.4(a)), (3.3.14) takes the form

$$\underline{\lambda}^1 : \Omega_X(\mathbb{G}) \xrightarrow{\simeq} \Omega(\mathcal{L}) = \Omega \otimes_{\mathcal{A}} \mathcal{L}. \tag{6.2.2}$$

Because of the completeness of the presheaf $U \longmapsto \Lambda^1(U, \mathbb{G})$ generating $\Omega_X(\mathbb{G})$, we have the canonical identification (see (1.2.8), (1.2.9) and (1.2.16))

$$\Lambda^1(U_\alpha, \mathbb{G}) \ni \theta \xrightarrow{\simeq} \widetilde{\theta} \in \Omega_X(\mathbb{G})(U_\alpha),$$

thus the local connection forms (ω_α) of ω correspond bijectively to the cochain $(\underline{\omega}_\alpha) \in C^0(\mathcal{U}, \Omega(\mathcal{L}))$, where $\underline{\omega}_\alpha = \underline{\lambda}^1(\widetilde{\omega}_\alpha)$. By the definition of $\underline{\lambda}^1$, Diagram 1.7, and the notation of $(\,\Diamond\,)$ on p. 104, we obtain

$$\underline{\omega}_\alpha = \underline{\lambda}^1(\widetilde{\omega}_\alpha) = \big(\underline{\lambda}^1_{U_\alpha}(\omega_\alpha)\big)^{\sim}. \tag{6.2.3}$$

We also recall that the cocycle of $\mathcal{P}$ is now $(\widetilde{g_{\alpha\beta}}) \in Z^1(\mathcal{U}, \mathcal{G})$, according to equality (4.3.7).

We shall show that (6.2.1) leads to

$$\underline{\omega}_\beta = \mathcal{A}d(\widetilde{g_{\alpha\beta}}^{-1}).\underline{\omega}_\alpha + \partial(\widetilde{g_{\alpha\beta}}), \tag{6.2.1$'$}$$

where $\mathcal{A}d$ and ∂ have been explicitly constructed in Example 3.3.6(a). Since the proof involves technical, though typical, computations, we give the necessary details.

Applying $\underline{\lambda}^1$ to both sides of (6.2.1), we first obtain

$$\begin{aligned} \underline{\omega}_\beta &= \underline{\lambda}^1(\widetilde{\omega}_\beta) = \underline{\lambda}^1\big((\mathrm{Ad}(g_{\alpha\beta}^{-1}).\omega_\alpha + g_{\alpha\beta}^{-1}.dg_{\alpha\beta})^{\sim}\big) \\ &= \underline{\lambda}^1\big((\mathrm{Ad}(g_{\alpha\beta}^{-1}).\omega_\alpha)^{\sim}\big) + \underline{\lambda}^1\big((g_{\alpha\beta}^{-1}.dg_{\alpha\beta})^{\sim}\big). \end{aligned} \tag{6.2.4}$$

Furthermore, using (1.2.17), the definition of ∂, and (3.3.17), we see that

$$\underline{\lambda}^1\big((g_{\alpha\beta}^{-1}.dg_{\alpha\beta})^{\sim}\big) = \big(\underline{\lambda}^1_{U_{\alpha\beta}}(g_{\alpha\beta}^{-1}.dg_{\alpha\beta})\big)^{\sim} = \big(\partial_{U_{\alpha\beta}}(g_{\alpha\beta})\big)^{\sim} = \partial(g_{\alpha\beta}). \tag{6.2.5}$$

On the other hand, (3.3.15) and the definition of the action δ, determined by Ad, give

$$(\mathrm{Ad}(g_{\alpha\beta}^{-1}).\omega_\alpha)^{\sim} = \big(\delta_{U_{\alpha\beta}}(g_{\alpha\beta}^{-1}, \omega_\alpha)\big)^{\sim} = \delta\big(\widetilde{g_{\alpha\beta}}^{-1}, \widetilde{\omega}_\alpha\big);$$

hence,

$$\underline{\lambda}^1\big((\mathrm{Ad}(g_{\alpha\beta}^{-1}).\omega_\alpha)^\sim\big) = (\underline{\lambda}^1 \circ \delta)\big(\widetilde{g_{\alpha\beta}}^{-1}, \widetilde{\omega}_\alpha\big).$$

However, as a result of $\underline{\lambda}^1 = (\underline{\mu}^1)^{-1}$ and ($\triangle$) on p. 108, we see that

$$\underline{\lambda}^1 \circ \delta = \delta' \circ (1 \times \underline{\lambda}^1).$$

Therefore, (6.2.3), the definition of $\mathcal{A}d$ via δ', and the general notation (3.3.9) imply that

$$\begin{aligned}\underline{\lambda}^1\big(\mathrm{Ad}(g_{\alpha\beta}^{-1}).\omega_\alpha)^\sim\big) &= \big(\delta' \circ (1 \times \underline{\lambda}^1)\big)\big(\widetilde{g_{\alpha\beta}}^{-1}, \widetilde{\omega}_\alpha\big)\\ &= \delta'\big(\widetilde{g_{\alpha\beta}}^{-1}, \underline{\lambda}^1_{U_\alpha}(\omega_\alpha)\big)\\ &= \delta'\big(\widetilde{g_{\alpha\beta}}^{-1}, \underline{\omega}_\alpha\big)\\ &= \mathcal{A}d(\widetilde{g_{\alpha\beta}}^{-1}).\underline{\omega}_\alpha.\end{aligned} \tag{6.2.6}$$

Substituting (6.2.5) and (6.2.6) in (6.2.4), we obtain (6.2.1′), which is precisely the compatibility condition (6.1.5) for $\rho = \mathcal{A}d$. Therefore, in virtue of Theorem 6.1.5, the 0-cochain $(\underline{\omega}_\alpha)$ yields a connection D on $\mathcal{P}$.

Conversely, a connection D on $\mathcal{P}$ leads to a connection on P by reversing the previous approach.

Summarizing, we have proved the following result, providing yet another (equivalent) definition of the classical connections on principal bundles, based on sheaf-theoretic methods.

6.2.1 Theorem. *Let $P \equiv (P, G, X, \pi_P)$ be a principal fiber bundle. Then the (infinitesimal) connections on P are in bijective correspondence with the sheaf-theoretic connections on the principal sheaf $\mathcal{P} \equiv (\mathcal{P}, \mathcal{G}, X, \pi)$ of germs of smooth sections of P.*

Note. For the sake of completeness let us remark that, in the context of the previous example and in analogy to (3.1.8),

$$\Omega(\mathcal{L})(U_\alpha) = (\Omega \otimes_{\mathcal{A}} \mathcal{L})(U_\alpha) = \Omega(U_\alpha) \otimes_{\mathcal{A}(U_\alpha)} \mathcal{L}(U_\alpha).$$

In fact, if $\dim \mathbb{G} = n$, then

$$\mathcal{L}(U_\alpha) \cong C^\infty(U_\alpha, \mathbb{G}) \cong C^\infty(U_\alpha, \mathbb{R})^n \cong \mathcal{A}(U_\alpha)^n \cong \mathcal{A}^n(U_\alpha);$$

consequently,

$$\begin{gathered}\Omega(U_\alpha)\otimes_{\mathcal{A}(U_\alpha)}\mathcal{L}(U_\alpha)\cong\Omega(U_\alpha)\otimes_{\mathcal{A}(U_\alpha)}\mathcal{A}(U_\alpha)^n\cong\\ \big(\Omega(U_\alpha)\otimes_{\mathcal{A}(U_\alpha)}\mathcal{A}(U_\alpha)\big)^n\cong\Omega(U_\alpha)^n\cong\Omega^n(U_\alpha)\equiv\\ \Omega^n|_{U_\alpha}(U_\alpha)\cong(\Omega|_{U_\alpha})^n(U_\alpha)\cong\big(\Omega|_{U_\alpha}\otimes_{\mathcal{A}|_{U_\alpha}}\mathcal{A}|_{U_\alpha}\big)^n(U_\alpha)\cong\\ \big(\Omega|_{U_\alpha}\otimes_{\mathcal{A}|_{U_\alpha}}\mathcal{A}^n|_{U_\alpha}\big)(U_\alpha)\cong(\Omega\otimes_{\mathcal{A}}\mathcal{L})(U_\alpha).\end{gathered}$$

We obtain the same identification if we take into account that

$$\Omega(U_\alpha)\cong\Lambda^1(U_\alpha,\mathbb{R})\cong C^\infty(U_\alpha,\mathbb{R})^n\cong\mathcal{A}^n(U_\alpha).$$

Let us apply the previous considerations to the special case of the *trivial bundle* $P := X\times G$ over X. Then there is a $\mathcal{G}$-equivariant isomorphism

$$F:\mathcal{P}\xrightarrow{\ \simeq\ }\mathcal{G},$$

generated by the presheaf isomorphism consisting of the $C^\infty(U,G)$-equivariant isomorphisms

$$F_U:\Gamma(U,U\times G)\ni\sigma\longmapsto s\in C^\infty(U,G).$$

Here the domain is the set of smooth sections of P over U, $s=\mathrm{pr}_2\circ\sigma$, and U is running in the topology $\mathfrak{T}_X$ of X. Hence, F is a global coordinate of $\mathcal{P}$. By (4.1.9), the global natural section $\sigma^{\mathrm{o}}\in\Gamma(X,X\times G)$, i.e., $\sigma^{\mathrm{o}}(x)=(x,e)$, determines the (global) natural section (with respect to F) $s^{\mathrm{o}}=\widetilde{\sigma^{\mathrm{o}}}\in\mathcal{P}(X)$. Indeed, for every $x\in X$,

$$F(s^{\mathrm{o}}(x))=F(\widetilde{\sigma^{\mathrm{o}}}(x))=F([\sigma^{\mathrm{o}}]_x)=[F_X\circ\sigma^{\mathrm{o}}]_x=[c_e]_x=\mathbf{1}(x),$$

where c_x is the constant map $X\ni x\mapsto e\in G$.

Now, the ***canonical flat connection*** ω^{o} of P is given by $\omega^{\mathrm{o}}=\mathrm{pr}_2^*\,\alpha$, if α is the Maurer-Cartan form of G. Hence, σ^{o} reduces the local connection forms of ω^{o} to the unique trivial form 0. Viewing this as the zero section of $\Omega(\mathcal{L})$, we obtain a connection D^{o} corresponding bijectively to ω^{o}. By the very construction of a connection on a principal sheaf from its local connection forms (see the proof of Theorem 6.1.5), we see that, for any $\tau\in\mathcal{P}(U)$,

$$D^{\mathrm{o}}(\tau)=\partial(g),$$

where $g\in\mathcal{G}(U)$ is uniquely determined by $\tau=s^{\mathrm{o}}\cdot g$. However, the equivariance of F yields $F(\tau)=F(s^{\mathrm{o}}\cdot g)=F(s^{\mathrm{o}})\cdot g=g$, from which we deduce

$$D^{\mathrm{o}}=\partial\circ F.$$

The previous remarks, along with Theorem 6.2.1, are summarized in the following:

6.2.2 Corollary. *The canonical flat connection ω° of the trivial bundle $P = X \times G$ over X and the corresponding connection D° (on the sheaf $\mathcal{P}$ of germs of smooth sections of P) coincide, within an isomorphism, with the Maurer-Cartan differential ∂ of the sheaf of germs $\mathcal{G}$ of the G-valued smooth maps on X.*

(b) Connections on projective limits

Here we combine Examples 3.3.6(c) and 4.1.9(b), whose notations and assumptions are applied hereafter. We further assume that each bundle P_i is endowed with a connection (form) ω^i so that, for every $j, i \in J$ with $j \geq i$, the connections ω^j and ω^i be $(p_{ji}, \rho_{ji}, id_X)$-*related*, i.e.

$$p_{ji}^{*}\omega^{i} = r_{ji} \cdot \omega^{j}.$$

As usual, the left-hand side is the pull-back of ω^i by p_{ji}^*, whereas the right-hand side is the 1-form given by

$$(r_{ji} \cdot \omega^{j})_p(u) := r_{ji}(\omega^{j}_{p}(u)); \qquad p \in P_j,\, u \in T_pP_j,$$

with $r_{ji} = T_e\rho_{ji} = d_e\rho_{ji}$ (: the ordinary differential of a smooth map).

If $\mathcal{U} = \{U_\alpha \subseteq X\}_{\alpha\in I}$ is a common trivializing open covering for all the bundles, and $(s^i_\alpha)_{\alpha\in I}$ are the natural sections of P_i over $\mathcal{U}$, then the previous $(p_{ji}, \rho_{ji}, id_X)$-relatedness condition is equivalent to the following formula, expressed in terms of the local connection forms $(\omega^i_\alpha)_{\alpha\in I}$ of each ω^i $(i \in J)$:

$$r_{ji} \cdot \omega^{j}_{\alpha} = Ad^{i}(h_{ji}^{-1}).\omega^{i}_{\alpha} + h_{ji}^{-1} \cdot dh_{ji}; \qquad \alpha \in I, \tag{6.2.7}$$

where the maps $h_{ji} \in C^{\infty}(U_\alpha, G_i)$ are determined by

$$p_{ji}(s^{j}_{\alpha}(x)) = s^{i}_{\alpha}(x) \cdot h_{ji}(x), \qquad x \in U_\alpha.$$

(For certain aspects of related connections we also refer to Vassiliou [124], [125]). As explained in the preceding Example 6.2(a), each $\mathbb{G}_i$-valued form ω^i_α determines the section

$$\underline{\omega}^{i}_{\alpha} \in \Omega(U_\alpha) \otimes_{\mathcal{A}(U_\alpha)} \mathcal{L}_i(U_\alpha) \cong \Omega(\mathcal{L}_i)(U_\alpha),$$

inducing a connection

$$D^i : \mathcal{P}_i \longrightarrow \Omega(\mathcal{L}_i) = \Omega \otimes_{\mathcal{A}} \mathcal{L}_i$$

on $\mathcal{P}_i$, for every $i \in J$. Following the construction of connections given in the proof of Theorem 6.1.5 (especially formula (6.1.6)), and taking into account (6.2.7) and the analog of (6.2.1′) for the sections $(\underline{\omega}^i_\alpha)_{i\in J}$, after some elementary calculations we check that $(D^i)_{i\in J}$ is a morphism of projective systems producing the connection

$$D := \varprojlim D_i \; : \; \mathcal{P} \cong \varprojlim \mathcal{P}_i \longrightarrow \varprojlim(\Omega \otimes_{\mathcal{A}} \mathcal{L}_i) \cong \Omega(\mathcal{L}).$$

More precisely, D is an $(\mathcal{A}d, \partial)$-connection, where $\mathcal{A}d$ and ∂ are the morphisms defined in Example 3.3.6(c).

Regarding connections on (Fréhet) bundles, which are projective limits of (Banach) principal bundles, we refer to Galanis [31].

(c) The sheaf of connections

Using the initial action of $\mathcal{G}$ on $\Omega(\mathcal{L})$, as defined by (3.3.5) (see also the notations (3.3.7)), we define a new action, namely

$$\begin{gathered} \mathcal{G} \times_X \Omega(\mathcal{L}) \longrightarrow \Omega(\mathcal{L}) : (g, \omega) \mapsto g \cdot \omega, \\ g \cdot \omega := \rho(g).\omega + \partial(g^{-1}) = \rho(g).\big(\omega - \partial(g)\big). \end{gathered} \tag{6.2.8}$$

Adopting the well-known terminology (for ordinary differential forms), the expression $g^{-1} \cdot \omega = \rho(g^{-1}).\omega + \partial(g)$ is called the ***gauge transform*** of ω by g.

Now, (6.2.8) induces the following right action of $\mathcal{G}$ on $\mathcal{P} \times_X \Omega(\mathcal{L})$:

$$\begin{gathered} (\mathcal{P} \times_X \Omega(\mathcal{L})) \times_X \mathcal{G} \longrightarrow \mathcal{P} \times_X \Omega(\mathcal{L}) : (p, \omega, g) \mapsto (p, \omega) \cdot g, \\ (p, \omega) \cdot g := (p \cdot g, g^{-1} \cdot \omega) = \big(p \cdot g, \rho(g^{-1}) \cdot \omega + \partial(g)\big). \end{gathered} \tag{6.2.9}$$

Accordingly, we define the equivalence relation:

$$(p, \omega) \sim (q, \theta) \qquad \Longleftrightarrow \qquad \exists | \;\; g \in \mathcal{G} \; : \; (q, \theta) = (p, \omega) \cdot g,$$

for pairs of elements projected to the same point of X.

Therefore, by Proposition 5.3.4 and Theorem 5.3.5, we obtain the sheaf

$$\mathcal{C}(\mathcal{P}) := \mathcal{P} \times^{\mathcal{G}}_X \Omega(\mathcal{L}), \tag{6.2.10}$$

whose structure type is the $\mathcal{A}$-module $\Omega(\mathcal{L})$

6.2.3 Definition. We call $\mathcal{C}(\mathcal{P})$ the ***sheaf of connections of*** $\mathcal{P}$.

The previous terminology is in accordance with A. Aragnol (see [1, Def. III.1.2]), who originally defined $\mathcal{C}(\mathcal{P})$ in the case of ordinary connections on a principal bundle P, with $\mathcal{P}$ now being the sheaf of germs of smooth sections of P.

The following result justifies Aragnol's terminology.

6.2.4 Theorem. *The connections of $\mathcal{P}$ are in bijective correspondence with the global sections of $\mathcal{C}(\mathcal{P})$.*

Proof. The result is a direct consequence of Theorem 5.3.9. Indeed, a global section of $\mathcal{C}(\mathcal{P})$ corresponds bijectively to a tensorial morphism of the form $D : \mathcal{P} \to \Omega(\mathcal{L})$. Taking into account Definition 5.3.8 and the action (6.2.8), we see that

$$D(p \cdot g) = g^{-1} \cdot D(p) := \rho(g^{-1}) \cdot D(p) + \partial(g), \qquad (p, g) \in \mathcal{P} \times_X \mathcal{G}.$$

Hence, D is a connection on $\mathcal{P}$. By the same token we prove the converse part of the statement. □

Theorem 6.2.4 provides an existence criterion, stated in the following obvious result:

6.2.5 Corollary. *A principal sheaf $\mathcal{P}$ admits a connection if and only if the sheaf of connections $\mathcal{C}(\mathcal{P})$ admits a global section.*

(d) The abelian case

For the purpose of later reference, we record here the particular case of a principal sheaf $\mathcal{P}$ with an *abelian* structure sheaf $\mathcal{G}$. Now, since ρ is the trivial representation (see Definition 3.3.4), a connection D on $\mathcal{P}$ is characterized by the property

$$D(p \cdot g) = D(p) + \partial(g), \qquad (p, g) \in \mathcal{P} \times_X \mathcal{G}.$$

Similarly, the local connection forms (ω_α) of D satisfy the compatibility condition

$$\omega_\beta = \omega_\alpha + \partial(g_{\alpha\beta})$$

over $U_{\alpha\beta}$, for all $\alpha, \beta \in I$.

6.3. Existence of connections

We shall show that the canonical local connections (D_α), defined in Example 6.1.2(b), provide a (global) connection D on $\mathcal{P}$ if the Atiyah class of $\mathcal{P}$, defined in the sequel, annihilates.

Before attaining our main objective, we note that, if $\mathcal{F}$ is an $\mathcal{A}$-*module* on which $\mathcal{G}$ acts from the left, then the isomorphism of Corollary 5.3.10, namely

$$\mathcal{H}om_{\mathcal{G}}(\mathcal{P},\mathcal{F}) \cong \mathcal{P} \times_X^{\mathcal{G}} \mathcal{F},$$

is an $\mathcal{A}$-isomorphism. This is a consequence of Proposition 5.4.5, in conjunction with Proposition 3.3.1 establishing the equivalence between representations and actions.

We consider the natural action of $\mathcal{G}$ on the left of $\Omega \otimes_{\mathcal{A}} \mathcal{F}$, induced by

$$g \cdot (\theta \otimes u) := \theta \otimes g \cdot u,$$

for every $\theta \otimes u \in \Omega_x \otimes_{\mathcal{A}_x} \mathcal{F}_x$, $g \in \mathcal{G}_x$, and $x \in X$.

6.3.1 Proposition. *Let Ω and $\mathcal{F}$ be $\mathcal{A}$-modules. Then, with respect to the previous action, we have the $\mathcal{A}$-isomorphisms*

$$\mathcal{H}om_{\mathcal{G}}(\mathcal{P},\Omega \otimes_{\mathcal{A}} \mathcal{F}) \cong \Omega \otimes_{\mathcal{A}} \big(\mathcal{P} \times_X^{\mathcal{G}} \mathcal{F}\big) \cong \Omega \otimes_{\mathcal{A}} \mathcal{H}om_{\mathcal{G}}(\mathcal{P},\mathcal{F}).$$

Proof. By the introductory comments we only need to prove the first isomorphism. Setting, for simplicity, $\mathcal{Q} = \mathcal{P} \times_X^{\mathcal{G}} \mathcal{F}$, we see that $\Omega \otimes_{\mathcal{A}} \mathcal{Q}$ is an $\mathcal{A}$-module of structure type $\Omega \otimes_{\mathcal{A}} \mathcal{F}$, with local coordinates

$$1 \otimes \widetilde{\Phi}_\alpha : \Omega|_{U_\alpha} \otimes_{\mathcal{A}|_{U_\alpha}} \mathcal{Q}|_{U_\alpha} \xrightarrow{\ \simeq\ } \Omega|_{U_\alpha} \otimes_{\mathcal{A}|_{U_\alpha}} \mathcal{F}|_{U_\alpha}, \qquad \alpha \in I.$$

Here $1 = 1_\Omega$ (restricted to U_α), and $\widetilde{\Phi}_\alpha : \mathcal{Q}|_{U_\alpha} \to \mathcal{F}|_{U_\alpha}$ is the coordinate of $\mathcal{Q}$ over U_α (see Theorem 5.3.5).

The proof now follows essentially those of Theorem 5.3.9 and Corollary 5.3.10, after an appropriate tensoring of the morphisms involved therein. More precisely, given any open $U \subseteq X$, we define the map

$$H_U : \mathrm{Hom}_{\mathcal{G}|_U}\big(\mathcal{P}|_U, \Omega|_U \otimes_{\mathcal{A}|_U} \mathcal{F}|_U\big) \longrightarrow (\Omega \otimes_{\mathcal{A}} \mathcal{Q})(U) \tag{6.3.1}$$

as follows: For a tensorial morphism f in the indicated domain, we set

$$H_U(f)(x) := \big((1 \otimes \widetilde{\Phi}_\alpha^{-1}) \circ f \circ s_\alpha\big)(x), \tag{6.3.2}$$

if $x \in U \cap U_\alpha$ and s_α is the natural section of $\mathcal{P}$ over U_α. Note that $\Omega|_U \otimes_{\mathcal{A}|_U} \mathcal{F}|_U = (\Omega \otimes_{\mathcal{A}} \mathcal{F})|_U$.

H_U is well defined. Indeed, for every $x \in U \cap U_{\alpha\beta}$, the tensoriality of f and Theorem 5.3.5 imply that

$$\begin{aligned}\big((1 \otimes \widetilde{\Phi}_\beta^{-1}) \circ f \circ s_\beta\big)(x) &= \big((1 \otimes \widetilde{\Phi}_\beta^{-1}) \circ f\big)\big(s_\alpha(x) \cdot g_{\alpha\beta}(x)\big) \\ &= \big(1 \otimes \widetilde{\Phi}_\beta^{-1}\big)\big(g_{\beta\alpha}(x) \cdot f(s_\alpha(x))\big) \\ &= \big(1 \otimes \widetilde{\Phi}_\beta^{-1}\big)\big((1 \otimes (\widetilde{\Phi}_\beta \circ \widetilde{\Phi}_\alpha^{-1}))(f(s_\alpha(x)))\big) \\ &= \big((1 \otimes \widetilde{\Phi}_\alpha^{-1}) \circ f \circ s_\alpha\big)(x).\end{aligned}$$

On the other hand, H_U is a bijection whose inverse is defined in the following way: If $\sigma \in (\Omega \otimes_{\mathcal{A}} \mathcal{Q})(U)$, then

$$H_U^{-1}(\sigma)(p) := \big(1 \otimes g_\alpha^{-1} \cdot \widetilde{\Phi}_\alpha\big)(\sigma(x)),$$

for every $p \in \mathcal{P}$ with $\pi(p) = x \in U \cap U_\alpha$ and g_α determined by $p = s_\alpha(x) \cdot g_\alpha$. The morphism $g_\alpha^{-1} \cdot \widetilde{\Phi}_\alpha$ is given by $(g_\alpha^{-1} \cdot \widetilde{\Phi}_\alpha)(q) := g_\alpha^{-1} \cdot \widetilde{\Phi}_\alpha(q)$, for every $q \in \mathcal{Q}$. Working in a way similar to the first part of the proof of Theorem 5.3.9, we check that $H_U^{-1}(\sigma)$ is a well defined tensorial morphism.

Identifying $\Omega \otimes_{\mathcal{A}} \mathcal{Q}$ with the sheaf of germs of its sections, we have that (H_U), U running the topology of X, is a presheaf morphism. The desired first isomorphism of the statement is generated by (H_U). □

Reverting to the context of connections, we replace $\mathcal{F}$ by $\Omega(\mathcal{L})$ and $\mathcal{Q}$ by the adjoint sheaf $\rho(\mathcal{P})$ (see Subsection 5.4(d)). Then Proposition 6.3.1 implies that

$$\begin{gathered}\mathcal{H}om_{\mathcal{G}}(\mathcal{P}, \Omega(\mathcal{L})) = \mathcal{H}om_{\mathcal{G}}(\mathcal{P}, \Omega \otimes_{\mathcal{A}} \mathcal{L}) \cong \\ \Omega \otimes_{\mathcal{A}} (\mathcal{P} \times_X^{\mathcal{G}} \mathcal{L}) = \Omega \otimes_{\mathcal{A}} \rho(\mathcal{P}).\end{gathered}$$

If we apply the analog of (3.3.4) for $\Omega \otimes_{\mathcal{A}} \rho(\mathcal{P})$, the last identification takes the form

$$\mathcal{H}om_{\mathcal{G}}(\mathcal{P}, \Omega(\mathcal{L})) \cong \Omega(\rho(\mathcal{P})) := \Omega \otimes_{\mathcal{A}} \rho(\mathcal{P}). \tag{6.3.3}$$

Moreover, if $f \in \mathcal{H}om_{\mathcal{G}}(\mathcal{P}, \Omega(\mathcal{L}))(U) \cong \mathrm{Hom}_{\mathcal{G}|_U}(\mathcal{P}|_U, \Omega(\mathcal{L})|_U)$, for any open $U \subseteq X$, the tensoriality of f is expressed by

$$f(p \cdot g) = \rho(g^{-1}).f(p), \qquad (p, g) \in \mathcal{P}|_U \times_U \mathcal{G}|_U. \tag{6.3.3$'$}$$

If $\mathcal{U}$ is a local frame of $\mathcal{P}$, we can form the cochain complex of $\mathcal{A}(X)$-modules

$$C^0\big(\mathcal{U}, \mathcal{H}om_{\mathcal{G}}(\mathcal{P}, \Omega(\mathcal{L}))\big) \xrightarrow{\delta^0} C^1\big(\mathcal{U}, \mathcal{H}om_{\mathcal{G}}(\mathcal{P}, \Omega(\mathcal{L}))\big) \xrightarrow{\delta^1} C^2\big(\mathcal{U}, \mathcal{H}om_{\mathcal{G}}(\mathcal{P}, \Omega(\mathcal{L}))\big) \xrightarrow{\delta^2} \cdots$$

from which the (Čech) cohomology groups (in fact, $\mathcal{A}(X)$-modules)

$$H^q\big(\mathcal{U}, \mathcal{H}om_{\mathcal{G}}(\mathcal{P}, \Omega(\mathcal{L}))\big) \quad \text{and} \quad H^q\big(X, \mathcal{H}om_{\mathcal{G}}(\mathcal{P}, \Omega(\mathcal{L}))\big)$$

are derived in the usual manner (see Subsection 1.6.1).

We now consider the canonical connections (D_α), with respect to a local frame $\mathcal{U}$. For every $(p,g) \in (\mathcal{P} \times_X \mathcal{G})|_{U_{\alpha\beta}} = \mathcal{P}|_{U_{\alpha\beta}} \times_{U_{\alpha\beta}} \mathcal{G}|_{U_{\alpha\beta}}$, (6.1.1) implies that

$$(D_\alpha - D_\beta)(p \cdot g) = \rho(g^{-1}).(D_\alpha - D_\beta)(p);$$

in other words,

$$D_\alpha - D_\beta \in \mathrm{Hom}_{\mathcal{G}|_{U_{\alpha\beta}}}\big(\mathcal{P}|_{U_{\alpha\beta}}, \Omega(\mathcal{L})|_{U_{\alpha\beta}}\big) \cong \mathcal{H}om_{\mathcal{G}}(\mathcal{P}, \Omega(\mathcal{L}))(U_{\alpha\beta}).$$

Therefore, we obtain the 1-cocycle,

$$(D_\alpha - D_\beta) \in Z^1\big(\mathcal{U}, \mathcal{H}om_{\mathcal{G}}(\mathcal{P}, \Omega(\mathcal{L}))\big), \tag{6.3.4}$$

called the ***Maurer-Cartan cocycle*** of $\mathcal{P}$ (with respect to $\mathcal{U}$). This determines the class $[(D_\alpha - D_\beta)]_{\mathcal{U}} \in H^1\big(\mathcal{U}, \mathcal{H}om_{\mathcal{G}}(\mathcal{P}, \Omega(\mathcal{L}))\big)$ and the corresponding cohomology class

$$\mathfrak{a}(\mathcal{P}) := [(D_\alpha - D_\beta)] \in H^1\big(X, \mathcal{H}om_{\mathcal{G}}(\mathcal{P}, \Omega(\mathcal{L}))\big). \tag{6.3.5}$$

Equivalently, in virtue of (6.3.3), we may write

$$\mathfrak{a}(\mathcal{P}) \in H^1\big(X, \Omega(\rho(\mathcal{P}))\big). \tag{6.3.5'}$$

6.3.2 Definition. The cohomology class $\mathfrak{a}(\mathcal{P})$ is called the ***Atiyah class*** of the principal sheaf $\mathcal{P}$.

The preceding terminology is inspired by J. L. Koszul's analogous *Atiyah class*, referring to an ordinary holomorphic principal bundle P (see [50, p. 119]). The class $\mathfrak{a}(\mathcal{P})$, as well as its classical analog, is the obstruction to the existence of connections, as shown in the next theorem.

The original idea of such an obstruction goes back to M. F. Atiyah ([6]), whose approach is based on the *extension theory* of vector bundles and their corresponding locally free sheaves (viz. vector sheaves). The resulting class, originally denoted by $a(P)$, is Koszul's *Atiyah obstruction class* $b(P)$ (see [50, p. 120]).

We would like to add that the approach via extensions cannot be applied to the present framework of arbitrary principal sheaves. The latter are not related with some natural vector sheaves, as in the case of principal bundles, whose corresponding vector bundles (tangent bundle, bundle of invariant vector fields etc.) and their associated sheaves of sections enable one to apply the extension mechanism. In contrast, the same mechanism works well in the case of arbitrary vector sheaves, as expounded in Mallios [62, Chapter VI, Sections 13–14].

We come to the following fundamental existence criterion:

6.3.3 Theorem. *A principal sheaf* $(\mathcal{P}, \mathcal{G}, X, \pi)$ *admits a connection if and only if* $\mathfrak{a}(\mathcal{P}) = 0$.

Proof. First assume that $\mathfrak{a}(\mathcal{P}) = 0$. Then there exists a local frame $(\mathcal{U}, (\phi_\alpha))$ of $\mathcal{P}$ and a 0-cochain $(f_\alpha) \in C^0\big(\mathcal{U}, \mathcal{H}om_{\mathcal{G}}(\mathcal{P}, \Omega(\mathcal{L}))\big)$ such that

$$(D_\alpha - D_\beta) = \delta^0\big((f_\alpha)\big).$$

Therefore, for every $\alpha, \beta \in I$ with $U_{\alpha\beta} \neq \emptyset$, $D_\alpha - D_\beta = f_\beta - f_\alpha$, or

$$D_\alpha + f_\alpha = D_\beta + f_\beta. \tag{6.3.6}$$

Setting, for the sake of convenience,

$$\mathcal{P}_\alpha := \mathcal{P}|_{U_\alpha} = \pi^{-1}(U_\alpha), \tag{6.3.7}$$

we define the mapping $D : \mathcal{P} \to \Omega(\mathcal{L})$ by

$$D|_{\mathcal{P}_\alpha} := D_\alpha + f_\alpha, \qquad \alpha \in I.$$

By (6.3.6), D is a well defined morphism of sheaves. On the other hand, for any $(p, g) \in \mathcal{P} \times_X \mathcal{G}$ with $\pi_X(p, g) = \pi(p) \in U_\alpha$, condition (6.1.1) for D_α, and the tensoriality of f_α (see (6.3.3′)) yield

$$\begin{aligned} D(p \cdot g) &= D_\alpha(p \cdot g) + f_\alpha(p \cdot g) \\ &= \rho(g^{-1}).\big(D_\alpha(p) + f_\alpha(p)\big) + \partial(g) \\ &= \rho(g^{-1}).D(p) + \partial(g); \end{aligned}$$

that is, D is a connection on $\mathcal{P}$.

Conversely, assume that $\mathcal{P}$ admits a connection D. If $\mathcal{U}$ is a local frame of $\mathcal{P}$, we define the connections $\bar{D}_\alpha := D|_{\mathcal{P}_\alpha}$, restrictions of D to the subsheaves $\mathcal{P}_\alpha$, $\alpha \in I$. Clearly, each $\bar{D}_\alpha$ does not necessarily coincide with the canonical local connection D_α defined by (6.1.2), henceforth the use of the bar on the restrictions of D. As a result, we obtain the equivariant morphisms

$$f_\alpha := D_\alpha - \bar{D}_\alpha \in \mathrm{Hom}_{\mathcal{G}|_{U_\alpha}}\big(\mathcal{P}|_{U_\alpha}, \Omega(\mathcal{L})|_{U_\alpha}\big); \qquad \alpha \in I,$$

which form the 0-cochain $(f_\alpha) \in C^0\big(\mathcal{P}, \mathcal{H}om_{\mathcal{G}}\left(\mathcal{P}, \Omega(\mathcal{L})\right)\big)$. Since, on $U_{\alpha\beta}$,

$$0 = \bar{D}_\alpha - \bar{D}_\beta = (D_\alpha - D_\beta) - (f_\beta - f_\alpha),$$

we have that $D_\alpha - D_\beta = f_\beta - f_\alpha \in \mathcal{H}om_{\mathcal{G}}\left(\mathcal{P}, \Omega(\mathcal{L})\right)(U_{\alpha\beta})$, from which it follows that

$$(D_\alpha - D_\beta) = \delta^0\big((f_\alpha)\big) \in \mathrm{im}(\delta^0).$$

Therefore, the Maurer-Cartan cocycle $(D_\alpha - D_\beta)$ is also a coboundary and $\mathfrak{a}(\mathcal{P}) = [(D_\alpha - D_\beta)] = 0$. This terminates the proof. □

Note. For the sake of completeness let us give another proof of the first part of the above theorem, using the local connection forms: Assuming that $\mathfrak{a}(\mathcal{P}) = 0$, we saw that $D_\alpha - D_\beta = f_\beta - f_\alpha$. Evaluating the morphism of sections, induced by the last equality, at s_β (restricted to $U_{\alpha\beta}$), we obtain $(D_\alpha - D_\beta)(s_\beta) = (f_\beta - f_\alpha)(s_\beta)$. Applying (4.3.3), (6.1.1′), (6.1.3) and the tensoriality of f, the previous equality transforms into

$$\partial(g_{\alpha\beta}) = f_\beta(s_\beta) - \rho(g_{\alpha\beta}^{-1}).f_\alpha(s_\alpha).$$

Consequently, the forms

$$\omega_\alpha := f_\alpha(s_\alpha) \in \Omega(\mathcal{L})(U_\alpha); \qquad \alpha \in I,$$

determine a cochain $(\omega_\alpha) \in C^0(\mathcal{U}, \Omega(\mathcal{L}))$ satisfying (6.1.5). Theorem 6.1.5 now ensures the existence of a connection on $\mathcal{P}$.

The Maurer-Cartan cocycle (6.3.4) can be related with the (coordinate) cocycle $(g_{\alpha\beta}) \in Z^1(\mathcal{U}, \mathcal{G})$ of $\mathcal{P}$ as follows: For each $U_{\alpha\beta} \neq \emptyset$, we consider the isomorphism (see (6.3.1) for $\mathcal{F} = \mathcal{L}$)

$$H_{U_{\alpha\beta}} : \mathrm{Hom}_{\mathcal{G}|_{U_{\alpha\beta}}}\big(\mathcal{P}|_{U_{\alpha\beta}}, \Omega(\mathcal{L})|_{U_{\alpha\beta}}\big) \longrightarrow \Omega(\rho(\mathcal{P}))(U_{\alpha\beta}).$$

Then, viewing $\rho(\mathcal{P})$ as the sheaf $\mathcal{P} \times_X^{\mathcal{G}} \mathcal{L}$ with coordinates $(\widetilde{\Phi}_\alpha)$, and applying the analog of (6.3.2) for the induced morphisms of sections, along with (6.1.1′) and (6.1.3), we see that

$$\begin{aligned} H_{U_{\alpha\beta}}(D_\alpha - D_\beta) &= \big((1 \otimes \widetilde{\Phi}_\beta^{-1}) \circ (D_\alpha - D_\beta)\big)(s_\beta) \\ &= \big(1 \otimes \widetilde{\Phi}_\beta^{-1}\big)(D_\alpha(s_\beta)) \\ &= \big(1 \otimes \widetilde{\Phi}_\beta^{-1}\big)\big(D_\alpha(s_\alpha \cdot g_{\alpha\beta})\big) \\ &= \big(1 \otimes \widetilde{\Phi}_\beta^{-1}\big)(\partial(g_{\alpha\beta})). \end{aligned} \tag{6.3.8}$$

We note that we obtain the same expression if we take $\widetilde{\Phi}_\alpha^{-1}$ and s_α (see the relevant argument in the proof of Proposition 6.3.1 showing that the isomorphism (6.3.1) is well defined). Therefore, the Maurer-Cartan cocycle $(D_\alpha - D_\beta)$ is identified, by means of the isomorphisms $(H_{U_{\alpha\beta}})$, with the cocycle

$$\big((1 \otimes \widetilde{\Phi}_\beta^{-1})(\partial(g_{\alpha\beta}))\big) \in Z^1\big(\mathcal{U}, \Omega(\rho(\mathcal{P}))\big)$$

(cf. also the classical complex analytic case in Atiyah [6, p. 190]).

Under the identifications (6.3.8), by *abuse of notation* one may write $(D_\alpha - D_\beta) \equiv (\partial(g_{\alpha\beta}))$. As a result,

$$\mathfrak{a}(\mathcal{P}) = [(D_\alpha - D_\beta)] \equiv [(\partial(g_{\alpha\beta}))].$$

However, this may lead to some confusion, since the differential of the cocycle $(g_{\alpha\beta})$, i.e., the *1-cochain* $\partial\big((g_{\alpha\beta})\big) = \big(\partial(g_{\alpha\beta})\big) \in C^1(\mathcal{U}, \Omega(\mathcal{L}))$ is *not* necessarily a cocycle, *unless* $\mathcal{G}$ *is abelian.*

From the preceding arguments we see that the sections

$$\bar{\boldsymbol{g}}_{\kappa\lambda} := \big(1 \otimes \widetilde{\Phi}_\lambda^{-1}\big)(\partial(g_{\kappa\lambda})); \qquad \kappa, \lambda \in I, \tag{6.3.9}$$

determine a cocycle $(\bar{\boldsymbol{g}}_{\alpha\beta}) \in Z^1\big(\mathcal{U}, \rho(\Omega(\mathcal{P}))\big)$ such that

$$[(\bar{\boldsymbol{g}}_{\alpha\beta})] \equiv \mathfrak{a}(\mathcal{P}).$$

Another way to prove that $(\bar{\boldsymbol{g}}_{\alpha\beta})$ is a cocycle is to "differentiate" the cocycle condition $g_{\alpha\gamma} = g_{\alpha\beta} \cdot g_{\beta\gamma}$ and then apply (6.3.9) along with (5.4.19). Similar calculations are used in the discussion before the proof of Corollary 6.3.4 below (see, in particular, (6.3.11)), where the reader can find more details.

Regarding the same cocycle, one may also consult two relevant discussions contained in Gunning [38, p. 100, and Appendix 1]. The first concerns the cocycle obtained from the differential of the coordinate cocycle of a Riemann surface. The second is related with the formalism of cohomology with coefficients in locally free analytic sheaves. The conclusions of the aforementioned Appendix can be easily adapted to the case of a principal sheaf admitting local coordinates.

Our previous considerations lead to the following:

6.3.4 Corollary. *A principal sheaf* $(\mathcal{P}, \mathcal{G}, X, \pi)$ *admits a connection if and only if* $[(\bar{\boldsymbol{g}}_{\alpha\beta})] \equiv [(\partial(g_{\alpha\beta}))] = 0$.

Although this is a restatement of Theorem 6.3.3, we intend to give another proof using the local connection forms and the local structure of $\Omega(\rho(\mathcal{P}))$. To prepare this we observe that the local coordinates of $\Omega(\rho(\mathcal{P}))$ are the isomorphisms

$$1 \otimes \widetilde{\Phi}_\alpha : \Omega(\rho(\mathcal{P}))|_{U_\alpha} = \Omega|_{U_\alpha} \otimes_{\mathcal{A}|_{U_\alpha}} \rho(\mathcal{P})|_{U_\alpha} \longrightarrow \Omega|_{U_\alpha} \otimes_{\mathcal{A}|_{U_\alpha}} \mathcal{L}|_{U_\alpha} = \Omega(\mathcal{L})|_{U_\alpha},$$

for all $\alpha \in I$. As in Theorem 6.3.3, $(\widetilde{\Phi}_\alpha)$ are the coordinates of $\rho(\mathcal{P}) \equiv \mathcal{P} \times_X^{\mathcal{G}} \mathcal{L}$. The change of coordinates over $U_{\alpha\beta}$ is the isomorphism

$$(1 \otimes \widetilde{\Phi}_\alpha) \circ (1 \otimes \widetilde{\Phi}_\beta^{-1}) = 1 \otimes (\widetilde{\Phi}_\alpha \circ \widetilde{\Phi}_\beta^{-1}) : \Omega(\mathcal{L})|_{U_{\alpha\beta}} \longrightarrow \Omega(\mathcal{L})|_{U_{\alpha\beta}},$$

whose explicit form is computed as follows: Given an element $\mathrm{w} \in \Omega(\mathcal{L})_x$, with $x \in U_{\alpha\beta}$, there is a $\sigma \in \Omega(U) \otimes_{\mathcal{A}(U)} \mathcal{L}(U)$ such that $\mathrm{w} = [\sigma]_x = \widetilde{\sigma}(x)$, for an open neighborhood U of x. The morphism $1 \otimes (\widetilde{\Phi}_\alpha \circ \widetilde{\Phi}_\beta^{-1})$ can be thought of as generated by the induced morphisms of sections (in full notation) $1 \otimes \overline{(\widetilde{\Phi}_\alpha \circ \widetilde{\Phi}_\beta^{-1})}_V$, for all open $V \subseteq U_{\alpha\beta}$. Then, setting $W := U \cap U_{\alpha\beta}$, we obtain the following analog of Diagram 1.6

$$\begin{array}{ccc}
\Omega(W) \otimes_{\mathcal{A}(W)} \mathcal{L}(W) & \xrightarrow{\;1 \otimes \overline{(\widetilde{\Phi}_\alpha \circ \widetilde{\Phi}_\beta^{-1})}_W\;} & \Omega(W) \otimes_{\mathcal{A}(W)} \mathcal{L}(W) \\
\Big\downarrow & & \Big\downarrow \\
\Omega(\mathcal{L})_x & \xrightarrow{\;1 \otimes (\widetilde{\Phi}_\alpha \circ \widetilde{\Phi}_\beta^{-1})_x\;} & \Omega(\mathcal{L})_x
\end{array}$$

Diagram 6.2

where the vertical arrows denote the corresponding canonical maps into germs. Therefore,

$$\begin{aligned}\big(1\otimes(\widetilde{\Phi}_\alpha\circ\widetilde{\Phi}_\beta^{-1})\big)(\mathrm{w}) &= \big(1\otimes(\widetilde{\Phi}_\alpha\circ\widetilde{\Phi}_\beta^{-1})\big)(\widetilde{\sigma}(x))\\ &= \Big(\big(1\otimes\overline{(\widetilde{\Phi}_\alpha\circ\widetilde{\Phi}_\beta^{-1})}_W\big)(\sigma)\Big)^{\sim}(x).\end{aligned}$$

On the other hand, if we assume that σ is a decomposable tensor of the form $\sigma=\theta\otimes\ell$, then (5.4.19) implies that

$$\begin{gathered}\big(1\otimes\overline{(\widetilde{\Phi}_\alpha\circ\widetilde{\Phi}_\beta^{-1})}_W\big)(\sigma)=\theta\otimes\overline{(\widetilde{\Phi}_\alpha\circ\widetilde{\Phi}_\beta^{-1})}_W(\ell)=\\ \theta\otimes\rho(g_{\alpha\beta})(\ell)=\big(1\otimes\rho(g_{\alpha\beta})\big)(\theta\otimes\ell)=\big(1\otimes\rho(g_{\alpha\beta})\big)(\sigma).\end{gathered}$$

The same equality holds, by linear extension, for every (not necessarily decomposable) tensor σ. Hence, combining the previous equalities with (3.3.7′), we conclude that

$$\begin{aligned}\big(1\otimes(\widetilde{\Phi}_\alpha\circ\widetilde{\Phi}_\beta^{-1})\big)(\mathrm{w}) &= \Big(\big(1\otimes\rho(g_{\alpha\beta})\big)(\sigma)\Big)^{\sim}(x)\\ &= \rho(g_{\alpha\beta}(x)).\mathrm{w},\end{aligned}\tag{6.3.10}$$

for every $\mathrm{w}\in\Omega(\mathcal{L})_x$ and $x\in X$.

The section-wise analog of (6.3.10) over $U_{\alpha\beta}$ is given by

$$\big(1\otimes(\widetilde{\Phi}_\alpha\circ\widetilde{\Phi}_\beta^{-1})\big)(\omega)=\rho(g_{\alpha\beta}).\omega,\qquad \omega\in\Omega(\mathcal{L})(U_{\alpha\beta}).\tag{6.3.11}$$

Indeed, for every $x\in U$, (6.3.10), convention (1.1.3), and (3.3.10) imply that

$$\begin{gathered}\Big(\big(1\otimes(\widetilde{\Phi}_\alpha\circ\widetilde{\Phi}_\beta^{-1})\big)(\omega)\Big)(x)\equiv\Big(\big(1\otimes\overline{(\widetilde{\Phi}_\alpha\circ\widetilde{\Phi}_\beta^{-1})}_{U_{\alpha\beta}}\big)(\omega)\Big)(x)=\\ \big(1\otimes(\widetilde{\Phi}_\alpha\circ\widetilde{\Phi}_\beta^{-1})\big)(\omega(x))=\rho(g_{\alpha\beta}(x)).\omega(x)=\big(\rho(g_{\alpha\beta}).\omega\big)(x),\end{gathered}$$

as claimed.

With the previous preparation, we are in a position to give the proof of the last statement.

Proof of Corollary 6.3.4. Assume that $\mathcal{P}$ admits a connection with corresponding connection forms (ω_α) over $\mathcal{U}$. We define the 1-forms (viz. sections)

$$\Theta_\alpha:=(1\otimes\widetilde{\Phi}_\alpha^{-1})(\omega_\alpha)\in\Omega(\rho(\mathcal{P}))(U_\alpha),\qquad \alpha\in I.$$

Based on (6.1.5), (6.3.11), (6.3.9), we find that

$$\begin{aligned}\Theta_\beta - \Theta_\alpha &= (1\otimes\widetilde{\Phi}_\beta^{-1})(\omega_\beta) - (1\otimes\widetilde{\Phi}_\alpha^{-1})(\omega_\alpha)\\ &= (1\otimes\widetilde{\Phi}_\beta^{-1})\big(\rho(g_{\alpha\beta})^{-1}.\omega_\alpha + \partial(g_{\alpha\beta})\big) - (1\otimes\widetilde{\Phi}_\alpha^{-1})(\omega_\alpha)\\ &= (1\otimes\widetilde{\Phi}_\beta^{-1})\big((1\otimes(\widetilde{\Phi}_\beta\circ\widetilde{\Phi}_\alpha^{-1})(\omega_\alpha)) + \partial(g_{\alpha\beta})\big) - (1\otimes\widetilde{\Phi}_\alpha^{-1})(\omega_\alpha)\\ &= (1\otimes\widetilde{\Phi}_\beta^{-1})(\partial(g_{\alpha\beta})) = \bar{\boldsymbol{g}}_{\alpha\beta};\end{aligned}$$

that is,

$$\Theta_\beta - \Theta_\alpha = \bar{\boldsymbol{g}}_{\alpha\beta}; \qquad \text{over} \quad U_{\alpha\beta}, \tag{6.3.12}$$

for every $\alpha,\beta\in I$. As a result, the cocycle $(\bar{\boldsymbol{g}}_{\alpha\beta})$ is the coboundary of the 0-cochain

$$(\Theta_\alpha) \in C^0\big(\mathcal{U},\Omega(\rho(\mathcal{P}))\big). \tag{6.3.13}$$

In other words, $(\bar{\boldsymbol{g}}_{\alpha\beta}) = \delta^0((\Theta_\alpha))$, if

$$\delta^0 : C^0\big(\mathcal{U},\Omega(\rho(\mathcal{P}))\big) \longrightarrow C^1\big(\mathcal{U},\Omega(\rho(\mathcal{P}))\big)$$

is the 0-coboundary operator. Consequently, $[(\bar{\boldsymbol{g}}_{\alpha\beta})] = 0$, as required.

Conversely, assume that $[(\bar{\boldsymbol{g}}_{\alpha\beta})] = 0$. Then (under an appropriate modification of the covering, if necessary) we may take $(\bar{\boldsymbol{g}}_{\alpha\beta}) \in Z^1(\mathcal{U},\Omega(\rho(\mathcal{P})))$; thus, there is a 0-cochain (Θ_α) of the form (6.3.13) satisfying (6.3.12). Reversing the arguments of the first part of the proof, we obtain a family (ω_α) yielding a connection D on $\mathcal{P}$. □

6.3.5 Corollary. *Connections on $\mathcal{P}$ are in bijective correspondence with cochains of 1-forms $(\Theta_\alpha) \in C^0\big(\mathcal{U},\Omega(\rho(\mathcal{P}))\big)$, such that the compatibility condition*

$$\Theta_\beta = \Theta_\alpha + \bar{\boldsymbol{g}}_{\alpha\beta}.$$

is satisfied over each $U_{\alpha\beta}$.

Proof. Working as in the proof of Corollary 6.3.4, we see that a connection $D \equiv (\omega_\alpha)$ determines the cochain (6.3.13) satisfying (6.3.12); hence, we obtain the condition of the statement.

Conversely, from a cochain $(\Theta_\alpha) \in C^0\big(\mathcal{U},\Omega(\rho(\mathcal{P}))\big)$ we define the cochain of 1-forms (ω_α) with

$$\omega_\alpha := (1\otimes\widetilde{\Phi}_\alpha)(\Theta_\alpha) \in \Omega(\rho(\mathcal{P}))(U_\alpha), \qquad \alpha\in I.$$

Therefore, (6.3.9) and (6.3.11) yield

$$\begin{aligned}\omega_\beta &= (1\otimes\widetilde{\Phi}_\beta)(\Theta_\beta) = (1\otimes\widetilde{\Phi}_\beta)(\Theta_\alpha) + (1\otimes\widetilde{\Phi}_\beta)(\bar{\boldsymbol{g}}_{\alpha\beta})\\ &= \big((1\otimes(\widetilde{\Phi}_\beta\circ\widetilde{\Phi}_\alpha^{-1}))\circ(1\otimes\widetilde{\Phi}_\alpha)\big)(\Theta_\alpha) + (1\otimes\widetilde{\Phi}_\beta)(\bar{\boldsymbol{g}}_{\alpha\beta})\\ &= \big(1\otimes(\widetilde{\Phi}_\beta\circ\widetilde{\Phi}_\alpha^{-1})\big)(\omega_\alpha) + (1\otimes\widetilde{\Phi}_\beta)(\bar{\boldsymbol{g}}_{\alpha\beta})\\ &= \rho(g_{\alpha\beta}^{-1}).\omega_\alpha + \partial(g_{\alpha\beta}).\end{aligned}$$

Hence, the cochain (ω_α) determines a connection on $\mathcal{P}$. □

6.3.6 Corollary. *A connection on $\mathcal{P}$ determines a global section*

$$\chi\in\Omega(\rho(\mathcal{P}))(X).$$

Proof. Let D be any connection on $\mathcal{P}$ with local connection forms (ω_α). By Corollary 6.3.5, there exists a cochain (Θ_α) satisfying (6.3.12). On the other hand, the existence of a connection, being equivalent to $[(\bar{\boldsymbol{g}}_{\alpha\beta})]=0$ (see Corollary 6.3.4), also implies the existence of a cochain $(\bar{\Theta}_\alpha)\in C^0\big(\mathcal{U},\Omega(\rho(\mathcal{P}))\big)$ such that

$$\bar{\Theta}_\beta-\bar{\Theta}_\alpha=\bar{\boldsymbol{g}}_{\alpha\beta};\qquad \alpha,\beta\in I, \tag{6.3.14}$$

(after refining all the coverings involved, if necessary, we can assume that all the cochains are taken over the same covering $\mathcal{U}$). Hence, (6.3.12) and (6.3.14) imply that

$$\Theta_\beta-\Theta_\alpha=\bar{\Theta}_\beta-\bar{\Theta}_\alpha. \tag{6.3.15}$$

Setting $\chi_\alpha:=\bar{\Theta}_\alpha-\Theta_\alpha$, $\alpha\in I$, (6.3.15) shows that we get a global section χ by gluing together all the χ_α's. □

In the sequel we assume that Ω is reflexive, a property already shared by the sheaf of germs of ordinary differential 1-forms on a differential manifold. More precisely, we shall say that Ω is a ***reflexive*** $\mathcal{A}$-module if

$$\Omega^{**}:=(\Omega^*)^*\cong\Omega,$$

where the dual module $\Omega^*:=\mathcal{H}om_{\mathcal{A}}(\Omega,\mathcal{A})$ is the sheaf of germs of $\mathcal{A}$-morphisms of Ω in $\mathcal{A}$ (see Subsection 1.3.5).

We can form the exact sequence of $\mathcal{A}$-modules

$$0\longrightarrow\rho(\mathcal{P})\xrightarrow{\;i\;}\rho(\mathcal{P})\oplus\Omega^*\xrightarrow{\;p\;}\Omega^*\longrightarrow 0, \tag{6.3.16}$$

where i and p are the natural injection and projection respectively. Then we obtain the following:

6.3.7 Corollary. *If Ω is reflexive, then every connection on $\mathcal{P}$ determines a splitting of the exact sequence (6.3.16).*

Proof. The reflexivity of Ω yields

$$\Omega(\rho(\mathcal{P})) = \Omega \otimes_{\mathcal{A}} \rho(\mathcal{P}) \cong \Omega^{**} \otimes_{\mathcal{A}} \rho(\mathcal{P}) \cong \mathcal{H}om_{\mathcal{A}}(\Omega^*, \rho(\mathcal{P})).$$

Consequently,

$$\Omega(\rho(\mathcal{P}))(X) \cong \mathcal{H}om_{\mathcal{A}}(\Omega^*, \rho(\mathcal{P}))(X) \cong \mathrm{Hom}_{\mathcal{A}}(\Omega^*, \rho(\mathcal{P})).$$

Thus, by Corollary 6.3.6, there exists an $\mathcal{A}$-morphism $\bar{h} \in \mathrm{Hom}_{\mathcal{A}}(\Omega^*, \rho(\mathcal{P}))$ corresponding to the given connection. The desired splitting is, evidently, $h := \bar{h} + id|_{\Omega^*}$. □

We close this section with a few comments on the structure of the ***set of connections*** on $\mathcal{P}$, denoted by $Conn(\mathcal{P})$.

Given two connections $D, D' \in Conn(\mathcal{P})$, working as in the case of the Maurer-Cartan cocycle (6.3.4), we see that (see also (6.3.3))

$$D - D' \in \mathrm{Hom}_{\mathcal{G}}(\mathcal{P}, \Omega(\mathcal{L})) \cong \mathcal{H}om_{\mathcal{G}}(\mathcal{P}, \Omega(\mathcal{L}))(X) \cong \Omega(\rho(\mathcal{P}))(X).$$

As a result, we conclude that:

6.3.8 Proposition. *The set of connections $Conn(\mathcal{P})$ is an affine space modelled on the $\mathcal{A}(X)$-module $\Omega(\rho(\mathcal{P}))(X)$.*

6.3.9 Remark. Let P be a smooth principal bundle over a (Hausdorff) paracompact base X and let $\mathcal{P}$ be the principal sheaf of sections of P (see Example 4.1.9(a)). As is well known, P admits connections, thus (by Theorem 6.2.1) so does $\mathcal{P}$. Therefore, in virtue of Theorem 6.3.3, $\mathfrak{a}(\mathcal{P}) = 0$.

However, the annihilation of $\mathfrak{a}(\mathcal{P})$, and the resulting existence of connections on $\mathcal{P}$ and P, can also be explained in the following way: the algebra sheaf $\mathcal{C}^\infty_X$ is fine and so is every $\mathcal{C}^\infty_X$-module. Hence $\Omega(\rho(\mathcal{P}))$, being a fine sheaf, is acyclic and $H^1\big(X, \Omega(\rho(\mathcal{P}))\big) = 0$.

6.4. Mappings of connections

Let $\mathcal{P} \equiv (\mathcal{P}, \mathcal{G}, X, \pi)$ and $\mathcal{P}' \equiv (\mathcal{P}', \mathcal{G}', X, \pi')$ be two principal sheaves, with structure sheaves $\mathcal{G} \equiv (\mathcal{G}, \rho, \mathcal{L}, \partial)$ and $\mathcal{G}' \equiv (\mathcal{G}', \rho', \mathcal{L}', \partial')$, respectively.

As we know (see Definition 4.2.1), a morphism of $\mathcal{P}$ into $\mathcal{P}'$ is determined by a quadruple $(f, \phi, \overline{\phi}, id_X)$, where $(\phi, \overline{\phi})$ is a morphism of Lie sheaves of groups, and the morphism of sheaves $f : \mathcal{P} \to \mathcal{P}'$ satisfies the equivariance property $f(p \cdot g) = f(p) \cdot \phi(g)$, for every $(p, g) \in \mathcal{P} \times_X \mathcal{G}$. Equivalently, for an open $U \subseteq X$, the induced morphism of sections satisfies $f(s \cdot g) = f(s) \cdot \phi(g)$, for all $(s, g) \in \mathcal{P}(U) \times \mathcal{G}(U)$.

6.4.1 Definition. Let $(f, \phi, \overline{\phi}, id_X)$ be a morphism of $\mathcal{P}$ into $\mathcal{P}'$. Two connections D and D', defined on $\mathcal{P}$ and $\mathcal{P}'$ respectively, are said to be $(f, \phi, \overline{\phi}, id_X)$***-related*** if

$$D' \circ f = (1 \otimes \overline{\phi}) \circ D, \tag{6.4.1}$$

as shown in the following commutative diagram.

$$\begin{array}{ccc} \mathcal{P} & \xrightarrow{\quad f \quad} & \mathcal{P}' \\ {\scriptstyle D}\downarrow & & \downarrow{\scriptstyle D'} \\ \Omega(\mathcal{L}) = \Omega \otimes_{\mathcal{A}} \mathcal{L} & \xrightarrow{\ 1 \otimes \overline{\phi}\ } & \Omega \otimes_{\mathcal{A}} \mathcal{L}' = \Omega(\mathcal{L}') \end{array}$$

DIAGRAM 6.3

In particular, if $\mathcal{P}$ and $\mathcal{P}'$ are $\mathcal{G}$-principal sheaves and $(\phi, \overline{\phi}) = (id_{\mathcal{G}}, id_{\mathcal{L}})$, then $(f, id_{\mathcal{G}}, id_{\mathcal{L}}, id_X)$-related connections are called ***f-conjugate***. In this case, (6.4.1) reduces to

$$D = D' \circ f. \tag{6.4.2}$$

We have already proved (Theorem 4.4.1) that a morphism $(f, \phi, \overline{\phi}, id_X)$ of $\mathcal{P}$ into $\mathcal{P}'$ determines a 0-cochain $(h_\alpha) \in C^0(\mathcal{U}, \mathcal{G}')$ satisfying equalities (4.4.1) and (4.4.2). As a result, we obtain the following criterion of relatedness with respect to a *given* morphism of principal sheaves.

6.4.2 Theorem. *Let $(f, \phi, \overline{\phi}, id_X)$ be a morphism of $\mathcal{P}$ into $\mathcal{P}'$. Assume that D and D' are two connections on $\mathcal{P}$ and $\mathcal{P}'$, respectively, with corresponding local connection forms (ω_α) and (ω'_α), over the same open covering $\mathcal{U}$ of X. Then D and D' are $(f, \phi, \overline{\phi}, id_X)$-related if and only if*

$$(1 \otimes \overline{\phi})(\omega_\alpha) = \rho'(h_\alpha^{-1}).\omega'_\alpha + \partial'(h_\alpha), \tag{6.4.3}$$

for every $\alpha \in I$.

Proof. Let D and D' be $(f, \phi, \overline{\phi}, id_X)$-related. Applying (6.1.4), (6.4.1), (4.4.1), and (6.1.1′) for D', we obtain

$$(1 \otimes \overline{\phi})(\omega_\alpha) = \big((1 \otimes \overline{\phi}) \circ D\big)(s_\alpha) =$$
$$(D' \circ f)(s_\alpha) = D'(s'_\alpha \cdot h_\alpha) = \rho'(h_\alpha^{-1}).\omega'_\alpha + \partial'(h_\alpha).$$

Conversely, assume that (6.4.3) is satisfied. For any $p \in \mathcal{P}$ with $\pi(p) = x \in U_\alpha$, we have that $p = s_\alpha(x) \cdot g_\alpha$, for a unique $g_\alpha \in \mathcal{G}$. Therefore (by Theorem 4.4.1),

$$f(p) = f(s_\alpha(x)) \cdot \phi(g_\alpha) = s'_\alpha(x) \cdot h_\alpha(x) \cdot \phi(g_\alpha),$$

and, in virtue of (6.4.3),

$$\begin{aligned}(D' \circ f)(p) &= D'\big(s'_\alpha(x) \cdot h_\alpha(x) \cdot \phi(g_\alpha)\big)\\ &= \big(\rho'(\phi(g_\alpha^{-1})) \cdot \rho'(h_\alpha(x)^{-1})\big).D'(s'_\alpha(x)) + \partial'\big(h_\alpha(x) \cdot \phi(g_\alpha)\big)\\ &= \rho'(\phi(g_\alpha^{-1})).\big(\rho'(h_\alpha(x)^{-1}).\omega'_\alpha(x) + \partial'(h_\alpha(x))\big) + \partial'(\phi(g_\alpha))\\ &= \rho'(\phi(g_\alpha^{-1})).\big((1 \otimes \overline{\phi})(\omega_\alpha(x))\big) + (\partial' \circ \phi)(g_\alpha).\end{aligned}$$

Hence, by Lemma 6.4.3 below and (3.4.2),

$$\begin{aligned}(D' \circ f)(p) &= (1 \otimes \overline{\phi})\big(\rho(g_\alpha^{-1}).\omega_\alpha(x)\big) + (1 \otimes \overline{\phi})(\partial(g_\alpha))\\ &= (1 \otimes \overline{\phi})\big(\rho(g_\alpha^{-1}).\omega_\alpha(x) + \partial(g_\alpha)\big)\\ &= (1 \otimes \overline{\phi}).(D(p)) = ((1 \otimes \overline{\phi}) \circ D)(p),\end{aligned}$$

which shows that D and D' are $(f, \phi, \overline{\phi}, id_X)$-related. □

In the inverse part of the previous proof we used the following general result:

6.4.3 Lemma. *Let $(\phi, \overline{\phi})$ be a morphism of Lie sheaves of groups of $\mathcal{G}$ into $\mathcal{G}'$. Then*

$$(1 \otimes \overline{\phi})\big(\rho(g).\mathrm{w}\big) = \rho'(\phi(g)).(1 \otimes \overline{\phi})(\mathrm{w}),$$

for every $g \in \mathcal{G}_x$, $\mathrm{w} \in \Omega(\mathcal{L})_x \cong \Omega_x \otimes_{\mathcal{A}_x} \mathcal{L}_x$, *and every $x \in X$.*

Proof. If $g = s(x)$ and $\mathrm{w} = [\sigma]_x$ for $s \in \mathcal{G}(U)$ and $\sigma \in \Omega(U) \otimes_{\mathcal{A}(U)} \mathcal{L}(U)$, with $x \in U$, applying (3.3.7′) and the interplay between sheaf morphisms and the induced morphisms of sections, we see that

$$\begin{aligned}(1\otimes\overline{\phi})\big(\rho(g).\mathrm{w}\big) &= (1\otimes\overline{\phi})\big([(1\otimes\rho(s))(\sigma)]_x\big)\\ &= \big[(1\otimes\overline{\phi})\big((1\otimes\rho(s))(\sigma)\big)\big]_x\\ &= \big[\big(1\otimes(\overline{\phi}\circ\rho(s))\big)(\sigma)\big]_x\\ \text{(see (3.4.1}')) \qquad &= \big[\big(1\otimes(\rho'(\phi(s))\circ\overline{\phi})\big)(\sigma)\big]_x\\ &= \big[\big((1\otimes\rho'(\phi(s)))\circ(1\otimes\overline{\phi})\big)(\sigma)\big]_x\\ &= \big[\big(1\otimes\rho'(\phi(s))\big)\big((1\otimes\overline{\phi})(\sigma)\big)\big]_x.\end{aligned}$$

But $\phi(g)=\phi(s(x))=\phi(s)(x)$ and $(1\otimes\overline{\phi})(\mathrm{w})=(1\otimes\overline{\phi})([\sigma]_x)=[(1\otimes\overline{\phi})(\sigma)]_x$. Therefore, using (3.3.7′) once more, we conclude that

$$\begin{aligned}(1\otimes\overline{\phi})(\rho(g).\mathrm{w}) &= \big[\big(1\otimes\rho'(\phi(s))\big)\big((1\otimes\overline{\phi})(\sigma)\big)\big]_x\\ &= \rho'(\phi(g)).\big((1\otimes\overline{\phi})(\mathrm{w})\big).\end{aligned}$$ □

We can also relate connections provided that we are given an appropriate cochain $(h_\alpha)\in C^0(\mathcal{U},\mathcal{G})$ and a morphism of Lie sheaves of groups. More precisely, we have:

6.4.4 Proposition. *Let $\mathcal{P}$ and $\mathcal{P}'$ be two principal sheaves equipped with the connections $D\equiv(\omega_\alpha)$ and $D'\equiv(\omega'_\alpha)$, respectively. Assume that we are given a morphism $(\phi,\overline{\phi})$ of $\mathcal{G}$ into $\mathcal{G}'$, and a 0-cochain $(h_\alpha)\in C^0(\mathcal{U},\mathcal{G}')$ satisfying equalities (4.4.2) and (6.4.3). Then there exists a unique morphism (of sheaves of sets) $f:\mathcal{P}\to\mathcal{P}'$ such that $(f,\phi,\overline{\phi},id_X)$ is a morphism of principal sheaves and the connections D and D' are $(f,\phi,\overline{\phi},id_X)$-related.*

Proof. The existence of the morphism $(f,\phi,\overline{\phi},id_X)$ was established in Theorem 4.4.1. Then equality (6.4.1) is a consequence of Theorem 6.4.2. □

In the particular case of $\mathcal{G}$-principal sheaves, combining Theorems 6.4.2 and 4.2.4, along with Proposition 6.4.4, we obtain:

6.4.5 Theorem. *Let $D\equiv(\omega_\alpha)$ and $D'\equiv(\omega'_\alpha)$ be two connections on the principal sheaves $\mathcal{P}\equiv(\mathcal{P},\mathcal{G},X,\pi)$ and $\mathcal{P}'\equiv(\mathcal{P}',\mathcal{G},X,\pi')$, respectively. Then the following conditions are equivalent:*

i) D and D' are f-conjugate, for a $\mathcal{G}$-(iso)morphism f of $\mathcal{P}$ onto $\mathcal{P}'$.

ii) There exists a 0-cochain $(h_\alpha)\in C^0(\mathcal{U},\mathcal{G})$ such that equalities

$$g'_{\alpha\beta}=h_\alpha\cdot g_{\alpha\beta}\cdot h_\beta^{-1}, \tag{6.4.4}$$

$$\omega_\alpha=\rho(h_\alpha^{-1}).\omega'_\alpha+\partial(h_\alpha) \tag{6.4.5}$$

hold over $U_{\alpha\beta}$ and U_α, respectively, for all $\alpha, \beta \in I$.

The next results provide examples of related connections. Their classical counterparts can be found in Kobayashi -Nomizu [49, Chap. II, Propositions 6.1 and 6.2].

6.4.6 Corollary. *Let $(f, \phi, \overline{\phi}, id_X)$ be a morphism of $\mathcal{P}$ into $\mathcal{P}'$. Then, for each connection $D \equiv (\omega_\alpha)$ on $\mathcal{P}$, there exists a unique $(f, \phi, \overline{\phi}, id_X)$-related connection D' on $\mathcal{P}'$.*

Proof. Motivated essentially by 6.4.2, we define the forms

$$\omega'_\alpha := \rho'(h_\alpha).(1 \otimes \overline{\phi})(\omega_\alpha) + \partial'(h_\alpha^{-1}), \qquad \alpha \in I. \tag{6.4.6}$$

The analog of (6.4.6) for ω_β, and the compatibility condition (6.1.5) yield

$$\begin{aligned}\omega'_\beta &= \rho'(h_\beta).(1 \otimes \overline{\phi})(\omega_\beta) + \partial'(h_\beta^{-1}) \\ &= \rho'(h_\beta).\big((1 \otimes \overline{\phi})(\rho(g_{\alpha\beta}^{-1}).\omega_\alpha) + (1 \otimes \overline{\phi})(\partial(g_{\alpha\beta}))\big) + \partial'(h_\beta^{-1}),\end{aligned}$$

or, by (3.4.2), the compatibility of the action with the $\mathcal{A}$-module structure (see Proposition 3.3.1), and the section-wise analog of Lemma 6.4.3,

$$\begin{aligned}\omega'_\beta &= \rho'(h_\beta).\big(\rho'(\phi(g_{\alpha\beta}^{-1})).(1 \otimes \overline{\phi})(\omega_\alpha) + \partial'(\phi(g_{\alpha\beta}))\big) + \partial'(h_\beta^{-1}) \\ &= \rho'(h_\beta).\big(\rho'(\phi(g_{\alpha\beta}^{-1})).(1 \otimes \overline{\phi})(\omega_\alpha)\big) + \rho'(h_\beta).\partial'(\phi(g_{\alpha\beta})) + \partial'(h_\beta^{-1}) \\ &= \rho'\big(h_\beta \cdot \phi(g_{\alpha\beta}^{-1})\big).(1 \otimes \overline{\phi})(\omega_\alpha) + \rho'(h_\beta).\partial'(\phi(g'_{\alpha\beta})) + \partial'(h_\beta^{-1}) \\ &= \rho'\big(h_\beta \cdot \phi(g_{\alpha\beta}^{-1})\big).(1 \otimes \overline{\phi})(\omega_\alpha) + \partial'\big(\phi(g_{\alpha\beta}).h_\beta^{-1}\big).\end{aligned}$$

Therefore, (4.4.2) and (6.4.6) transform the preceding into

$$\begin{aligned}\omega'_\beta &= \rho'\big((g'_{\alpha\beta})^{-1} \cdot h_\alpha\big).(1 \otimes \overline{\phi})(\omega_\alpha) + \partial'\big(h_\alpha^{-1} \cdot g'_{\alpha\beta}\big) \\ &= \rho'\big((g'_{\alpha\beta})^{-1} \cdot h_\alpha\big).(1 \otimes \overline{\phi})(\omega_\alpha) + \rho'((g'_{\alpha\beta})^{-1}).\partial'(h_\alpha^{-1}) + \partial'(g'_{\alpha\beta}) \\ &= \rho'((g'_{\alpha\beta})^{-1}).\big(\rho'(h_\alpha).(1 \otimes \overline{\phi})(\omega_\alpha) + \partial'(h_\alpha^{-1})\big) + \partial'(g'_{\alpha\beta}) \\ &= \rho'((g'_{\alpha\beta})^{-1}).\omega'_\alpha + \partial'(g'_{\alpha\beta});\end{aligned}$$

that is, the 0-cochain (ω'_α) determines a connection D' on $\mathcal{P}'$, as a consequence of Theorem 6.1.5.

On the other hand, in virtue of Proposition 3.3.5, equality (6.4.6) takes the form

$$\begin{aligned}\omega'_\alpha &= \rho'(h_\alpha).(1\otimes\overline{\phi})(\omega_\alpha) + \partial'(h_\alpha^{-1})\\ &= \rho'(h_\alpha).(1\otimes\overline{\phi})(\omega_\alpha) - \rho'(h_\alpha).\partial'(h_\alpha)\\ &= \rho'(h_\alpha).\big((1\otimes\overline{\phi})(\omega_\alpha) - \partial'(h_\alpha)\big),\end{aligned}$$

from which we get (6.4.3), for all $\alpha \in I$. As a consequence, D and D' are $(f,\phi,\overline{\phi}, id_X)$-related.

Finally, the uniqueness of $D' \equiv (\omega'_\alpha)$ is proved as follows: If there is a connection $\overline{D} \equiv (\overline{\omega}_\alpha)$, which is also $(f,\phi,\overline{\phi}, id_X)$-related with D, then (6.4.3) implies that $(\omega'_\alpha) = (\overline{\omega}_\alpha)$. Hence, for any $p \in \mathcal{P}_x$ with $x \in U_\alpha$, we write $p = s_\alpha(x)\cdot g$ (for a unique $g \in \mathcal{G}_x$) and

$$\overline{D}(p) = \overline{D}(s_\alpha(x)\cdot g) = \rho(g^{-1}).\overline{D}(s_\alpha(x)) + \partial(g) =$$
$$\rho(g^{-1}).\overline{\omega}_\alpha(x) + \partial(g) = \rho(g^{-1}).\omega'_\alpha(x) + \partial(g) = D'(p). \qquad \square$$

6.4.7 Corollary. *Let $(f,\phi,\overline{\phi}, id_X)$ be a morphism of $\mathcal{P}$ into $\mathcal{P}'$ such that $\overline{\phi}$ is an isomorphism of Lie algebra $\mathcal{A}$-modules. Then every connection $D' \equiv (\omega'_\alpha)$ on $\mathcal{P}'$ induces a unique connection D on $\mathcal{P}$, which is $(f,\phi,\overline{\phi}, id_X)$-related with D'.*

Proof. It suffices to take $D := (1\otimes\overline{\phi}^{-1})\circ D'\circ f$. $\square$

6.4.8 Remark. It is immediately checked that the local connection forms (ω_α) of the connection D, defined in Corollary 6.4.7, are given by

$$\omega_\alpha = (1\otimes\overline{\phi})^{-1}\big(\rho'(h_\alpha^{-1}).\omega'_\alpha + \partial'(h_\alpha)\big), \tag{6.4.7}$$

as a result of equality (4.4.1) defining h_α. In the case of an isomorphism of Lie sheaves of groups $(\phi,\overline{\phi})$, setting $g_\alpha := \phi^{-1}(h_\alpha)$ and applying (3.4.2), together with the section-wise analog of Lemma 6.4.3, we transform (6.4.7) into

$$\omega_\alpha = \rho(g_\alpha^{-1}).\big((1\otimes\overline{\phi})^{-1}(\omega'_\alpha)\big) + \partial(g_\alpha). \tag{6.4.7$'$}$$

Thus one can equivalently define the connection D of Corollary 6.4.7 starting with the 0-cochain (ω_α), given by (6.4.7) or (6.4.7′), and working as in the proof of Corollary 6.4.6.

6.5. The pull-back of a connection

The present short section deals with the connection induced on the pull-back –by a continuous map– of a principal sheaf equipped with a connection. This important construction adds another (non trivial) example to the list of Section 6.2.

Taking up the notations of Example 4.1.9(c), we consider a fixed continuous map $f : Y \to X$ and a principal sheaf $\mathcal{P} \equiv (\mathcal{P}, \mathcal{G}, X, \pi)$. We have already seen that $f^*(\mathcal{P}) \equiv (f^*(\mathcal{P}), f^*(\mathcal{G}), Y, \pi^*)$ is a principal sheaf with structure sheaf $f^*(\mathcal{G}) \equiv \big(f^*(\mathcal{G}), \rho^*, f^*(\mathcal{L}), \partial^*\big)$, where ρ^* and ∂^* are defined by (3.5.14) and (3.5.6), respectively (see also Theorem 3.5.4).

Assume now that $\mathcal{P}$ is provided with a connection $D : \mathcal{P} \to \Omega(\mathcal{L})$. Then we define the morphism of sheaves

$$D^* := \boldsymbol{\tau} \circ f^*(D), \tag{6.5.1}$$

as pictured in the following diagram

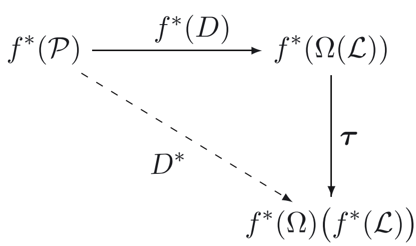

Diagram 6.4

where $\boldsymbol{\tau}$ is the isomorphism of $f^*(\mathcal{A})$-modules defined by Lemma 3.5.1.

6.5.1 Proposition. *The morphism $D^* : f^*(\mathcal{P}) \to f^*(\Omega)\big(f^*(\mathcal{L})\big)$ is a connection on $f^*(\mathcal{P})$, called the* ***pull-back connection*** *of D.*

Proof. We prove the property (6.1.1) by essentially repeating the proof of Theorem 3.5.4. As a matter of fact, for any $(y,p) \in f^*(\mathcal{P})_y = \{y\} \times \mathcal{P}_{f(y)}$ and $(y,g) \in f^*(\mathcal{G})_y = \{y\} \times \mathcal{G}_{f(y)}$, we check that

$$D^*\big((y,p)\cdot(y,g)\big) = D^*(y, p\cdot g)$$

$$
\begin{aligned}
&= \boldsymbol{\tau}(y, D(p\cdot g))\\
&= \boldsymbol{\tau}\big(y, \rho(g^{-1}).D(p) + \partial(g)\big)\\
&= \boldsymbol{\tau}\big(y, \rho(g^{-1}).D(p)\big) + \boldsymbol{\tau}\big(y, \partial(g)\big)\\
&= \boldsymbol{\tau}\big(y, \Delta(g^{-1}, D(p))\big) + \partial^*(y,g)\\
&= \Delta^*\big((y,g^{-1}), \boldsymbol{\tau}(y, D(p))\big) + \partial^*(y,g)\\
&= \rho^*(y,g^{-1}).D^*(y,p) + \partial^*(y,g).
\end{aligned}
$$

Since $(y, g^{-1}) = (y,g)^{-1}$, we conclude the proof. □

Note. Analogously to the identifications $\partial^*(x,g) \equiv (x, \partial(g))$, and $\rho^*(x,g) \equiv (x, \rho(g))$ (see (3.5.7′), (3.5.15′)), we may write

(6.5.2) $$D^*(y,p) \equiv (y, D(p)),$$

which simplifies complex computations.

Let us find the local connection forms (ω^*_α) of D^*, over the local frame $(\mathcal{V}, (\phi^*_\alpha))$ of $f^*(\mathcal{P})$, where

$$\mathcal{V} = \big\{V_\alpha := f^{-1}(U_\alpha) \mid U_\alpha \in \mathcal{U}\big\} \quad \text{and} \quad \phi^*_\alpha := f^*(\phi_\alpha),$$

if $(\mathcal{U}, (\phi_\alpha))$ is a local frame of $\mathcal{P}$. In virtue of (4.1.11), we find, for every $y \in Y$:

$$
\begin{aligned}
\omega^*_\alpha(y) &= D^*(s^*_\alpha)(y) = D^*\big(y, s_\alpha(f(y))\big)\\
&= \boldsymbol{\tau}\big(y, D(s_\alpha(f(y)))\big) = \boldsymbol{\tau}\big(y, \omega_\alpha(f(y))\big).
\end{aligned}
$$

Using the adjunction map $f^*_{U_\alpha} : (\Omega \otimes_{\mathcal{A}} \mathcal{L})(U_\alpha) \to f^*(\Omega \otimes_{\mathcal{A}} \mathcal{L})(V_\alpha)$ (see also (1.4.2)), the above expression yields

(6.5.3) $$\omega^*_\alpha = \boldsymbol{\tau}\big(f^*_{U_\alpha}(\omega_\alpha)\big) \equiv f^*_{U_\alpha}(\omega_\alpha).$$

We close the section by showing that the connections D and D^* are related in a sense generalizing that of Definition 6.4.1. Since this is the only case where we use this generalization, we have taken a slightly informal, less detailed approach.

According to Remark 4.2.2(2) and equality (4.2.2), we define the following morphisms over the continuous map f:

$$\begin{aligned} f^*_{\mathcal{P}} &: f^*(\mathcal{P}) \longrightarrow \mathcal{P} \\ f^*_{\mathcal{G}} &: f^*(\mathcal{G}) \longrightarrow \mathcal{G} \\ f^*_{\mathcal{L}} &: f^*(\mathcal{L}) \longrightarrow \mathcal{P} \\ f^*_{\Omega} &: f^*(\Omega) \longrightarrow \Omega \\ f^*_{\otimes} &: f^*(\Omega(\mathcal{L})) \longrightarrow \Omega(\mathcal{L}) \end{aligned}$$

i.e., the projections of the indicated fiber products (pull-backs) to the second factor.

By Corollary (3.5.5), $(f^*_{\mathcal{G}}, f^*_{\mathcal{L}})$ is a morphism between the Lie sheaves of groups $f^*(\mathcal{G})$ and $\mathcal{G}$. Analogously, $(f^*_{\mathcal{P}}, f^*_{\mathcal{G}}, f^*_{\mathcal{L}}, f)$ is a morphism of principal sheaves, with respect to the previous Lie sheaves of groups (see also Proposition 4.2.5). Finally, the identification $\boldsymbol{\tau}$ implies that $f^*_{\otimes} = f^*_{\Omega} \otimes f^*_{L} \circ \boldsymbol{\tau}$.

As a result, (6.5.1) leads at once to the commutative diagram

$$\begin{array}{ccc} f^*(\mathcal{P}) & \xrightarrow{\quad f^*_{\mathcal{P}} \quad} & \mathcal{P} \\ {\scriptstyle D^*}\big\downarrow & & \big\downarrow{\scriptstyle D} \\ f^*(\Omega)\big(f^*(\mathcal{L})\big) & \xrightarrow{\quad f^*_{\Omega} \otimes f^*_{\mathcal{L}} \quad} & \Omega(\mathcal{L}) \end{array}$$

Diagram 6.5

which means that D^* and D are $\big(f^*_{\mathcal{P}}, f^*_{\mathcal{G}}, f^*_{\mathcal{L}}, f\big)$-related, according to the obvious generalization of Definition 6.4.1 and Diagram 6.3.

6.6. The moduli sheaf of connections

Here we examine the behavior of the connections on a principal sheaf $\mathcal{P}$, with regard to the gauge transformations of $\mathcal{P}$. This will allow us to group the connections of $\mathcal{P}$ in certain equivalence classes.

We assume that the set of connections is not empty, $Conn(\mathcal{P}) \neq \emptyset$, a fact ensured, e.g., by the conditions of Section 6.3. Based on Definitions 5.3.11 and 6.4.1, we give the following:

6.6.1 Definition. Two connections $D, D' \in Conn(\mathcal{P})$ are said to be ***gauge equivalent*** if they are conjugate by means of a gauge transformation; in other words, if there is an $f \in GA(\mathcal{P})$ such that $D' = D \circ f$.

It is also customary to write

$$D' = f^*D := D \circ f, \tag{6.6.1}$$

but f^*D should not be confused with the pull-back connection $f^*(D)$ already defined by (6.5.1).

Let (ω_α) and (ω'_α) be the respective local connection forms of D and D', over a local frame $\mathcal{U}$. If τ *is the tensorial morphism corresponding to* f (see Proposition 5.3.12 and equality (5.3.24)), then condition (6.6.1) is equivalent to

$$\omega_\alpha = \rho\big(\tau(s_\alpha)^{-1}\big).\omega'_\alpha + \partial(\tau(s_\alpha)), \tag{6.6.2}$$

in virtue of Theorem 6.4.5. This is the case, since, as is readily checked, the sections (h_α) in (6.4.5) (defined by (4.4.6) for $s_\alpha = s'_\alpha$) now coincide with $\tau(s_\alpha)$.

Definition 6.6.1 induces the following equivalence relation on $Conn(\mathcal{P})$:

$$D \sim D' \quad \Longleftrightarrow \quad \exists\ f \in GA(\mathcal{P})\ :\ D' = f^*D := D \circ f. \tag{6.6.3}$$

The same relation can be obtained via the natural action

$$Conn(\mathcal{P}) \times GA(\mathcal{P}) \longrightarrow Conn(\mathcal{P}) : (D, f) \mapsto f^*D. \tag{6.6.4}$$

In this case we check that f^*D is indeed a connection, since

$$\begin{aligned} (f^*D)(p.g) &= D(f(p \cdot g)) = D(f(p) \cdot g) \\ &= \rho(g^{-1}).D(f(p)) + \partial(g) \\ &= \rho(g^{-1}).(f^*D)(p) + \partial(g), \end{aligned}$$

for every $(p, g) \in \mathcal{P} \times_X \mathcal{G}$.

It is clear that an equivalence class $[D]$, with respect to (6.6.3), coincides with the orbit of D, with respect to (6.6.4), i.e.,

$$[D] = \mathcal{O}_D := \big\{\, D \circ f \mid f \in GA(\mathcal{P}) \big\}.$$

6.6.2 Definition. The ***moduli space*** of (the connections of) $\mathcal{P}$ is the space

$$M(\mathcal{P}) := Conn(\mathcal{P})/GA(\mathcal{P}) = \bigcup_{D\in Conn(\mathcal{P})} \mathcal{O}_D.$$

Our objective is to relate $M(\mathcal{P})$ with the global sections of a sheaf constructed by an appropriate action of the sheaf of groups $\mathcal{H}om_{\mathrm{ad}}(\mathcal{P},\mathcal{G})$ on the sheaf of connections $\mathcal{C}(\mathcal{P})$, the latter being described in Example 6.2(c).

By Corollary 5.3.6, $\mathcal{C}(\mathcal{P})$ can be thought of as being generated by the presheaf

$$U \longmapsto Q(U) := \big(\mathcal{P}(U)\times\Omega(\mathcal{L})(U)\big)/\mathcal{G}(U),$$

where the quotient is defined with respect to the action (6.2.9), localized over each U. On the other hand, $\mathcal{H}om_{\mathrm{ad}}(\mathcal{P},\mathcal{G})$ is generated by the presheaf

$$U \longmapsto \mathrm{Hom}_{\mathrm{ad}}(\mathcal{P}|_U,\mathcal{G}|_U),$$

as discussed after Corollary 5.3.10.

Now, for each open $U\subseteq X$, we define the map

$$\delta_U : Q(U)\times \mathrm{Hom}_{\mathrm{ad}}(\mathcal{P}|_U,\mathcal{G}|_U) \longrightarrow Q(U)$$

by setting

$$\delta_U([(s,\omega)],\tau) \equiv [(s,\omega)]\cdot\tau := \big[\big(s,\ \rho(\tau(s)^{-1}).\omega + \partial(\tau(s))\big)\big]. \tag{6.6.5}$$

First we check that δ_U is *well defined*: If $[(s',\omega')] = [(s,\omega)]$, then (by the definition of the equivalence in $Q(U)$ induced by (6.2.9)), there is a (unique) $g\in\mathcal{G}(U)$ such that $s' = s\cdot g$ and $\omega' = \rho(g^{-1}).\omega + \partial(g)$. Therefore, (5.3.22) implies that

$$\begin{aligned}\rho(\tau(s')^{-1}).\omega' &= \rho\big(\tau(s\cdot g)^{-1}\big).\big(\rho(g^{-1}).\omega + \partial(g)\big)\\ &= \rho\big(g^{-1}\cdot\tau(s)^{-1}\cdot g\big).\big(\rho(g^{-1}).\omega + \partial(g)\big)\\ &= \rho\big(g^{-1}\cdot\tau(s)^{-1}\big).\omega + \rho\big(g^{-1}\cdot\tau(s)^{-1}\cdot g\big).\partial(g).\end{aligned} \tag{6.6.6}$$

Similarly, using Proposition 3.3.5, we find that

$$\begin{aligned}\partial(\tau(s')) &= \partial(\tau(s\cdot g)) = \partial(g^{-1}\cdot\tau(s)\cdot g)\\ &= \rho(g^{-1}).\partial(g^{-1}\cdot\tau(s)) + \partial(g)\\ &= \rho(g^{-1}).\big(\rho(\tau(s)^{-1}).\partial(g^{-1}) + \partial(\tau(s))\big) + \partial(g)\\ &= -\rho\big(g^{-1}\cdot\tau(s)^{-1}\cdot g\big).\partial(g) + \rho(g^{-1}).\partial(\tau(s)) + \partial(g).\end{aligned} \tag{6.6.7}$$

Hence, (6.6.6) and (6.6.7), along, once again, with the local form of (6.2.9), imply that

$$\begin{aligned}
[(s',\omega')]\cdot\tau &= \big[\big(s',\, \rho(\tau(s')^{-1}).\omega' + \partial(\tau(s'))\big)\big]\\
&= \big[\big(s\cdot g,\, \rho(g^{-1}).\big(\rho(\tau(s)^{-1}).\omega + \partial(\tau(s))\big) + \partial(g)\big)\big]\\
&= \big[\big(s,\, \rho(\tau(s)^{-1}).\omega + \partial(\tau(s))\big)\big]\\
&= [(s,\omega)]\cdot\tau,
\end{aligned}$$

which proves the claim.

Next, we check that δ_U defines an *action*: For any $[(s,\omega)] \in Q(U)$ and $\tau, \tau' \in \mathrm{Hom}_{\mathrm{ad}}(\mathcal{P}|_U, \mathcal{G}|_U)$, we have that

$$\begin{aligned}
[(s,\omega)]\cdot(\tau\cdot\tau') &= \big[\big(s,\, \rho((\tau\cdot\tau')(s)^{-1}).\omega + \partial((\tau\cdot\tau')(s))\big)\big]\\
&= \big[\big(s,\, (\rho(\tau'(s)^{-1})\cdot\rho(\tau(s)^{-1})).\omega +\\
&\qquad\qquad + \rho(\tau'(s)^{-1}).\partial(\tau(s)) + \partial(\tau'(s))\big)\big]\\
&= \big[\big(s,\, \rho(\tau'(s)^{-1}).(\rho(\tau(s)^{-1}).\omega + \partial(\tau(s))) + \partial(\tau'(s))\big)\big]\\
&= \big[\big(s,\, \rho(\tau(s)^{-1}).\omega + \partial(\tau(s))\big)\big]\cdot\tau'\\
&= ([(s,\omega)]\cdot\tau)\cdot\tau'.
\end{aligned}$$

On the other hand, the identity (neutral) element of $\mathrm{Hom}_{\mathrm{ad}}(\mathcal{P}|_U, \mathcal{G}|_U)$ is the tensorial morphism $\tau_{\mathrm{o}} = \mathbf{1}\circ\pi$, where $\mathbf{1}$ is the identity section of $\mathcal{G}$ and π the projection of $\mathcal{P}$. Then $\partial(\tau_{\mathrm{o}}) = \partial(\mathbf{1}) = 0$ and $\rho(\tau_{\mathrm{o}}^{-1}) = \rho(\mathbf{1}) = id_{\mathcal{L}}$. Thus (6.6.5) yields $[(s,\omega)]\cdot\tau_{\mathrm{o}} = [(s,\omega)]$, by which we complete the verification of the properties of an action.

Finally, it is not difficult to show that the family (δ_U), with U running in the topology of X, is a presheaf morphism generating an action

(6.6.8) $$\delta : \mathcal{C}(\mathcal{P}) \times_X \mathcal{H}om_{\mathrm{ad}}(\mathcal{P},\mathcal{G}) \longrightarrow \mathcal{C}(\mathcal{P}).$$

Therefore, following the general construction of Section 5.3, we obtain the quotient sheaf

(6.6.9) $$\mathcal{M}(\mathcal{P}) := \mathcal{C}(\mathcal{P})/\mathcal{H}om_{\mathrm{ad}}(\mathcal{P},\mathcal{G}).$$

As a matter fact, $\mathcal{M}(\mathcal{P})$ is generated by the presheaf

(6.6.10) $$U \longmapsto \mathcal{C}(\mathcal{P})(U)/\mathrm{Hom}_{\mathrm{ad}}(\mathcal{P}|_U, \mathcal{G}|_U).$$

6.6.3 Definition. The sheaf $\mathcal{M}(\mathcal{P})$ is called the ***moduli sheaf*** of (the connections of) the principal sheaf $\mathcal{P}$.

The action δ induces the action on global sections

$$\begin{aligned}\bar{\delta}_X : \mathcal{C}(\mathcal{P})(X) \times \mathcal{H}om_{\mathrm{ad}}(\mathcal{P},\mathcal{G})(X) &\longrightarrow \mathcal{C}(\mathcal{P})(X) \\ \bar{\delta}_X(S,\tau) &:= S\cdot\tau,\end{aligned} \tag{6.6.11}$$

where the section on the right-hand side is clearly defined by

$$(S\cdot\tau)(x) := S(x)\cdot\tau(x) = \delta(S(x),\tau(x)), \qquad x\in X. \tag{6.6.12}$$

Thus, as usual, we define an obvious equivalence relation on $\mathcal{C}(\mathcal{P})(X)$ in virtue of which we obtain a new quotient space, namely

$$\mathcal{C}(\mathcal{P})(X)/\mathcal{H}om_{\mathrm{ad}}(\mathcal{P},\mathcal{G})(X). \tag{6.6.13}$$

6.6.4 Lemma. *There exists a bijection*

$$\mu : M(\mathcal{P}) := Conn(\mathcal{P})/GA(\mathcal{P}) \xrightarrow{\ \simeq\ } \mathcal{C}(\mathcal{P})(X)/\mathcal{H}om_{\mathrm{ad}}(\mathcal{P},\mathcal{G})(X).$$

Proof. Given a class $[D]\in M(\mathcal{P})$, we set $\mu([D]) := [S]$, where $S\in\mathcal{C}(\mathcal{P})$ is the global section of the sheaf of connections corresponding uniquely to D, in virtue of Theorem 6.2.4. Although we use the same symbol for both equivalence classes, we should carefully distinguish them, since the first one is induced by the action (6.6.4), whereas the second is induced by (6.6.11), the latter being in turn generated by (6.6.5).

To proceed to the main part of the proof, we need to find an explicit expression of S in terms of D. Localizing the general construction of Theorem 5.3.9, adapted to the case of $\mathcal{C}(\mathcal{P})$, S is obtained by gluing together the local sections S_α defined by

$$S_\alpha(x) = \big[\big(s_\alpha(x), D(s_\alpha(x))\big)\big] = \big[\big(s_\alpha(x),\omega_\alpha(x)\big)\big]; \qquad x\in U_\alpha, \tag{6.6.14}$$

where the last equivalence class is now determined by the action (6.2.9).

Conversely, given a section $S\in\mathcal{C}(\mathcal{P})(X)$, the corresponding connection D is determined by the formula

$$D(p) = g_\alpha^{-1}\cdot\widetilde{\Phi}_\alpha(S(x)),$$

where $\pi(p)=x\in U_\alpha$, $p=s_\alpha(x)\cdot g_\alpha$, and $\widetilde{\Phi}_\alpha : \mathcal{C}(\mathcal{P})|_{U_\alpha} \xrightarrow{\simeq} \Omega(\mathcal{L})|_{U_\alpha}$. Then, taking $p=s_\alpha(x)$, the above formula yields

$$\omega_\alpha(x) = D(s_\alpha(x)) = \mathbf{1}(x)\cdot\widetilde{\Phi}_\alpha(S(x));$$

that is,

$$\omega_\alpha = \widetilde{\Phi}_\alpha \circ S|_{U_\alpha}.$$

The map μ is well defined. Indeed, if D' is another connection such that $D' \sim D$, there exists $f \in GA(\mathcal{P})$ satisfying $D' = f^*D = D \circ f$. Hence, considering the sections S and S' corresponding to D and D', respectively, as well as the equivariant morphism $\tau \in GA(\mathcal{P})$ corresponding to $f \in \mathrm{Hom}_{\mathrm{ad}}(\mathcal{P}, \mathcal{G}) \cong \mathcal{H}om_{\mathrm{ad}}(\mathcal{P}, \mathcal{G})(X)$, equalities (6.6.14), (6.6.2), (6.6.5) and (6.6.12) imply that

$$\begin{aligned} S'(x) &= S'_\alpha(x) = \big[\big(s_\alpha(x), \omega'_\alpha(x)\big)\big] \\ &= \big[\big(s_\alpha(x), (\rho(\tau(s_\alpha)^{-1}).\omega_\alpha + \partial(\tau(s_\alpha))(x)\big)\big] \\ &= \big[\big(s_\alpha(x), \omega_\alpha(x)\big)\big] \cdot \tau(x) = S(x) \cdot \tau(x) \\ &= (S \cdot \tau)(x), \end{aligned}$$

for every $x \in U_\alpha$, and similarly for every $x \in X$. This proves that $S' \sim S$; hence, $\mu([D]) = [S]$ is independent of the choice of representatives.

Now assume that $[S'] = \mu([D']) = \mu([D]) = [S]$. Then there exists a morphism τ such that $S' = S \cdot \tau$. Working as above, in a reverse sense, we see that

$$\big[\big(s_\alpha(x), \omega'_\alpha(x)\big)\big] = \big[\big(s_\alpha(x), \big(\rho(\tau(s_\alpha)^{-1}).\omega_\alpha + \partial(\tau(s_\alpha))\big)(x)\big)\big],$$

for every $x \in U_\alpha$. Applying the definition of the equivalence induced by (6.2.9), we obtain

$$\omega'_\alpha = \rho(\tau(s_\alpha)^{-1}).\omega_\alpha + \partial(\tau(s_\alpha)),$$

for all $\alpha \in I$. Hence, (6.6.2) yields $D \sim D'$, i.e., $[D'] = [D]$, which means that μ is an injection.

Finally, given $[S] \in \mathcal{C}(\mathcal{P})(X)/\mathcal{H}om_{\mathrm{ad}}(\mathcal{P}, \mathcal{G})(X)$, the unique D corresponding to S determines the class $[D]$ with $\mu([D]) = [S]$. This shows that μ is a surjection and completes the proof. □

Since $\mathcal{M}(\mathcal{P})$ is generated by the presheaf (6.6.10), there is a canonical map (see (1.2.8))

$$\rho_U : \mathcal{C}(\mathcal{P})(U)/\mathrm{Hom}_{\mathrm{ad}}(\mathcal{P}|_U, \mathcal{G}|_U) \longrightarrow \mathcal{M}(\mathcal{P})(U),$$

for every open $U \subseteq X$. Therefore, we obtain the following result connecting the moduli space with the moduli sheaf.

6.6.5 Theorem. *The moduli space $M(\mathcal{P})$ is canonically mapped into the global sections $\mathcal{M}(\mathcal{P})(X)$ of the moduli sheaf by means of $\boldsymbol{\mu} := \rho_X \circ \mu$, also shown in the following diagram.*

$$\begin{array}{ccc} M(\mathcal{P}) & \xrightarrow{\ \mu\ } & \mathcal{C}(\mathcal{P})(X)/\mathcal{H}om_{\mathrm{ad}}(\mathcal{P},\mathcal{G})(X) \\ & \searrow^{\boldsymbol{\mu}} & \Big\downarrow \rho_X \\ & & \mathcal{M}(\mathcal{P})(X) \end{array}$$

Diagram 6.6

Note. In virtue of Corollary 5.3.14, we see that Lemma 6.6.4 implies the following sequence of identifications:

$$\begin{aligned} M(\mathcal{P}) &\cong \mathcal{C}(\mathcal{P})(X)/\mathcal{H}om_{\mathrm{ad}}(\mathcal{P},\mathcal{G})(X) \\ &\cong \mathcal{C}(\mathcal{P})(X)/\big(\mathcal{G}\mathcal{A}(\mathcal{P})\big)(X) \\ &\cong \mathcal{C}(\mathcal{P})(X)/ad(\mathcal{P})(X), \end{aligned}$$

which give other interpretations of the moduli space.

6.7. Classification of principal sheaves with connections and abelian structure group

This section deals with the hypercohomological classification of principal sheaves equipped with connections, under the assumption that the structure group is an abelian Lie sheaf of groups.

For the sake of brevity, a $\mathcal{G}$-principal sheaf with abelian $\mathcal{G}$ is called an ***abelian principal sheaf***, a term explaining the (short) running head of the section.

We first give the following, fairly general:

6.7.1 Definition. A pair $(\mathcal{P}, D)$, where $\mathcal{P}$ is any (not necessarily abelian) principal sheaf and D a connection on $\mathcal{P}$, is said to be ***equivalent*** with $(\mathcal{P}', D')$, symbolically $(\mathcal{P}, D) \sim (\mathcal{P}', D')$, if there exists a $\mathcal{G}$-isomorphism $f : \mathcal{P} \to \mathcal{P}'$ such that D and D' are f-conjugate (see Definition 6.4.1).

Now assume that $\mathcal{G}$ is an abelian Lie sheaf of groups. Then the compatibility condition (6.1.5) between the local connection forms (ω_α) of a connection D on $\mathcal{P}$, reduces to

$$\omega_\beta = \omega_\alpha + \partial(g_{\alpha\beta}). \tag{6.7.1}$$

Moreover, equalities (6.4.4) and (6.4.5), expressing the equivalence of two pairs $(\mathcal{P}, D)$ and $(\mathcal{P}', D')$, take the respective forms

$$g'_{\alpha\beta} = h_\alpha \cdot g_{\alpha\beta} \cdot h_\beta^{-1}, \tag{6.7.2}$$

$$\omega_\alpha = \omega'_\alpha + \partial(h_\alpha). \tag{6.7.3}$$

We note that, although (6.7.2) looks identical to (6.4.4), the commutativity of $\mathcal{G}$ plays a significant rôle, as we shall see soon.

Extending the notation (4.6.1) to include connections, we denote by

$$\boldsymbol{P}_{\mathcal{G}}(X)^D \tag{6.7.4}$$

the set of equivalence classes derived from Definition 6.7.1. On the other hand,

$$\check{\boldsymbol{H}}^1\big(X, \mathcal{G} \xrightarrow{\partial} \Omega(\mathcal{L})\big) \tag{6.7.5}$$

stands for the (Čech) 1-dimensional ***hypercohomology group*** with respect to the 2-term complex $\mathcal{G} \xrightarrow{\partial} \Omega(\mathcal{L})$ (see Brylinski [17, p. 21], Mallios [62, Vol. I, p. 224], and the brief commentary following the next statement).

With these notations in mind, we state the main result of this section.

6.7.2 Theorem. *If $\mathcal{G}$ is an abelian Lie sheaf of groups, then*

$$\boldsymbol{P}_{\mathcal{G}}(X)^D \cong \check{\boldsymbol{H}}^1\big(X, \mathcal{G} \xrightarrow{\partial} \Omega(\mathcal{L})\big).$$

For the reader's convenience, before the proof, we recall the highlights of Čhech hypercohomology, referring for complete details to [17] and [62] and their references to the subject. Here we mainly follow the terminology and notations of the second source.

Let $(X, \mathcal{A})$ be a fixed algebraized space and $\mathcal{U} = (U_\alpha)$ an open covering of X. We also assume that we are given a complex of $\mathcal{A}$-modules

$$\mathcal{E}^\bullet \equiv (\mathcal{E}^\bullet, d) = (\mathcal{E}^m, d^m)_{m\in\mathbb{Z}}.$$

Motivated by the particular considerations needed in the proof of Theorem 6.7.2, we further assume that $\mathcal{E}^m = 0$, for every $m < 0$. Fixing an $m \in \mathbb{Z}$, we may consider the chain complex

$$C^\bullet(\mathcal{U}, \mathcal{E}^m) = \big(C^n(\mathcal{U}, \mathcal{E}^m),\, \delta = \{\delta^{n,m}\}\big)_{n \in \mathbb{Z}_0^+},$$

where each $\delta^{n,m} : C^n(\mathcal{U}, \mathcal{E}^m) \longrightarrow C^{n+1}(\mathcal{U}, \mathcal{E}^m)$ is the usual coboundary operator (see Subsection 1.6.1). Varying both n and m, we form a double complex of $\mathcal{A}$-modules

$$C^\bullet\big(\mathcal{U}, \mathcal{E}^\bullet, \delta,\, d\big) = \big(\{C^n(\mathcal{U}, \mathcal{E}^m)\}_{(n,m) \in \mathbb{Z}_0^+ \times \mathbb{Z}},\, \delta,\, d\big).$$

This situation is illustrated in the next diagram:

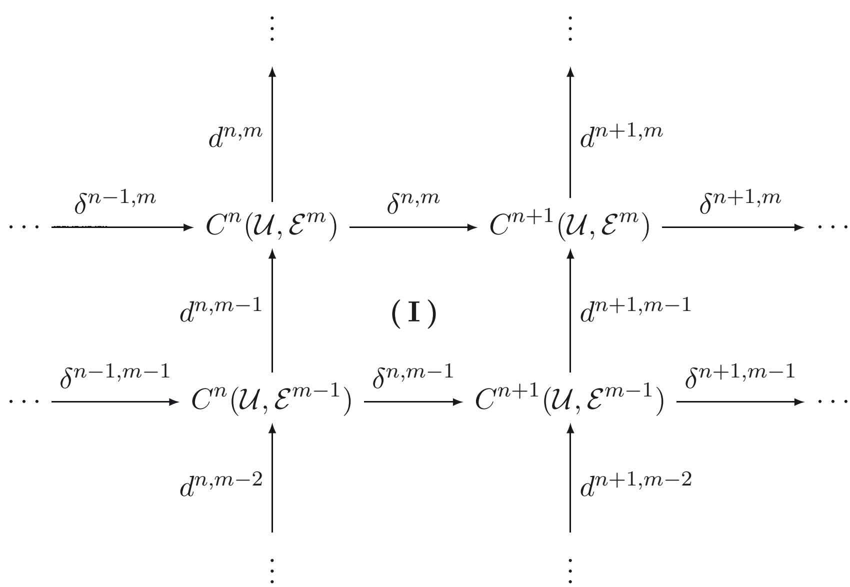

Diagram 6.7

By definition, all the sub-diagrams like (I) are commutative. The vertical operators between various cochains are induced by the corresponding differentials of the initial complex $\mathcal{E}$.

A double complex, as above, gives rise to an ordinary complex of $\mathcal{A}$-modules

$$\big\{\mathsf{tot}\big(C^\bullet(\mathcal{U}, \mathcal{E}^\bullet)\big), D\big\} = (\mathcal{S}^p, D^p)_{p \in \mathbb{Z}},$$

whose elements are defined by the relations

$$\mathcal{S}^p := \bigoplus_{n+m=p} C^n(\mathcal{U}, \mathcal{E}^m),$$

$$D^p := \sum_{n+m=p} \delta^{n,m} + (-1)^n d^{n,m} : \mathcal{S}^p \longrightarrow \mathcal{S}^{p+1},$$

for every $p \in \mathbb{Z}$. Then the ***Čech hypercohomology of $\mathcal{U}$ with respect to $\mathcal{E}^\bullet$*** is defined to be the cohomology of the (total) complex $\mathsf{tot}\big(C^\bullet(\mathcal{U}, \mathcal{E}^\bullet)\big)$. In other words, the corresponding hypercohomology groups are

$$\check{\boldsymbol{H}}^p(\mathcal{U}, \mathcal{E}^\bullet) = \ker(D^p)/\mathrm{im}(D^{p-1}), \qquad p \in \mathbb{Z}.$$

Therefore, as in the case of the ordinary cohomology, the ***p-dimensional Čech hypercohomology group of X with respect to $\mathcal{E}^\bullet$*** is

$$\check{\boldsymbol{H}}^p(X, \mathcal{E}^\bullet) := \varinjlim_{\mathcal{U}} \check{\boldsymbol{H}}^p(\mathcal{U}, \mathcal{E}^\bullet),$$

where the inductive limit is taken over all the proper open coverings of X.

Based on the idea of [62, Chap. VI, Theorem 18.2], we now proceed to the proof.

Proof of Theorem 6.7.2. Since, in our case, we are dealing only with the 1-dimensional hypercohomology with respect to the 2-term complex $\mathcal{E}^\bullet := \{\mathcal{G} \xrightarrow{\partial} \Omega(\mathcal{L})\}$, it suffices to consider the next diagram, derived from a part of Diagram 6.7 under the necessary modifications.

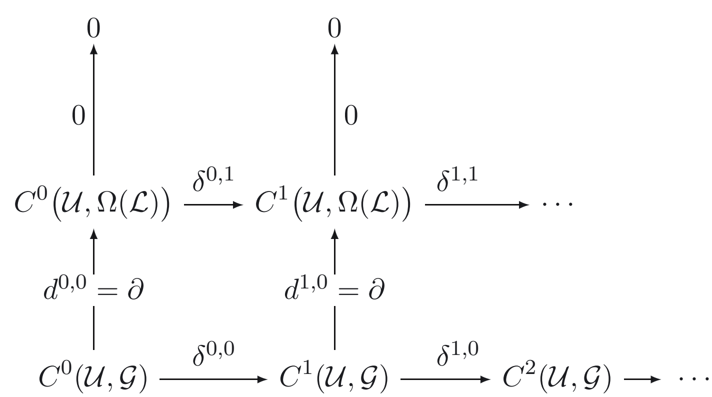

Diagram 6.8

The horizontal morphisms of the diagram are the usual coboundary operators and the vertical ones are induced by ∂, whence the notation applied therein. As a result, we can form the total complex

$$\mathcal{S}^0 \xrightarrow{D^0} \mathcal{S}^1 \xrightarrow{D^1} \mathcal{S}^2 \xrightarrow{D^2} \cdots$$

where we have set

$$\begin{aligned} \mathcal{S}^0 &:= C^0(\mathcal{U},\mathcal{G}) \\ \mathcal{S}^1 &:= C^1(\mathcal{U},\mathcal{G}) \oplus C^0\big(\mathcal{U},\Omega(\mathcal{L})\big) \\ \mathcal{S}^2 &:= C^2(\mathcal{U},\mathcal{G}) \oplus C^1\big(\mathcal{U},\Omega(\mathcal{L})\big) \\ D^0 &:= \delta^{0,0} + \partial \\ D^1 &:= (\delta^{1,0} - \partial) + \delta^{0,1}. \end{aligned}$$

By a routine computation we verify that

$$\ker(D^1) = \ker(\delta^{1,0} - \partial) \oplus \ker(\delta^{0,1}), \tag{6.7.6}$$

$$\operatorname{im}(D^0) = \operatorname{im}(\delta^{0,0}) \oplus \operatorname{im}(\partial). \tag{6.7.7}$$

Therefore,

$$\check{\boldsymbol{H}}^1\big(\mathcal{U},\mathcal{G} \xrightarrow{\partial} \Omega(\mathcal{L})\big) = \ker(D^1)/\operatorname{im}(D^0) = \frac{\ker(\delta^{1,0} - \partial) \oplus \ker(\delta^{0,1})}{\operatorname{im}(\delta^{0,0}) \oplus \operatorname{im}(\partial)}.$$

We now take any pair $(\mathcal{P}, D)$. The principal sheaf $\mathcal{P}$ determines a cocycle $(g_{\alpha\beta}) \in Z^1(\mathcal{U},\mathcal{G}) \subseteq C^1(\mathcal{U},\mathcal{G})$, while D defines the local connection forms $(\omega_\alpha) \in C^0(\mathcal{U},\Omega(\mathcal{L}))$ satisfying (6.7.1). The last equality implies that

$$\partial\big((g_{\alpha\beta})\big) = (\omega_\beta - \omega_\alpha) = \delta^{0,1}\big((\omega_\alpha)\big). \tag{6.7.8}$$

Hence, applying D^1 to the pair $\big((g_{\alpha\beta}),(\omega_\alpha)\big)$, and taking into account (6.7.6), (6.7.8), along with the cocycle condition of $(g_{\alpha\beta})$, we see that

$$\begin{aligned} D^1\big((g_{\alpha\beta}),(\omega_\alpha)\big) &= \big(\delta^{1,0}\big((g_{\alpha\beta})\big) - \partial\big((g_{\alpha\beta})\big)\big) + \delta^{0,1}\big((\omega_\alpha)\big) \\ &= \delta^{1,0}\big((g_{\alpha\beta})\big) = g_{\alpha\beta} \cdot g_{\beta\gamma} \cdot g_{\gamma\alpha} = 0. \end{aligned}$$

(Recall that $\mathcal{G}$ is abelian.) This shows that $\big((g_{\alpha\beta}),(\omega_\alpha)\big) \in \ker(D^1)$, thus we obtain the class

$$\big[\big((g_{\alpha\beta}),(\omega_\alpha)\big)\big]_{\mathcal{U}} \in \check{\boldsymbol{H}}^1\big(\mathcal{U},\mathcal{G} \xrightarrow{\partial} \Omega(\mathcal{L})\big)$$

and the corresponding class $\big[\big((g_{\alpha\beta}),(\omega_\alpha)\big)\big] \in \check{\boldsymbol{H}}^1\big(X, \mathcal{G} \xrightarrow{\partial} \Omega(\mathcal{L})\big)$.

This procedure allows us to define the map

$$\Phi : \boldsymbol{P}_{\mathcal{G}}(X)^D \ni [(\mathcal{P}, D)] \longmapsto \big[\big((g_{\alpha\beta}),(\omega_\alpha)\big)\big] \in \check{\boldsymbol{H}}^1\big(X, \mathcal{G} \xrightarrow{\partial} \Omega(\mathcal{L})\big),$$

which establishes the identification of the statement after proving the following facts:

i) Φ is well defined. Assume that $(\mathcal{P}, D)$ and $(\mathcal{P}', D')$ are equivalent. Taking local frames over the same open covering of the base space, equality (6.7.2) and the commutativity of $\mathcal{G}$ imply that

$$g'_{\alpha\beta} = h_\alpha \cdot g_{\alpha\beta} \cdot h_\beta^{-1} = (h_\alpha \cdot h_\beta^{-1}) \cdot g_{\alpha\beta},$$

from which we obtain

$$(g'_{\alpha\beta}) \cdot (g_{\alpha\beta}^{-1}) = \delta^{0,0}\big((h_\alpha^{-1})\big). \tag{6.7.9}$$

Similarly, equality (6.7.3), in conjunction with Proposition 3.3.5, yields

$$(\omega'_\alpha - \omega_\alpha) = \big(-\partial(h_\alpha)\big) = \partial\big((h_\alpha^{-1})\big). \tag{6.7.10}$$

Hence, to prove our claim, it suffices to show that

$$\big[\big((g_{\alpha\beta}),(\omega_\alpha)\big)\big]_{\mathcal{U}} = \big[\big((g'_{\alpha\beta}),(\omega'_\alpha)\big)\big]_{\mathcal{U}} \in \ker(D^1)/\operatorname{im}(D^0),$$

or, equivalently,

$$\big((g'_{\alpha\beta}),(\omega'_\alpha)\big) - \big((g_{\alpha\beta}),(\omega_\alpha)\big) = \big((g'_{\alpha\beta} - g_{\alpha\beta}),(\omega'_\alpha - \omega_\alpha)\big) \in \operatorname{im}(D^0).$$

The last inclusion is true, since (6.7.9), (6.7.10) and the definition of D^0 (see also (6.7.7)), along with the commutativity of $\mathcal{G}$ (whence the *equivalent use of multiplicative and additive notations*), yield

$$\begin{aligned}
\big((g'_{\alpha\beta} - g_{\alpha\beta}),(\omega'_\alpha - \omega_\alpha)\big) &= \big((g'_{\alpha\beta}) \cdot (g_{\alpha\beta}^{-1}),(\omega'_\alpha - \omega_\alpha)\big) \\
&= \big(\delta^{0,0}((h_\alpha^{-1})), \partial((h_\alpha^{-1}))\big) \\
&= \big(\delta^{0,0}, \partial\big)\big((h_\alpha^{-1})\big) \\
&= D^0\big((h_\alpha^{-1})\big).
\end{aligned}$$

Note that if we consider local frames over different coverings of the base, then we obtain equal classes in the inductive limit, working as in the first part of the proof of Theorem 4.6.2.

ii) Φ *is injective.* This is proved by similar arguments to those of *i)*, but in a reverse way. In this respect we also refer to the proof of Theorem 4.6.2.

iii) Φ *is surjective.* To this end let us choose an arbitrary class in $\check{\boldsymbol{H}}^1(X, \mathcal{G} \xrightarrow{\partial} \Omega(\mathcal{L}))$, derived from a class of the form

$$\big[\big((g_{\alpha\beta}), (\omega_\alpha)\big)\big]_{\mathcal{U}}, \quad \text{where} \quad \big((g_{\alpha\beta}), (\omega_\alpha)\big) \in C^1(\mathcal{U}, \mathcal{G}) \oplus C^0(\mathcal{U}, \Omega(\mathcal{L})).$$

Applying the definition of the hypercohomology group (over $\mathcal{U}$) and that of D^1, we have that

$$\begin{aligned} 0 &= D^1\big((g_{\alpha\beta}), (\omega_\alpha)\big) = (\delta^{1,0}, \partial)\big((g_{\alpha\beta})\big) + \delta^{0,1}\big((\omega_\alpha)\big) \\ &= \delta^{1,0}\big((g_{\alpha\beta})\big) + \big(-\partial\big((g_{\alpha\beta})\big) + \delta^{0,1}\big((\omega_\alpha)\big)\big), \end{aligned}$$

from which, together with (6.7.2), we obtain the equalities

$$\delta^{1,0}\big((g_{\alpha\beta})\big) = 0, \tag{6.7.11}$$

$$\partial\big((g_{\alpha\beta})\big) = \delta^{0,1}\big((\omega_\alpha)\big). \tag{6.7.12}$$

But (6.7.11) implies that $g_{\alpha\beta} \cdot g_{\beta\gamma} = g_{\alpha\gamma}$, i.e., $(g_{\alpha\beta})$ is a cocycle belonging to $Z^1(\mathcal{U}, \mathcal{G})$, which determines a $\mathcal{G}$-principal sheaf $\mathcal{P}$ with cocycle $(g_{\alpha\beta})$ (see Theorem 4.5.1).

On the other hand, (6.7.12) yields

$$\partial\big((g_{\alpha\beta})\big) = \big(\partial(g_{\alpha\beta})\big) = \delta^{0,1}\big((\omega_\alpha)\big) = (\omega_\beta - \omega_\alpha),$$

or, $\omega_\beta = \omega_\alpha + \partial(g_{\alpha\beta})$, for every $\alpha, \beta \in I$. The last equality is precisely (6.7.1), showing that the cochain (ω_α) determines a connection D on $\mathcal{P}$ (see Theorem 6.1.5). Therefore, $\Phi\big([(\mathcal{P}, D)]\big) = \big[\big((g_{\alpha\beta}), (\omega_\alpha)\big)\big]$. The proof of the theorem is now accomplished. □

In particular, taking as $\mathcal{G}$ the abelian sheaf of groups $\mathcal{A}^\bullet$ (see Example 3.3.6(b)), we get the following result, which will be used in the classification of Maxwell fields in Section 7.2 (see also [62, Vol. II, p. 94]).

6.7.3 Corollary. *The following isomorphism holds true:*

$$\boldsymbol{P}_{\mathcal{A}^\bullet}(X)^D \cong \check{\boldsymbol{H}}^1\big(X, \mathcal{A}^\bullet \xrightarrow{\widetilde{\partial}} \Omega\big).$$

Chapter 7

Connections on vector and associated sheaves

There are many advantages to developing a theory in the most general context possible.

R. Hartshorne [41, p. 59]

L' effort de synthèse correspond à un effort métaphysique ... Et c' est de répondre à un besoin de l' esprit: le besoin d' unification est un besoin fondamental de l' esprit.

R. Thom [122, p. 131]

Our intention is to define connections on vector sheaves (also called $\mathcal{A}$-connections) using the general theory of Chapter 6. However, in order to have a clear motive for our general approach, we begin with a short account of $\mathcal{A}$-connections without reference to connections on principal sheaves. Subsequently we show that $\mathcal{A}$-connections on a vector sheaf $\mathcal{E}$ are fully determined by the connections of the principal sheaf of frames $\mathcal{P}(\mathcal{E})$ and vice-versa.

The chapter closes with a discussion about connections induced on principal and vector sheaves, associated with a given principal sheaf endowed with a connection.

7.1. Connections on vector sheaves

Throughout the present section we consider a fixed differential triad $(\mathcal{A}, d, \Omega)$ and a vector sheaf $\mathcal{E} \equiv (\mathcal{E}, \pi_{\mathcal{E}}, X)$ of rank n.

With the notations of Section 5.1, we give the following basic definition (cf. Mallios [62, Chap. VI]).

7.1.1 Definition. An ***$\mathcal{A}$-connection*** on the vector sheaf $\mathcal{E}$ is a $\mathbb{K}$-linear morphism

$$\nabla : \mathcal{E} \longrightarrow \Omega(\mathcal{E}) := \mathcal{E} \otimes_{\mathcal{A}} \Omega,$$

satisfying the ***Leibniz-Koszul*** condition

$$\nabla(a \cdot u) = a \cdot \nabla(u) + u \otimes da, \qquad (a, u) \in \mathcal{A} \times_X \mathcal{E}. \tag{7.1.1}$$

Since $\mathcal{E} \otimes_{\mathcal{A}} \Omega \cong \Omega \otimes_{\mathcal{A}} \mathcal{E}$, the above expression of $\Omega(\mathcal{E})$ conforms with the general notation (3.3.4)

To obtain the section-wise analog of (7.1.1), we notice that, given two $\mathcal{A}$-modules $\mathcal{S}$, $\mathcal{T}$, and two sections $s \in \mathcal{S}(U)$, $t \in \mathcal{T}(U)$, equality

$$s(x) \otimes t(x) = \widetilde{(s \otimes t)}(x) \equiv (s \otimes t)^{\sim}(x)$$

is verified for every $x \in U$ (recall the notation ($\Diamond$) on p. 104). This is a direct consequence of the representation of an element of the stalk as the germ of appropriate sections. Thus the morphism of sections induced by ∇ satisfies

$$\nabla(\alpha \cdot s) = \alpha \cdot \nabla(s) + \widetilde{s \otimes d\alpha}; \qquad (\alpha, s) \in \mathcal{A}(U) \times \mathcal{E}(U), \tag{7.1.1a}$$

for every open $U \subseteq X$.

Conversely, (7.1.1a) implies (7.1.1) according to the comments (1.2.15′). Thus (7.1.1) and (7.1.1a) can be used interchangeably as the Koszul-Leibniz condition of ∇.

If $\mathcal{U} \equiv ((U_\alpha), (\psi_\alpha))$ is a local frame of $\mathcal{E}$, we know that $\mathcal{E}(U_\alpha)$ is a free $\mathcal{A}(U_\alpha)$-module of rank n, endowed with the natural basis $e^\alpha = (e_i^\alpha)_{1 \leq i \leq n}$

(see (5.1.3′) and Proposition 5.1.2). Therefore, analogously to what has been said in the note following Theorem 6.2.1, one has the identification

$$\mathcal{E}(U_\alpha) \otimes_{\mathcal{A}(U_\alpha)} \Omega(U_\alpha) \cong (\mathcal{E} \otimes_{\mathcal{A}} \Omega)(U_\alpha). \tag{7.1.2}$$

As a result, over each U_α, (7.1.1a) can be written as

$$\nabla(\alpha \cdot s) = \alpha \cdot \nabla(s) + s \otimes d\alpha, \qquad (\alpha, s) \in \mathcal{A}(U) \times \mathcal{E}(U). \tag{7.1.1b}$$

Hence, one infers that

> if the local frame of $\mathcal{E}$ is a basis for the topology of X, then (7.1.1), (7.1.1a) and (7.1.1b) are equivalent.

Reconsider a local frame $\mathcal{U}$ as before and the natural bases (e^α). The properties of the tensor product (for their classical analogs see, e.g., Greub [34, pp. 7–8]) and the identification (7.1.2) allow us to write

$$\nabla(e_j^\alpha) = \sum_{i=1}^{n} e_i^\alpha \otimes \omega_{ij}^\alpha; \qquad 1 \leq j \leq n, \tag{7.1.3}$$

where $\omega_{ij}^\alpha \in \Omega(U_\alpha)$ are uniquely determined sections. The previous n^2 elements form an $n \times n$ matrix

$$\omega^\alpha := \big(\omega_{ij}^\alpha\big) \in M_n(\Omega(U_\alpha)), \tag{7.1.4}$$

called the ***local connection matrix*** of ∇, with respect to the basis e^α.

Using e^α, any section $s \in \mathcal{E}(U_\alpha)$ can be written in the form

$$s = \sum_{i=1}^{n} s_i^\alpha \cdot e_i^\alpha; \qquad s_i^\alpha \in \mathcal{A}(U_\alpha), \tag{7.1.5}$$

thus the Leibniz-Koszul condition (7.1.1b) and equality (7.1.3) yield

$$\begin{aligned}\nabla(s) &= \nabla\Big(\sum_{j=1}^{n} s_j^\alpha \cdot e_j^\alpha\Big) \\ &= \sum_{j=1}^{n} s_j^\alpha \cdot \nabla(e_j^\alpha) + \sum_{j=1}^{n} e_j^\alpha \otimes ds_j^\alpha\end{aligned}$$

$$= \sum_{j=1}^{n} s_j^\alpha \cdot \Big(\sum_{i=1}^{n} e_i^\alpha \otimes \omega_{ij}^\alpha \Big) + \sum_{i=1}^{n} e_i^\alpha \otimes ds_i^\alpha$$
$$= \sum_{i=1}^{n} \Big(\sum_{j=1}^{n} e_i^\alpha \otimes (s_j^\alpha \cdot \omega_{ij}^\alpha) + e_i^\alpha \otimes ds_i^\alpha \Big)$$
$$= \sum_{i=1}^{n} e_i^\alpha \otimes \Big(\sum_{j=1}^{n} s_j^\alpha \cdot \omega_{ij}^\alpha + ds_i^\alpha \Big);$$

that is,

$$\nabla(s) = \sum_{i=1}^{n} e_i^\alpha \otimes \Big(\sum_{j=1}^{n} s_j^\alpha \cdot \omega_{ij}^\alpha + ds_i^\alpha \Big). \tag{7.1.6}$$

The previous calculations show that the restriction of ∇ to $\mathcal{E}|_{U_\alpha}$ is completely determined by the connection matrix ω^α. As one expects, the entire ∇ will be determined by the family of all local matrices $(\omega^\alpha)_{\alpha \in I}$, under suitable conditions. In fact, we prove the following:

7.1.2 Lemma. *Let ∇ be an $\mathcal{A}$-connection on a vector sheaf $\mathcal{E}$ of rank n. Then the local connection matrices $(\omega^\alpha)_{\alpha \in I}$ of ∇, with respect to a local frame $\mathcal{U} \equiv ((U_\alpha), (\psi_\alpha))$, satisfy the compatibility condition (alias local gauge equivalence)*

$$\omega^\beta = \mathrm{Ad}(\psi_{\alpha\beta}^{-1})(\omega^\alpha) + \widetilde{\partial}(\psi_{\alpha\beta}), \tag{7.1.7}$$

over each $U_{\alpha\beta} \neq \emptyset$.

Before the proof we clarify that each coordinate transformation $\psi_{\alpha\beta} := \psi_\alpha \circ \psi_\beta^{-1}$ is now identified, in virtue of (5.1.6), with the transition matrix $\big(g_{ij}^{\alpha\beta}\big) \in \mathrm{GL}(n, \mathcal{A}(U_{\alpha\beta}))$, while

$$\mathrm{Ad}(\psi_{\alpha\beta}^{-1})(\omega^\alpha) := \mathrm{Ad}_{U_{\alpha\beta}}(\psi_{\alpha\beta}^{-1})(\omega^\alpha) = \psi_{\alpha\beta} \cdot \omega^\alpha \cdot \psi_{\alpha\beta}^{-1}$$

(see also the notations preceding (3.2.12)). The last term represents a matrix multiplication, where ω^α is restricted to $U_{\alpha\beta}$. Finally, $\widetilde{\partial}$ is now the induced morphism of sections over $U_{\alpha\beta}$, which (due to the completeness of the presheaves involved) identifies with (3.2.9).

Proof. Analogously to (7.1.3), the local connection matrix ω^β is given by

$$\nabla(e_j^\beta) = \sum_{i=1}^{n} e_i^\beta \otimes \omega_{ij}^\beta ; \qquad 1 \leq j \leq n, \tag{7.1.8}$$

with respect to the natural basis e^β of $\mathcal{E}(U_\beta)$. Thus, working over $U_{\alpha\beta}$, (5.1.6′), (7.1.1b) and (7.1.3) transform the left-hand side of (7.1.8) into

$$\begin{aligned}\nabla(e_j^\beta) &= \nabla\Big(\sum_{i=1}^{n} g_{ij}^{\alpha\beta} \cdot e_i^\alpha\Big) \\ &= \sum_{i=1}^{n}\Big(g_{ij}^{\alpha\beta}\cdot\Big(\sum_{k=1}^{n} e_k^\alpha \otimes \omega_{ki}^\alpha\Big) + e_i^\alpha \otimes dg_{ij}^{\alpha\beta}\Big) \\ &= \sum_{i=1}^{n}\sum_{k=1}^{n} e_k^\alpha \otimes \big(\omega_{ki}^\alpha \cdot g_{ij}^{\alpha\beta}\big) + \sum_{k=1}^{n} e_k^\alpha \otimes dg_{kj}^{\alpha\beta} \\ &= \sum_{k=1}^{n} e_k^\alpha \otimes \Big(\sum_{i=1}^{n} \omega_{ki}^\alpha \cdot g_{ij}^{\alpha\beta} + dg_{kj}^{\alpha\beta}\Big).\end{aligned} \tag{7.1.9}$$

Similarly, reapplying (5.1.6′), the right-hand side of (7.1.8) is transformed into

$$\begin{aligned}\sum_{i=1}^{n} e_i^\beta \otimes \omega_{ij}^\beta &= \sum_{i=1}^{n}\Big(\sum_{k=1}^{n} g_{ki}^{\alpha\beta}\cdot e_k^\alpha\Big)\otimes \omega_{ij}^\beta \\ &= \sum_{k=1}^{n} e_k^\alpha \otimes \Big(\sum_{i=1}^{n} g_{ki}^{\alpha\beta}\cdot \omega_{ij}^\beta\Big).\end{aligned} \tag{7.1.10}$$

Therefore, substituting (7.1.9) and (7.1.10) in (7.1.8), we get the equalities

$$\sum_{i=1}^{n} g_{ki}^{\alpha\beta}\cdot\omega_{ij}^\beta = \sum_{i=1}^{n} \omega_{ki}^\alpha \cdot g_{ij}^{\alpha\beta} + dg_{kj}^{\alpha\beta}; \qquad j = 1, \ldots, n, \tag{7.1.11}$$

which, taken altogether, lead to the matrix equality (see also (3.1.10))

$$\big(g_{ij}^{\alpha\beta}\big)\cdot\big(\omega_{ij}^\beta\big) = \big(\omega_{ij}^\alpha\big)\cdot\big(g_{ij}^{\alpha\beta}\big) + d_{U_{\alpha\beta}}\big(\big(g_{ij}^{\alpha\beta}\big)\big).$$

In virtue of (7.1.4) and (5.1.6), the preceding equality turns into

$$\psi_{\alpha\beta}\cdot\omega^\beta = \omega^\alpha\cdot\psi_{\alpha\beta} + d_{U_{\alpha\beta}}\psi_{\alpha\beta},$$

which, by the definition of $\widetilde{\partial}$ (see (3.2.9)), yields (7.1.7). □

The converse of Lemma 7.1.2 is stated in the following form:

7.1.3 Lemma. *Let $\mathcal{E}$ be a vector sheaf of rank n with a local frame $\mathcal{U} \equiv ((U_\alpha), (\psi_\alpha))$. If*

$$\omega^\alpha := \left(\omega^\alpha_{ij}\right) \in M_n(\Omega(U_\alpha)); \qquad \alpha \in I,$$

is a family of local matrices satisfying the compatibility condition (7.1.7), then there exists a unique $\mathcal{A}$-connection on $\mathcal{E}$ whose local connection matrices coincide with the given (ω^α).

Proof. We are motivated by equality (7.1.6), which should necessarily be satisfied if we wish the statement to be true. Thus, for a fixed $\alpha \in I$, we define the family of mappings

$$\nabla^\alpha_U : \mathcal{E}(U) \longrightarrow \Omega(\mathcal{E})(U) \cong \mathcal{E}(U) \otimes_{\mathcal{A}(U)} \Omega(U),$$

for all open $U \subseteq U_\alpha$, given by

$$\nabla^\alpha_U(s) = \sum_{i=1}^n e^\alpha_i \otimes \Big(\sum_{j=1}^n s^\alpha_j \cdot \omega^\alpha_{ij} + ds^\alpha_i \Big), \tag{7.1.12}$$

for every $s \in \mathcal{E}(U)$ expressed correspondingly by (7.1.5). Note that, for the sake of simplicity, in (7.1.12) we omit the explicit mention of the restrictions involved; namely, we write e^α_i and ω^α_{ij} instead of $e^\alpha_i|_U$ and $\omega^\alpha_{ij}|_U$.

We verify the following facts:

i) ∇^α_U is a $\mathbb{K}$-linear morphism. This is clear from the $\mathbb{K}$-linearity of d and the $\mathcal{A}(U_\alpha)$-linearity of the other operators.

ii) ∇^α_U satisfies (7.1.1b). Indeed, for any $a \in \mathcal{A}(U)$ and $s \in \mathcal{E}(U)$, in virtue of (7.1.5) and the Leibniz condition of d, we obtain

$$\begin{aligned}
\nabla^\alpha_U(a \cdot s) &= \sum_{i=1}^n e^\alpha_i \otimes \Big(\sum_{j=1}^n (a \cdot s^\alpha_j) \cdot \omega^\alpha_{ij} + d(a \cdot s^\alpha_i) \Big) \\
&= \sum_{i=1}^n e^\alpha_i \otimes \Big(a \cdot \Big(\sum_{j=1}^n s^\alpha_j \cdot \omega^\alpha_{ij} + ds^\alpha_i \Big) + s^\alpha_i \cdot da \Big) \\
&= a \cdot \sum_{i=1}^n e^\alpha_i \otimes \Big(\sum_{j=1}^n s^\alpha_j \cdot \omega^\alpha_{ij} + ds^\alpha_i \Big) + \sum_{i=1}^n (s^\alpha_i \cdot e^\alpha_i) \otimes da \\
&= a \cdot \nabla^\alpha_U(s) + s \otimes da.
\end{aligned}$$

iii) For every open $V \subseteq U$, the diagram

$$\begin{array}{ccc} \mathcal{E}(U) & \xrightarrow{\nabla_U^\alpha} & \Omega(\mathcal{E})(U) \\ \big\downarrow & & \big\downarrow \\ \mathcal{E}(V) & \xrightarrow{\nabla_V^\alpha} & \Omega(\mathcal{E})(V) \end{array}$$

Diagram 7.1

is commutative, with the vertical arrows denoting the natural restrictions of sections.

As a consequence, the family (∇_U^α), for all open $U \subseteq U_\alpha$, is a presheaf morphism generating an $\mathcal{A}|_{U_\alpha}$-connection $\nabla^\alpha : \mathcal{E}|_{U_\alpha} \to \Omega(\mathcal{E})|_{U_\alpha}$. Varying $\alpha \in I$, we obtain a family of local connections, which glued together define an $\mathcal{A}$-connection ∇ on $\mathcal{E}$, provided that ∇^α and ∇^β coincide on $\mathcal{E}|_{U_{\alpha\beta}}$.

To verify the last condition, we take an arbitrary $u \in \mathcal{E}_x$ with $x \in U_{\alpha\beta}$. Then there is an $s \in \mathcal{E}(U)$, for some open $U \subseteq U_{\alpha\beta}$, such that $s(x) = u$. Since $\nabla^\alpha(u) = (\nabla_U^\alpha(s))\tilde{}\,(x) \equiv (\nabla_U^\alpha(s))\,(x)$, it suffices to show that (again omitting restrictions)

$$\nabla_U^\alpha(s) = \nabla_U^\beta(s), \tag{7.1.13}$$

for every $s \in \mathcal{E}(U)$ and every $U \subseteq U_{\alpha\beta}$. To this end, we first observe that (7.1.5) and its analog for the basis e^β imply that (over U)

$$\sum_{i=1}^n s_i^\alpha \cdot e_i^\alpha = \sum_{i=1}^n s_i^\beta \cdot e_i^\beta,$$

from which, as in the classical linear algebra, we obtain

$$s_i^\alpha = \sum_{j=1}^n s_j^\beta \cdot g_{ij}^{\alpha\beta}. \tag{7.1.14}$$

Now, applying (5.1.6′) to the analog of (7.1.12) for ∇_U^β, we have that

$$\begin{aligned} \nabla_U^\beta(s) &= \sum_{i=1}^n \Big(\Big(\sum_{k=1}^n g_{ki}^{\alpha\beta} \cdot e_k^\alpha \Big) \otimes \Big(\sum_{j=1}^n s_j^\beta \cdot \omega_{ij}^\beta + ds_i^\beta \Big) \Big) \\ &= \sum_{k=1}^n e_k^\alpha \otimes \Big(\sum_{j=1}^n s_j^\beta \cdot \Big(\sum_{i=1}^n g_{ki}^{\alpha\beta} \cdot \omega_{ij}^\beta \Big) + \sum_{i=1}^n g_{ki}^{\alpha\beta} \cdot ds_i^\beta \Big) \end{aligned}$$

or, taking into account (7.1.7) in its equivalent form (7.1.11),

$$\begin{aligned}\nabla_U^\beta(s) &= \sum_{k=1}^n e_k^\alpha \otimes \Big(\sum_{j=1}^n s_j^\beta \cdot \Big(\sum_{i=1}^n \omega_{ki}^\alpha \cdot g_{ij}^{\alpha\beta} + dg_{kj}^{\alpha\beta}\Big) + \sum_{i=1}^n g_{ki}^{\alpha\beta} \cdot ds_i^\beta\Big) \\ &= \sum_{k=1}^n e_k^\alpha \otimes \Big(\sum_{i=1}^n \omega_{ki}^\alpha \cdot \Big(\sum_{j=1}^n s_j^\beta \cdot g_{ij}^{\alpha\beta}\Big) + \sum_{j=1}^n s_j^\beta \cdot dg_{kj}^{\alpha\beta} \\ &\qquad\qquad + \sum_{i=1}^n g_{ki}^{\alpha\beta} \cdot ds_i^\beta\Big).\end{aligned}$$

Hence, in virtue of (7.1.14) and the result of its differentiation,

$$\begin{aligned}\nabla_U^\beta(s) &= \sum_{k=1}^n e_k^\alpha \otimes \Big(\sum_{i=1}^n s_i^\alpha \cdot \omega_{ki}^\alpha + d\Big(\sum_{i=1}^n g_{ki}^{\alpha\beta} \cdot s_i^\beta\Big)\Big) \\ &= \sum_{k=1}^n e_k^\alpha \otimes \Big(\sum_{i=1}^n s_i^\alpha \cdot \omega_{ki}^\alpha + ds_k^\alpha\Big),\end{aligned}$$

from which we get (7.1.13) by an obvious substitution of indices ($k \to i$ and $i \to j$).

The connection matrices of ∇ are precisely the given ω_α, $\alpha \in I$, by the very construction of ∇, i.e., by applying (7.1.12) for $U = U_\alpha$, for every $\alpha \in I$.

Finally, the uniqueness of the $\mathcal{A}$-connection having as local matrices the given family (ω^α) is immediately checked by elementary calculations. □

The previous lemmata, combined together, prove the following:

7.1.4 Theorem. *Let $\mathcal{E} \equiv (\mathcal{E}, \pi, X)$ be a vector sheaf of rank n and let $\mathcal{U}$ be a local frame with corresponding transition transformations $(\psi_{\alpha\beta})$. Then every $\mathcal{A}$-connection ∇ on $\mathcal{E}$ corresponds bijectively to a family of local matrix forms $\{\omega^\alpha \in M_n(\Omega)(U_\alpha) \,|\, \alpha \in I\}$ satisfying the compatibility condition (7.1.7).*

7.1.5 Remarks. 1) Using (3.1.4) and (3.1.5), (ω^α) can be thought of as a 0-cochain with coefficients in the sheaf $\mathcal{M}_n(\Omega)$, i.e., $(\omega^\alpha) \in C^0(\mathcal{U}, \mathcal{M}_n(\Omega))$.

2) To prepare our next result, we express (7.1.7) in an equivalent form involving the adjoint representation (3.2.12′). Namely, by the section analog of (3.2.15) and the general definition (3.3.10), we have that

$$\mathcal{A}d(\psi_{\alpha\beta}^{-1}).\omega^\alpha = \delta_n(\psi_{\alpha\beta}^{-1}, \omega^\alpha),$$

where δ_n is the morphism (action) of sections induced by (3.2.13). Therefore, (3.2.14) and the definition of δ_n (along with the completeness of the presheaves involved) give:

$$\begin{aligned}\mathcal{A}d(\psi_{\alpha\beta}^{-1}).\omega^{\alpha} &= \delta_n(\psi_{\alpha\beta}^{-1}, \omega^{\alpha}) \equiv \delta_{n,U}(\psi_{\alpha\beta}^{-1}, \omega^{\alpha}) \\ &= \mathrm{Ad}_{U_{\alpha\beta}}(\psi_{\alpha\beta}^{-1})(\omega^{\alpha}) =: \mathrm{Ad}_{U_{\alpha\beta}}(\psi_{\alpha\beta}^{-1})(\omega^{\alpha}).\end{aligned}$$

As a result, (7.1.7) takes the form

$$\omega^{\beta} = \mathcal{A}d(\psi_{\alpha\beta}^{-1}).\omega^{\alpha} + \widetilde{\partial}(\psi_{\alpha\beta}). \tag{7.1.7'}$$

We are now in a position to relate $\mathcal{A}$-connections on vector sheaves with the general theory of connections on principal sheaves. As one expects, the link between the two aspects is the sheaf of frames, studied in Section 5.2.

As we have seen, if $\mathcal{E} \equiv (\mathcal{E}, \pi_{\mathcal{E}}, X)$ is a vector sheaf of rank n and $\mathcal{P}(\mathcal{E}) \equiv (\mathcal{P}(\mathcal{E}), \mathcal{GL}(n, \mathcal{A}), X, \pi)$ denotes its principal sheaf of frames, then the cocycle $(\psi_{\alpha\beta})$ of $\mathcal{E}$ (relative to a local frame $(\mathcal{U}, (\psi_{\alpha}))$ coincides –up to isomorphism– with the cocycle $(g_{\alpha\beta})$ of $\mathcal{P}(\mathcal{E})$, over $(\mathcal{U}, (\Phi_{\alpha}))$ (see Corollary 5.2.3). Thus,

$$\mathcal{GL}(n, \mathcal{A})(U_{\alpha\beta}) \ni g_{\alpha\beta} \equiv \psi_{\alpha\beta} \equiv \big(g_{ij}^{\alpha\beta}\big) \in \mathrm{GL}(n, \mathcal{A}(U_{\alpha\beta})). \tag{7.1.15}$$

7.1.6 Theorem. *There is a bijective correspondence between the $\mathcal{A}$-connections on a vector sheaf $\mathcal{E}$ and the connections on the principal sheaf of frames $\mathcal{P}(\mathcal{E})$.*

Proof. Let ∇ be an $\mathcal{A}$-connection on $\mathcal{E}$. Then ∇ determines the local connection matrices $\omega^{\alpha} \in M_n(\Omega(U_{\alpha}))$, $\alpha \in I$. By (3.1.6) and (3.1.8), each matrix ω^{α} determines the form

$$\omega_{\alpha} := \lambda_{U_{\alpha}}^{1}(\omega^{\alpha}) \in \Omega(U_{\alpha}) \otimes_{\mathcal{A}(U_{\alpha})} M_n(\mathcal{A}(U_{\alpha})) \cong (\Omega \otimes_{\mathcal{A}} \mathcal{M}_n(\mathcal{A}))(U_{\alpha}).$$

Thus, restricted to $U_{\alpha\beta}$, (7.1.7′) yields

$$\begin{aligned}\omega_{\beta} = \lambda_{U_{\alpha\beta}}^{1}(\omega^{\beta}) &= \lambda_{U_{\alpha\beta}}^{1}\big(\mathcal{A}d(\psi_{\alpha\beta}^{-1}).\omega^{\alpha} + \widetilde{\partial}(\psi_{\alpha\beta})\big) \\ &= \lambda_{U_{\alpha\beta}}^{1}\big(\delta_{n,U_{\alpha\beta}}(\psi_{\alpha\beta}^{-1}, \omega^{\alpha})\big) + \lambda_{U_{\alpha\beta}}^{1}\big(\widetilde{\partial}(\psi_{\alpha\beta})\big)\end{aligned}$$

or, taking into account the analog of Diagram 3.2 for $\lambda_{U_{\alpha\beta}}^{1}$ (: the inverse of $\mu_{U_{\alpha\beta}}^{1}$), (7.1.7) and its equivalent form (7.1.7′), as well as (3.2.15′) and (3.2.17),

$$\begin{aligned}\omega_{\beta} &= \delta_{n',U_{\alpha\beta}}(\psi_{\alpha\beta}^{-1}, \lambda_{U_{\alpha\beta}}^{1}(\omega^{\alpha})) + \lambda_{U_{\alpha\beta}}^{1}\big(\widetilde{\partial}(\psi_{\alpha\beta})\big) \\ &= \mathcal{A}d(g_{\alpha\beta}^{-1}).\omega_{\alpha} + \partial(g_{\alpha\beta});\end{aligned}$$

that is, we arrive at the compatibility condition (6.1.5), adapted to the data of the principal sheaf $\mathcal{P}(\mathcal{E})$. In virtue of Theorem 6.1.5, the cochain $(\omega_\alpha) \in C^0(\mathcal{U}, \Omega(\mathcal{M}_n(\mathcal{A})))$ provides $\mathcal{P}(\mathcal{E})$ with a connection D.

Conversely, a connection D determines the local connection forms (ω_α) satisfying the above compatibility condition. Then, reversing the previous procedure, we see that the forms (ω_α) define the cochain of matrices (ω^α), with $\omega^\alpha := \mu^1_U(\omega_\alpha)$, satisfying (7.1.7). Hence, in virtue of Lemma 7.1.3, we obtain a connection ∇ on $\mathcal{E}$.

The desired bijectivity is a consequence of the uniqueness of the connections corresponding to a given family of local connection forms or matrices, ensured again by Theorem 6.1.5 or Lemma 7.1.3, respectively. □

7.2. Related $\mathcal{A}$-connections

Here we examine the behavior of $\mathcal{A}$-connections, when the vector sheaves carrying them are linked together by appropriate morphisms. Our investigation is inspired by that of Section 6.4.

7.2.1 Definition. Let $\mathcal{E} \equiv (\mathcal{E}, \pi, X)$ and $\mathcal{E}' \equiv (\mathcal{E}', \pi', X)$ be vector sheaves endowed with the $\mathcal{A}$-connections ∇ and ∇', respectively. If $f : \mathcal{E} \to \mathcal{E}'$ is an $\mathcal{A}$-morphism, then ∇ and ∇' are said to be f***-related*** if

$$\nabla' \circ f = (f \otimes 1_\Omega) \circ \nabla.$$

If f is an $\mathcal{A}$-isomorphism, then ∇ and ∇' are called f***-conjugate***.

Equivalently, we have the following commutative diagram, which is the vector sheaf analog of Diagram 6.3.

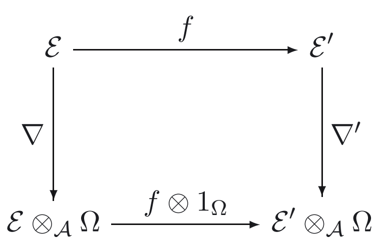

Diagram 7.2

We wish to express the previous situation in local terms, analogously to Theorem 6.4.2. To this end we specify that $\text{rank}(\mathcal{E}) = m$ and $\text{rank}(\mathcal{E}') = n$.

As we have seen in Theorem 5.1.6, a morphism $f : \mathcal{E} \to \mathcal{E}'$ is completely determined by a unique family of $\mathcal{A}|_{U_\alpha}$-morphisms $h_\alpha : \mathcal{A}^m|_{U_\alpha} \to \mathcal{A}^n|_{U_\alpha}$, $\alpha \in I$, satisfying (5.1.9). Moreover, f and (h_α) are connected by (5.1.18).

According to (5.1.10), h_α can be identified with a matrix

$$h^\alpha := \left(h^\alpha_{ij}\right) \in M_{m\times n}(\mathcal{A}(U_\alpha)); \qquad i = 1, \ldots, m;\ j = 1, \ldots, n.$$

Here $M_{m\times n}(\mathcal{A}(U_\alpha)) \cong \mathcal{M}_{m\times n}(\mathcal{A})(U_\alpha)$, where the matrix sheaf $\mathcal{M}_{m\times n}(\mathcal{A})$ is generated by the (complete) presheaf $U \mapsto M_{m\times n}(U)$. The entries of h^α are determined by

$$f(e^\alpha_i) = \sum_{j=1}^{n} h^\alpha_{ji} \cdot {}^{\backprime}e^\alpha_j; \qquad i = 1 \ldots m, \tag{7.2.1}$$

where (e^α_i), $1 \leq i \leq m$, and $({}^{\backprime}e^\alpha_j)$, $1 \leq j \leq n$, are the natural bases of $\mathcal{E}(U_\alpha)$ and $\mathcal{E}'(U_\alpha)$, respectively.

By the same token, the equivalent conditions (5.1.9) and (5.1.9′) are written in the matrix form

$$h^\alpha \cdot \psi_{\alpha\beta} = \psi'_{\alpha\beta} \cdot h^\beta, \tag{7.2.2}$$

with $\psi_{\alpha\beta} \equiv \left(g^{\alpha\beta}_{ij}\right)$ and $\psi'_{\alpha\beta} \equiv \left({}^{\backprime}g^{\alpha\beta}_{ij}\right)$.

With these notations we prove:

7.2.2 Theorem. *Let $\mathcal{E}$ and $\mathcal{E}'$ be two vector sheaves with* $\mathrm{rank}(\mathcal{E}) = m$ *and* $\mathrm{rank}(\mathcal{E}') = n$*, equipped with the $\mathcal{A}$-connections ∇ and ∇', respectively. If ∇ and ∇' are f-related, then there is a 0-cochain $(h^\alpha) \in C^0(\mathcal{U}, \mathcal{M}_{m\times n}(\mathcal{A}))$ satisfying (7.2.2) and*

$$h^\alpha \cdot \omega^\alpha = {}^{\backprime}\omega^\alpha \cdot h^\alpha + dh^\alpha; \qquad \alpha \in I, \tag{7.2.3}$$

where (ω^α) and $({}^{\backprime}\omega^\alpha)$ are the connection matrices of ∇ and ∇'.

Conversely, if (h^α) is a 0-cochain of $m \times n$ matrices satisfying (7.2.2) and (7.2.3), then there exists a unique morphism $f : \mathcal{E} \to \mathcal{E}'$ under which ∇ and ∇' are f-related.

Proof. If $f : \mathcal{E} \to \mathcal{E}'$ is a given morphism, then equalities (7.2.1) and (7.2.2) are satisfied. Applying (7.2.1), the analog of (7.1.3) for ∇', and bearing in mind (7.1.2), we have that

$$
\begin{aligned}
(\nabla' \circ f)(e_i^\alpha) &= \sum_{j=1}^{n} \big(h_{ji}^\alpha \cdot \nabla'({}^{\backprime}e_j^\alpha) + {}^{\backprime}e_j^\alpha \otimes dh_{ji}^\alpha\big) \\
&= \sum_{j=1}^{n} \Big(h_{ji}^\alpha \cdot \Big(\sum_{k=1}^{n} {}^{\backprime}e_k^\alpha \otimes {}^{\backprime}\omega_{kj}^\alpha\Big) + {}^{\backprime}e_j^\alpha \otimes dh_{ji}^\alpha\Big) \\
&= \sum_{k=1}^{n}\sum_{j=1}^{n} {}^{\backprime}e_k^\alpha \otimes \big(h_{ji}^\alpha \cdot {}^{\backprime}\omega_{kj}^\alpha\big) + \sum_{k=1}^{n} {}^{\backprime}e_k^\alpha \otimes dh_{ki}^\alpha \\
&= \sum_{k=1}^{n} {}^{\backprime}e_k^\alpha \otimes \Big(\sum_{j=1}^{n} {}^{\backprime}\omega_{kj}^\alpha \cdot h_{ji}^\alpha + dh_{ki}^\alpha\Big).
\end{aligned}
\tag{7.2.4}
$$

Similarly,

$$
\begin{aligned}
&\big((f \otimes 1_\Omega) \circ \nabla\big)(e_i^\alpha) = (f \otimes 1_\Omega)\Big(\sum_{l=1}^{m} e_l^\alpha \otimes \omega_{li}^\alpha\Big) = \\
&\sum_{l=1}^{m}\Big(\sum_{k=1}^{n} h_{kl}^\alpha \cdot {}^{\backprime}e_k^\alpha\Big) \otimes \omega_{li}^\alpha = \sum_{k=1}^{n} {}^{\backprime}e_k^\alpha \otimes \Big(\sum_{l=1}^{m} h_{kl}^\alpha \cdot \omega_{li}^\alpha\Big).
\end{aligned}
\tag{7.2.5}
$$

Hence, if ∇ and ∇' are f-related, then (7.2.4) and (7.2.5) yield (7.2.3), by the usual properties of the tensor product (cf. their classical counterparts in Greub [34, pp. 7–8]).

Conversely, a 0-cochain (h^α), as in the statement, determines a cochain of $\mathcal{A}|_{U_\alpha}$-morphisms (h_α) satisfying (5.1.9). Then, by Theorem 5.1.6, we obtain a morphism $f : \mathcal{E} \to \mathcal{E}'$ satisfying (5.1.8). As before, equalities (7.2.1), (7.2.4) and (7.2.5) are valid; hence, together with (7.2.3), they imply that ∇ and ∇' are f-related. □

7.2.3 Corollary. *Let $\mathcal{E}$ and $\mathcal{E}'$ be vector sheaves of the same rank, say n. Two $\mathcal{A}$-connections ∇ and ∇' are related by an $\mathcal{A}$-isomorphism of $\mathcal{E}$ onto $\mathcal{E}'$ if and only if there exists a 0-cochain $(h^\alpha) \in C^0(\mathcal{U}, \mathcal{GL}(n, \mathcal{A}))$ such that*

$$
\begin{aligned}
\psi'_{\alpha\beta} &= h^\alpha \cdot \psi_{\alpha\beta} \cdot (h^\beta)^{-1}, \\
\omega^\alpha &= \mathcal{A}d\big((h^\alpha)^{-1}\big)\,.{}^{\backprime}\omega^\alpha + \widetilde{\partial}(h^\alpha).
\end{aligned}
\tag{7.2.6}
$$

We note that the first equality of the statement is a variation of (7.2.2). The meaning of the first term on right-hand side of (7.2.6) is explained in Remark 7.1.5(2).

We shall link related $\mathcal{A}$-connections with related connections on principal sheaves of frames. Prior to this, we need the next technical extension of Definition 5.6.1.

7.2.4 Definition. Let $\ell : \mathcal{A}^m \to \mathcal{A}^n$ be an $\mathcal{A}$-morphism and $(\phi, \overline{\phi})$ a morphism between the Lie sheaves of groups $(\mathcal{GL}(m, \mathcal{A}), \mathcal{M}_m(\mathcal{A}), \mathcal{A}d, \widetilde{\partial})$ and $(\mathcal{GL}(n, \mathcal{A}), \mathcal{M}_n(\mathcal{A}), \mathcal{A}d, \widetilde{\partial})$. We say that ℓ is ***compatible*** with $(\phi, \overline{\phi})$ if the following conditions are satisfied:

$$\ell(a \cdot g) = \ell(a) \cdot \phi(g); \qquad (a, g) \in \mathcal{A}^m \times_X \mathcal{GL}(m, \mathcal{A}), \tag{7.2.7}$$

$$\ell(a \cdot h) = \ell(a) \cdot \overline{\phi}(h); \qquad (a, h) \in \mathcal{A}^m \times_X \mathcal{M}_m(\mathcal{A}), \tag{7.2.8}$$

$$d(\ell^\alpha) = 0, \tag{7.2.9}$$

where ℓ^α denotes the matrix representing the restriction of ℓ to $\mathcal{A}^m|_{U_\alpha}$, and $d : \mathcal{M}_{m\times n}(\mathcal{A}) \to \mathcal{M}_{m\times n}(\Omega)$ is the matrix sheaf extension of d given by (3.1.13). Furthermore, if $\overline{\overline{\phi}} : \mathcal{M}_m(\Omega) \to \mathcal{M}_n(\Omega)$ is the unique morphism determined by $\overline{\phi}$, after the identification (3.1.7), it is required that the condition

$$\ell^\alpha \cdot \theta = \overline{\overline{\phi}}(\theta) \cdot \ell^\alpha, \qquad \theta \in M_m(\Omega(U_\alpha)) \tag{7.2.10}$$

be fulfilled for every ℓ^α, as before, and every $U_\alpha \in \mathfrak{T}_X$.

$$\begin{array}{ccc}
\mathcal{GL}(m, \mathcal{A}) \times_X \mathcal{A}^m & \xrightarrow{\ \phi \times \ell\ } & \mathcal{GL}(n, \mathcal{A}) \times_X \mathcal{A}^n \\
\gamma_m \downarrow & (\mathbf{I}) & \downarrow \gamma_n \\
\mathcal{A}^m & \xrightarrow{\ \ell\ } & \mathcal{A}^n
\end{array}$$

$$\begin{array}{ccc}
\mathcal{M}_m(\mathcal{A}) \times_X \mathcal{A}^m & \xrightarrow{\ \overline{\phi} \times \ell\ } & \mathcal{M}_n(\mathcal{A}) \times_X \mathcal{A}^n \\
\gamma_m \downarrow & (\mathbf{II}) & \downarrow \gamma_n \\
\mathcal{A}^m & \xrightarrow{\ \ell\ } & \mathcal{A}^n
\end{array}$$

Diagrams 7.3

A few comments, regarding the previous definition, are appropriate here. Firstly, equalities (7.2.7) and (7.2.8) express the equivariance of ℓ with respect to the natural actions of matrices on $\mathcal{A}^m$ and $\mathcal{A}^n$. Schematically, we have the commutative Diagrams 7.3, where γ_m and γ_n are the aforementioned actions. For convenience, these actions are represented by "$\cdot$" in (7.2.7) and (7.2.8).

Secondly, (7.2.9) –needed in the next basic theorem– is automatically satisfied in the case of vector bundles (see, for instance, Vassiliou [126, Theorem 4.3]), where, after the necessary localizations, we obtain constant maps, hence their differential is annihilated. Here, the elements of ℓ^α do not necessarily belong to $\mathbb{K}$, whence this extra condition.

Thirdly, $\overline{\overline{\phi}}$ figuring in (7.2.10) is the morphism of sections induced by the sheaf morphism $\overline{\overline{\phi}}$, the construction of the latter being shown in the next diagram.

$$\begin{array}{ccc} \Omega \otimes_{\mathcal{A}} \mathcal{M}_m(\mathcal{A}) & \xleftarrow{\ \lambda^1\ } & \mathcal{M}_m(\Omega) \\ {\scriptstyle 1\otimes\overline{\phi}}\downarrow & & \downarrow{\scriptstyle \overline{\overline{\phi}}} \\ \Omega \otimes_{\mathcal{A}} \mathcal{M}_n(\mathcal{A}) & \xrightarrow{\ \mu^1\ } & \mathcal{M}_n(\Omega) \end{array}$$

Diagram 7.4

We now come to the following result:

7.2.5 Theorem. *Let $\mathcal{E}$ and $\mathcal{E}'$ be two vector sheaves with* $\operatorname{rank}(\mathcal{E}) = m$ *and* $\operatorname{rank}(\mathcal{E}') = n$, *equipped with the $\mathcal{A}$-connections ∇ and ∇', respectively. Assume that $(f, \phi, \overline{\phi}, id_X)$ is a morphism between the sheaves of frames $\mathcal{P}(\mathcal{E})$ and $\mathcal{P}(\mathcal{E}')$, $\ell : \mathcal{A}^m \to \mathcal{A}^n$ is an $\mathcal{A}$-morphism compatible with $(\phi, \overline{\phi})$, and $F : \mathcal{E} \to \mathcal{E}'$ is the unique morphism of vector sheaves induced by $(f, \phi, \overline{\phi}, id_X)$ and ℓ, as in Corollary 5.6.3. We denote by D and D' the connections on $\mathcal{P}(\mathcal{E})$ and $\mathcal{P}(\mathcal{E}')$ corresponding (by Theorem 7.1.6) in a unique way to ∇ and ∇', respectively. Then the following conditions are equivalent:*

i) D and D' are $(f, \phi, \overline{\phi}, id_X)$-related.

ii) ∇ and ∇' are F-related.

Proof. We shall prove that *i) $\Rightarrow$ ii)* by applying Theorem 7.2.2. To this end we need to express F in terms of a family $h_\alpha : \mathcal{A}^m|_{U_\alpha} \to \mathcal{A}^n|_{U_\alpha}$ of

$\mathcal{A}|_{U_\alpha}$-morphisms with corresponding matrices $h^\alpha \in M_{m\times n}(\mathcal{A}(U_\alpha))$, relative to a common open covering $\mathcal{U} = (U_\alpha)$ of X, over which the local frames of $\mathcal{E}$ and $\mathcal{E}'$ are defined.

Setting $\widetilde{\mathcal{E}} = (\mathcal{P}(\mathcal{E})\times_X \mathcal{A}^m)/\mathcal{GL}(m,\mathcal{A})$ and $\widetilde{\mathcal{E}'} = (\mathcal{P}(\mathcal{E}')\times_X \mathcal{A}^n)/\mathcal{GL}(n,\mathcal{A})$, we have shown (see Proposition 5.5.1, Corollary 5.5.2 and the ensuing comments) that there are isomorphisms $R : \widetilde{\mathcal{E}} \xrightarrow{\simeq} \mathcal{E}$ and $R' : \widetilde{\mathcal{E}'} \xrightarrow{\simeq} \mathcal{E}'$, obtained by gluing, respectively, the local isomorphisms

$$R_\alpha := \psi_\alpha^{-1} \circ \underline{\Phi}_\alpha : \widetilde{\mathcal{E}}|_{U_\alpha} \xrightarrow{\simeq} \mathcal{E}_{U_\alpha},$$
$$R'_\alpha := (\psi'_\alpha)^{-1} \circ \underline{\Phi}'_\alpha : \widetilde{\mathcal{E}'}|_{U_\alpha} \xrightarrow{\simeq} \mathcal{E}'_{U_\alpha},$$

for all $\alpha \in I$, where $\psi_\alpha : \mathcal{E}|_{U_\alpha} \to \mathcal{A}^m|_{U_\alpha}$ and $\underline{\Phi}_\alpha : \widetilde{\mathcal{E}}|_{U_\alpha} \to \mathcal{A}^m|_{U_\alpha}$ are the respective coordinates of $\mathcal{E}$ and $\widetilde{\mathcal{E}}$ over U_α. Analogously we have the coordinates $\psi'_\alpha : \mathcal{E}'|_{U_\alpha} \to \mathcal{A}^n|_{U_\alpha}$ and $\underline{\Phi}'_\alpha : \widetilde{\mathcal{E}'}|_{U_\alpha} \to \mathcal{A}^n|_{U_\alpha}$ giving R'_α.

Specializing Proposition 5.6.2 to the case of sheaves of frames as in Corollary 5.6.3, we see that $(f, \phi, \overline{\phi}, id_X)$ and ℓ induce a morphism of vector sheaves, say $\widetilde{F} : \widetilde{\mathcal{E}} \to \widetilde{\mathcal{E}'}$, generated by the local morphisms

$$\widetilde{F}_U : \big(\mathcal{P}(\mathcal{E})(U) \times \mathcal{A}^m(U)\big)/\mathcal{GL}(m,\mathcal{A})(U) \longrightarrow \big(\mathcal{P}(\mathcal{E}')(U) \times \mathcal{A}^n(U)\big)/\mathcal{GL}(n,\mathcal{A})(U),$$

for all $U \in \mathfrak{T}_X$, defined, in turn, by

$$\widetilde{F}_U([(\sigma, s)]_U) := [(f(\sigma), \ell(s))]_U,$$

where f and ℓ now denote the induced morphisms of sections. The desired morphism F is clearly given by $F = (R')^{-1} \circ \widetilde{F} \circ R$, as pictured in the following diagram.

$$\begin{array}{ccc}
\mathcal{P}(\mathcal{E}) \times_X \mathcal{A}^m & \xrightarrow{\ f \times \ell\ } & \mathcal{P}(\mathcal{E}') \times_X \mathcal{A}^n \\
\downarrow{\scriptstyle \kappa} & & \downarrow{\scriptstyle \kappa'} \\
\widetilde{\mathcal{E}} & \xrightarrow{\ \widetilde{F}\ } & \widetilde{\mathcal{E}'} \\
\downarrow{\scriptstyle R_\alpha} & & \downarrow{\scriptstyle R'_\alpha} \\
\mathcal{E} & \xrightarrow{\ F\ } & \mathcal{E}'
\end{array}$$

Diagram 7.5

(Compare with Diagrams 5.8, 5.9, where F is actually identified with $\widetilde{F}$.)

Therefore, by (5.1.8) and the definition of R_α, R'_α, we see that

$$h_\alpha := \psi'_\alpha \circ F \circ \psi_\alpha^{-1} = \underline{\Phi}'_\alpha \circ \widetilde{F} \circ \underline{\Phi}_\alpha^{-1}.$$

We determine the matrix h^α of the morphism h_α by computing the induced morphism of sections on the natural basis of $\mathcal{A}^m(U_\alpha)$. To prepare for this, let us find $h_\alpha(s)$, for an *arbitrary* section $s \in \mathcal{A}^m(U_\alpha)$. Taking into account that $s = \tilde{s}$ (after the identification of $\mathcal{A}^m$ with the sheaf of germs of its sections), convention (1.1.3), Diagram 1.7, equality (1.2.17), and the definition of $\underline{\Phi}_\alpha$, $\underline{\Phi}'_\alpha$, we see that

$$\begin{aligned} h_\alpha(s) &\equiv \overline{(h_\alpha)_{U_\alpha}(s)} = \overline{(\underline{\Phi}'_\alpha \circ \widetilde{F} \circ \underline{\Phi}_\alpha^{-1})_{U_\alpha}(\tilde{s})} \\ &= \overline{(\underline{\Phi}'_\alpha \circ \widetilde{F})_{U_\alpha}}\big(\widetilde{\underline{\Phi}_{\alpha,U_\alpha}^{-1}(s)}\big) \\ &= \big((\underline{\Phi}'_{\alpha,U_\alpha} \circ \widetilde{F}_{U_\alpha} \circ \underline{\Phi}_{\alpha,U_\alpha}^{-1})(s)\big)^{\sim} \\ &\equiv (\underline{\Phi}'_{\alpha,U_\alpha} \circ \widetilde{F}_{U_\alpha} \circ \underline{\Phi}_{\alpha,U_\alpha}^{-1})(s). \end{aligned}$$

Therefore, if (σ_α) and (σ'_α) are the natural sections of $\mathcal{P}(\mathcal{E})$ and $\mathcal{P}(\mathcal{E}')$, respectively, the definition of $\underline{\Phi}_{\alpha,U_\alpha}$, $\underline{\Phi}'_{\alpha,U_\alpha}$ (see Theorem 5.3.2) implies that

$$h_\alpha(s) = \big(\underline{\Phi}'_{\alpha,U_\alpha} \circ \widetilde{F}_{U_\alpha}\big)\big([(\sigma_\alpha, s)]_{U_\alpha}\big) = \underline{\Phi}'_{\alpha,U_\alpha}\big([(f(\sigma_\alpha), \ell(s))]_{U_\alpha}\big).$$

But $f(\sigma_\alpha) = \sigma'_\alpha \cdot g_\alpha$, for a unique $g_\alpha \in \mathcal{GL}(n, \mathcal{A})(U_\alpha)$; hence,

$$\begin{aligned} h_\alpha(s) &= \underline{\Phi}'_{\alpha,U_\alpha}([(\sigma_\alpha \cdot g_\alpha,\, \ell(s))]_{U_\alpha}) \\ &= \underline{\Phi}'_{\alpha,U_\alpha}([(\sigma_\alpha,\, g_\alpha \cdot \ell(s))]_{U_\alpha}) \\ &= g_\alpha \cdot \ell(s), \end{aligned}$$

or, identifying g_α with an element of $\mathrm{Aut}_{\mathcal{A}|_{U_\alpha}}(\mathcal{A}^n|_{U_\alpha})$ (after (5.1.14); see also the discussion following Diagram 5.5),

$$h_\alpha(s) = (g_\alpha \circ \ell)(s),$$

thus obtaining $h_\alpha = g_\alpha \circ \ell$. The last equality leads to its matrix analog

$$h^\alpha = g^\alpha \cdot \ell^\alpha, \tag{7.2.11}$$

where g^α is the matrix of g_α, and ℓ^α is the matrix of the restriction of ℓ to the subsheaf $\mathcal{A}^m|_{U_\alpha}$.

Now assume that D and D' are $(f, \phi, \overline{\phi}, id_X)$-related. Then, by Theorem 6.4.2, their local connection forms satisfy (6.4.3), which, under the present data, reads

$$(1 \otimes \overline{\phi})(\omega_\alpha) = \mathcal{A}d(g_\alpha^{-1}).\omega'_\alpha + \partial(g_\alpha), \qquad \alpha \in I. \tag{7.2.12}$$

Let ω^α (resp. ${}^{\backprime}\omega^\alpha$) be the matrix corresponding to ω_α (resp. ω'_α) by (3.1.6). Then, localizing Diagram 7.4 (over U_α) and working as in the proof of Theorem 7.1.6 (especially applying Diagram 3.2 and equality (3.2.17)), we transform (7.2.12) into the equivalent equality

$$g^\alpha \cdot \overline{\overline{\phi}}(\omega^\alpha) = {}^{\backprime}\omega^\alpha \cdot g^\alpha + dg^\alpha.$$

Multiplying both members of the preceding by ℓ^α (from the right), and taking into account (7.2.9) and (7.2.10), we obtain

$$g^\alpha \cdot \ell^\alpha \cdot \omega^\alpha = g^\alpha \cdot \overline{\overline{\phi}}(\omega^\alpha) \cdot \ell^\alpha =$$
$${}^{\backprime}\omega^\alpha \cdot g^\alpha \cdot \ell^\alpha + (dg^\alpha) \cdot \ell^\alpha = {}^{\backprime}\omega^\alpha \cdot g^\alpha \cdot \ell^\alpha + d(g^\alpha \cdot \ell^\alpha),$$

or, by (7.2.11),

$$h^\alpha \cdot \omega^\alpha = {}^{\backprime}\omega^\alpha \cdot h^\alpha + dh^\alpha, \qquad \alpha \in I.$$

This is exactly (7.2.3), which, in virtue of Theorem 7.2.2, shows that ∇ and ∇' are F-related.

Conversely, assume that ∇ and ∇' are F-related. Then, reversing the previous arguments concerning the local connection forms and matrices, we have that (7.2.3) implies (7.2.12); thus, by Theorem 6.4.5, D and D' are $(f, \phi, \overline{\phi}, id_X)$-related. □

The fiber bundle analog of the previous result is proved in Vassiliou [126, Theorem 4.3].

Conversely to the Theorem 7.2.5, we examine related connections starting with an isomorphism between vector sheaves and then applying the isomorphism between the corresponding sheaves of frames. More precisely, we state the following:

7.2.6 Theorem. *Let $F : \mathcal{E} \to \mathcal{E}'$ be an $\mathcal{A}$-isomorphism of vector sheaves and let $f \equiv (f, id_{\mathcal{GL}(n,\mathcal{A})}, id_{\mathcal{M}_n(\mathcal{A})}, id_X)$ be the $\mathcal{GL}(n, \mathcal{A})$-isomorphism between the corresponding principal sheaves of frames determined by Proposition 5.6.5.*

If ∇, ∇' are $\mathcal{A}$-connections on $\mathcal{E}$ and $\mathcal{E}'$, respectively, and D, D' the corresponding connections on the sheaves of frames, then the following conditions are equivalent:

i) ∇ and ∇' are F-conjugate.

ii) D and D' are f-conjugate.

Proof. We can reproduce the proof of Theorem 7.2.5, since, according to Proposition 5.6.5, f is the unique morphism inducing F. Note that in the present case, $\ell = id|_{\mathcal{A}^n}$. □

We close this section with the classification of vector sheaves of rank 1 equipped with connections, by combining the previous theorems with Corollary 6.7.3. First we introduce the following convenient terminology.

7.2.7 Definition. A ***line sheaf*** is a vector sheaf of rank 1. A pair $(\mathcal{E}, \nabla)$, where $\mathcal{E}$ is a line sheaf and ∇ an $\mathcal{A}$-connection on it, is called a ***Maxwell field***.

Line sheaves are classified by $\boldsymbol{\Phi}^1_{\mathcal{A}}(X) \cong H^1(X, \mathcal{A}^{\bullet})$, as a consequence of Theorem 5.1.8. However, due to their particular rank, their classification can be strengthened by including connections.

In analogy to Definition 6.7.1, we have:

7.2.8 Definition. Two Maxwell fields $(\mathcal{E}, \nabla)$ and $(\mathcal{E}', \nabla')$ are said to be ***equivalent*** if there is an isomorphism of line sheaves $F : \mathcal{E} \to \mathcal{E}'$ such that ∇ and ∇' are F-conjugate. The set of the resulting equivalence classes is denoted by

$$\boldsymbol{\Phi}^1_{\mathcal{A}}(X)^{\nabla}.$$

7.2.9 Theorem. *Maxwell fields are classified by*

$$\boldsymbol{\Phi}^1_{\mathcal{A}}(X)^{\nabla} \cong \check{\boldsymbol{H}}^1\big(X, \mathcal{A}^{\bullet} \xrightarrow{\tilde{\partial}} \Omega\big).$$

Proof. The result will follow from Corollary 6.7.3 and the bijection

$$\chi : \boldsymbol{\Phi}^1_{\mathcal{A}}(X)^{\nabla} \longrightarrow \boldsymbol{P}_{\mathcal{A}^{\bullet}}(X)^D \,:\, [(\mathcal{E}, \nabla)] \mapsto [(\mathcal{P}(\mathcal{E}), D)],$$

if $\mathcal{P}(\mathcal{E}) \equiv (\mathcal{P}(\mathcal{E}), \mathcal{GL}(1, \mathcal{A}) = \mathcal{A}^{\bullet}, X, \tilde{\pi})$ is the sheaf of frames of $\mathcal{E}$ and D the connection corresponding bijectively to ∇ (see Theorem 7.1.6).

The map χ is well defined, since $[(\mathcal{E}, \nabla)] = [(\mathcal{E}', \nabla')]$ implies the existence of a line sheaf isomorphism $F : \mathcal{E} \to \mathcal{E}'$ so that ∇ and ∇' be F-conjugate.

Then, by Theorem 7.2.6, there is an isomorphism $f \equiv (f, id_{\mathcal{A}^\bullet}, id_{\mathcal{A}}, id_X)$ of $\mathcal{P}(\mathcal{E})$ onto $\mathcal{P}(\mathcal{E}')$ such that D and D' are f-conjugate. Hence, by Definition 6.7.1, $[(\mathcal{P}(\mathcal{E}), D)] = [(\mathcal{P}(\mathcal{E}'), D')]$.

Furthermore, χ is 1–1. Indeed, if

$$[(\mathcal{P}(\mathcal{E}), D)] = \chi([(\mathcal{E}, \nabla)]) = \chi([(\mathcal{E}', \nabla')]) = [(\mathcal{P}(\mathcal{E}'), D')],$$

there exists an $\mathcal{A}^\bullet$-isomorphism $f : \mathcal{P}(\mathcal{E}) \to \mathcal{P}(\mathcal{E}')$ such that D and D' are f-conjugate. Then, by Theorem 7.2.5, there is an isomorphism $F : \mathcal{E} \to \mathcal{E}'$ such that ∇ and ∇' are F-conjugate; thus, $[(\mathcal{E}, \nabla)] = [(\mathcal{E}', \nabla')]$.

Finally, let $[(\mathcal{P}, D)]$ be any class in the range of χ, where $\mathcal{P}$ is an arbitrary principal sheaf of the form $\mathcal{P} \equiv (\mathcal{P}, \mathcal{A}^\bullet, X, \pi)$. In virtue of Proposition 5.2.5, there exists a line sheaf such that $\mathcal{P} \cong \mathcal{P}(\mathcal{E})$. If $f \equiv (f, id_{\mathcal{A}^\bullet}, id_{\mathcal{A}}, id_X)$ realizes the previous equivalence and D' is the unique connection on $\mathcal{P}(\mathcal{E})$ which is f-conjugate with D (see Corollary 6.4.6), then $[(\mathcal{P}, D)] = [(\mathcal{P}(\mathcal{E}), D')]$. On the other hand, D' determines bijectively a connection ∇ on $\mathcal{E}$. As a result, $\chi([(\mathcal{E}, \nabla)]) = [(\mathcal{P}(\mathcal{E}), D')] = [(\mathcal{P}, D)]$, which proves that χ is onto. This completes the proof. □

Note. A direct proof of the previous theorem (without recurrence to principal sheaves) is given in Mallios [62, Vol. II, p. 175]. In the latter treatise, line sheaves are denoted by $\mathcal{L}$, a notation reserved here for the sheaves of Lie algebras $\mathcal{L}$ attached to Lie sheaves of groups, as used systematically from Chapter 3 onwards.

7.3. Associated connections

This section deals with connections induced on certain associated sheaves studied in Chapter 5. The sheaves under consideration carry the structure of either a principal or a vector sheaf, so they can be provided with connections.

In what follows, we fix a principal sheaf $\mathcal{P} \equiv (\mathcal{P}, \mathcal{G}, X, \pi)$, where $\mathcal{G} \equiv (\mathcal{G}, \rho_{\mathcal{G}}, \mathcal{L}_{\mathcal{G}}, \partial_{\mathcal{G}})$. Since more than one Lie sheaves of groups will occur, the particular components of each one of them are marked by an appropriate index.

The first case of interest is that of Section 5.4(a). More explicitly, we assume that $\mathcal{H} \equiv (\mathcal{H}, \rho_{\mathcal{H}}, \mathcal{L}_{\mathcal{H}}, \partial_{\mathcal{H}})$ is a second Lie sheaf of groups and $(\phi, \overline{\phi})$ a morphism of $\mathcal{G}$ into $\mathcal{H}$. As we have seen in Proposition 5.4.1,

$$\phi(\mathcal{P}) \equiv (\phi(\mathcal{P}), \mathcal{H}, X, \bar{\pi}), \quad \text{with} \quad \phi(\mathcal{P}) \cong \mathcal{P} \times_X^{\mathcal{G}} \mathcal{H},$$

is a principal sheaf associated with $\mathcal{P}$ by $(\phi, \overline{\phi})$. Moreover, $\mathcal{P}$ and $\phi(\mathcal{P})$ are linked together by a canonical morphism

$$(\varepsilon, \phi, \overline{\phi}, id_X) : (\mathcal{P}, \mathcal{G}, X, \pi) \longrightarrow (\phi(\mathcal{P}), \mathcal{H}, X, \bar{\pi}).$$

We recall that in the construction of $\phi(\mathcal{P})$ only (the morphism of sheaves of groups) $\phi : \mathcal{P} \to \mathcal{H}$ is involved. However, $\overline{\phi}$ is now indispensable in order to provide $\phi(\mathcal{P})$ with connections derived from (and related with) the connections of $\mathcal{P}$.

Before inducing connections on $\phi(\mathcal{P})$, for convenience, we restate Lemma 6.4.3 in the following section-wise version.

7.3.1 Lemma. *If U is any open subset of X, then equality*

$$(1 \otimes \overline{\phi})\big(\rho_{\mathcal{G}}(g).\theta\big) = \rho_{\mathcal{H}}(\phi(g)).(1 \otimes \overline{\phi})(\theta) \tag{7.3.1}$$

holds for every $g \in \mathcal{G}(U)$ and $\theta \in \Omega(\mathcal{L}_{\mathcal{G}})(U) = (\Omega \otimes_{\mathcal{A}} \mathcal{L}_{\mathcal{G}})(U)$.

Proof. For every $x \in U$, based on the interplay between morphisms and the induced morphisms of sections (see convention (1.1.3)), equality (3.3.10) and Lemma 6.4.3 imply that

$$\begin{aligned}\big((1 \otimes \overline{\phi})(\rho_{\mathcal{G}}(g).\theta)\big)\,(x) &= (1 \otimes \overline{\phi})\big(\rho_{\mathcal{G}}(g(x)).\theta(x)\big) \\ &= \rho_{\mathcal{H}}\big(\phi(g(x))\big).(1 \otimes \overline{\phi})(\theta(x)) \\ &= \rho_{\mathcal{H}}\big(\phi(g)(x)\big).\big((1 \otimes \overline{\phi})(\theta)\big)(x) \\ &= \big(\rho_{\mathcal{H}}(\phi(g)).(1 \otimes \overline{\phi})(\theta)\big)(x).\end{aligned}$$

□

7.3.2 Proposition. *Each connection $D_{\mathcal{P}}$ on $\mathcal{P}$ induces a unique connection $D_{\phi(\mathcal{P})}$ on $\phi(\mathcal{P})$ such that $D_{\mathcal{P}}$ and $D_{\phi(\mathcal{P})}$ are $(\varepsilon, \phi, \overline{\phi}, id_X)$-related; that is,*

$$D_{\phi(\mathcal{P})} \circ \varepsilon = (1 \otimes \overline{\phi}) \circ D_{\mathcal{P}},$$

as pictured in the commutative Diagram 7.6 on the next page.

Proof. If (ω_α) are the local connection forms of $D_{\mathcal{P}}$, we set

$$\omega_\alpha^{\phi(\mathcal{P})} := (1 \otimes \overline{\phi})(\omega_\alpha), \quad \alpha \in I. \tag{7.3.2}$$

Using the compatibility condition (6.1.5), Lemma 7.3.1, and equalities (3.4.2) and (5.4.7), we check that

$$\begin{aligned}\omega_\beta^{\phi(\mathcal{P})} &= (1 \otimes \overline{\phi})\big(\rho_{\mathcal{G}}(g_{\alpha\beta}^{-1}).\omega_\alpha + \partial_{\mathcal{G}}(g_{\alpha\beta})\big) \\ &= \rho_{\mathcal{H}}\big(\phi(g_{\alpha\beta}^{-1})\big).(1 \otimes \overline{\phi})(\omega_\alpha) + \partial_{\mathcal{H}}(\phi(g_{\alpha\beta})) \\ &= \rho_{\mathcal{H}}\big(\big(g_{\alpha\beta}^{\phi(\mathcal{P})}\big)^{-1}\big).\omega_\alpha^{\phi(\mathcal{P})} + \partial_{\mathcal{H}}\big(g_{\alpha\beta}^{\phi(\mathcal{P})}\big).\end{aligned}$$

$$\begin{array}{ccc} \mathcal{P} & \xrightarrow{\ \varepsilon\ } & \phi(\mathcal{P}) \\ {\scriptstyle D_{\mathcal{P}}}\downarrow & & \downarrow{\scriptstyle D_{\phi(\mathcal{P})}} \\ \Omega\otimes_{\mathcal{A}}\mathcal{L}_G & \xrightarrow{\ 1\otimes\overline{\phi}\ } & \Omega\otimes_{\mathcal{A}}\mathcal{L}_H \end{array}$$

Diagram 7.6

Thus the local forms $\big(\omega_\alpha^{\phi(\mathcal{P})}\big)$ satisfy the compatibility condition (6.1.5) and determine a connection $D_{\phi(\mathcal{P})}$ on $\phi(\mathcal{P})$.

To prove that $D_{\mathcal{P}}$ and $D_{\phi(\mathcal{P})}$ are $(\varepsilon, \phi, \overline{\phi}, id_X)$-related, we apply Theorem 6.4.2, in particular equality (6.4.3). The latter now takes the form

$$(7.3.2') \qquad (1\otimes\overline{\phi})(\omega_\alpha) = \rho_{\mathcal{H}}(h_\alpha^{-1}).\omega_\alpha^{\phi(\mathcal{P})} + \partial_{\mathcal{H}}(h_\alpha); \qquad \alpha\in I,$$

where each section $h_\alpha \in \mathcal{H}(U_\alpha)$ is determined by $\varepsilon(s_\alpha) = s_\alpha^{\phi(\mathcal{P})}\cdot h_\alpha$ (: the analog of (4.4.1)). However, (5.4.6) implies that $h_\alpha = \mathbf{1}_{\mathcal{H}}|_{U_\alpha}$, thus (7.3.2′) reduces to (7.3.2), and the connections are related as desired.

The uniqueness of a connection $D_{\phi(\mathcal{P})}$ with the property of the statement is guaranteed by Corollary 6.4.6. □

The second case we consider is related with the construction of Section 5.4(c). Here, instead of the morphism $(\phi,\overline{\phi})$ from $\mathcal{G}$ into $\mathcal{H}$, we consider a morphism of Lie sheaves of groups of the particular form

$$(7.3.3) \qquad (\varphi,\overline{\varphi}) : (\mathcal{G},\rho,\mathcal{L},\partial) \longrightarrow \big(\mathcal{GL}(n,\mathcal{A}), \mathcal{A}d, \mathcal{M}_n(\mathcal{A}), \partial\equiv\widetilde{\partial}\big).$$

Note the use of the typefaces $\varphi,\overline{\varphi}$ to distinguish the morphism of the present case from that of the arbitrary $(\phi,\overline{\phi})$ treated before.

Since φ is a representation of $\mathcal{G}$ in $\mathcal{A}^n$, we obtain the vector sheaf

$$(7.3.4) \qquad \mathcal{E}_\varphi := \varphi(\mathcal{P}) \cong \mathcal{P}\times_X^{\mathcal{G}}\mathcal{A}^n,$$

associated with $\mathcal{P}$ by φ, and the natural morphism of principal sheaves

$$(F_{\mathcal{P}},\varphi,\overline{\varphi},id_X) : (\mathcal{P},\mathcal{G},X,\pi) \longrightarrow (\mathcal{P}(\mathcal{E}_\varphi), \mathcal{GL}(n,\mathcal{A}), X, \tilde{\pi}),$$

determined by Lemma 5.5.3. We would like to recall that in the construction of Lemma 5.5.3 we have only used the morphism of sheaves of groups φ.

The previous considerations lead to:

7.3.3 Proposition. *Let $D_{\mathcal{P}}$ be any connection on $\mathcal{P}$. Then there is a unique connection $D_{\mathcal{P}(\mathcal{E}_\varphi)}$ on the sheaf of frames $\mathcal{P}(\mathcal{E}_\varphi)$, which is $(F_{\mathcal{P}}, \varphi, \overline{\varphi}, id_X)$-related with $D_{\mathcal{P}}$; that is,*

$$D_{\mathcal{P}(\mathcal{E}_\varphi)} \circ F_{\mathcal{P}} = (1 \otimes \overline{\varphi}) \circ D_{\mathcal{P}}.$$

Proof. If (ω_α) are the local connection forms of $D_{\mathcal{P}}$, we set

$$\omega_\alpha^{\mathcal{P}(\mathcal{E}_\varphi)} := (1 \otimes \overline{\phi})(\omega_\alpha), \quad \alpha \in I. \tag{7.3.5}$$

Working as in Proposition 7.3.2 and taking into account equality (5.5.9), we verify that the local forms (7.3.5) determine a connection $D_{\mathcal{P}(\mathcal{E}_\varphi)}$ as in the statement. □

The previous result is illustrated in the following commutative diagram.

$$\begin{array}{ccc}
\mathcal{P} & \xrightarrow{\quad F_{\mathcal{P}} \quad} & \mathcal{P}(\mathcal{E}_\varphi) \\
{\scriptstyle D_{\mathcal{P}}}\downarrow & & \downarrow{\scriptstyle D_{\mathcal{P}(\mathcal{E}_\varphi)}} \\
\Omega \otimes_{\mathcal{A}} \mathcal{L} & \xrightarrow{\quad 1 \otimes \overline{\varphi} \quad} & \Omega \otimes_{\mathcal{A}} \mathcal{M}_n(\mathcal{A})
\end{array}$$

Diagram 7.7

As in the first part of the present section, the same morphism (7.3.3) also determines the principal sheaf (see (5.5.3))

$$\mathcal{P}_\varphi \equiv (\mathcal{P}_\varphi, \mathcal{GL}(n, \mathcal{A}), X, \pi_\varphi), \quad \text{with} \quad \mathcal{P}_\varphi = \mathcal{P} \times_X^{\mathcal{G}} \mathcal{GL}(n, \mathcal{A}),$$

and the $\mathcal{GL}(n, \mathcal{A})$-isomorphism of principal sheaves (see Theorem 5.5.5)

$$\theta \equiv (\theta, id_{\mathcal{GL}(n,\mathcal{A})}, id_{\mathcal{M}_n(\mathcal{A})}, id_X) : \mathcal{P}_\varphi \longrightarrow \mathcal{P}(\mathcal{E}_\varphi)$$

satisfying equality (5.5.12), i.e., $\theta \circ \varepsilon = F_{\mathcal{P}}$.

As a consequence, we obtain:

7.3.4 Proposition. *Let $D_{\mathcal{P}}$ be any connection on a principal sheaf $\mathcal{P}$. We denote by D_φ and $D_{\mathcal{P}(\mathcal{E}_\varphi)}$ the connections induced on $\mathcal{P}_\varphi$ and $\mathcal{P}(\mathcal{E}_\varphi)$, respectively, in virtue of Propositions 7.3.2 and 7.3.3. Then D_φ and $D_{\mathcal{P}(\mathcal{E}_\varphi)}$ are θ-conjugate; that is,*

$$D_\varphi = D_{\mathcal{P}(\mathcal{E}_\varphi)} \circ \theta.$$

Equivalently, we have the commutative diagram:

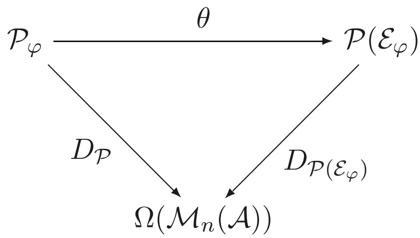

DIAGRAM 7.8

Proof. In virtue of Theorem 6.4.5, D_φ and $D_{\mathcal{P}(\mathcal{E}_\varphi)}$ are θ-conjugate if and only if their local connection forms satisfy equality

$$\omega_\alpha^{\mathcal{P}_\varphi} = \mathcal{A}d(h_\alpha^{-1}).\omega_\alpha^{\mathcal{P}(\mathcal{E}_\varphi)} + \partial(h_\alpha); \qquad \alpha \in I, \tag{7.3.6}$$

where the local sections $h_\alpha \in \mathcal{GL}(n, \mathcal{A})(U_\alpha)$ are determined by the analog of (4.4.6), namely

$$\theta(s_\alpha^{\mathcal{P}_\varphi}) = s_\alpha^{\mathcal{P}(\mathcal{E}_\varphi)} \cdot h_\alpha.$$

However, the preceding equality, combined with (5.5.15), implies that $h_\alpha = \mathbf{1}_{\mathcal{GL}(n,\mathcal{A})}|_{U_\alpha}$, thus (7.3.6) turns into

$$\omega_\alpha^{\mathcal{P}_\varphi} = \omega_\alpha^{\mathcal{P}(\mathcal{E}_\varphi)}, \qquad \alpha \in I. \tag{7.3.7}$$

Therefore, the θ-conjugation of the statement reduces to the verification of the last equality. This is indeed true, since both sides of (7.3.7) coincide with $(1 \otimes \overline{\varphi})(\omega_\alpha)$, according to (7.3.2) (for $\phi = \varphi$, $\phi(\mathcal{P}) = \mathcal{P}_\varphi$) and (7.3.5). $\square$

Chapter 8

Curvature

Curvature is the simplest local measure of geometric properties. Curvature is therefore a good first step toward a more comprehensive picture of the space-time in question.

C. Misner–K. Thorne–J. Wheeler [76, § 14.1: "Curvature as a tool for understanding physics", p. 334]

Curvature is another fundamental geometric notion. This chapter deals first with the curvature of connections on principal sheaves and then moves on to the particular cases of connections on vector and associated sheaves. In the language of physics, the curvature is the *field strength* of a gauge potential (viz. connection). It is precisely the former which manifests the presence of the latter, henceforth the importance of the curvature, both in geometry and physics.

In our abstract approach, the curvature of a connection can be defined if we assume the existence of a differential (of order 1) on $\Omega(\mathcal{L})$, extending –in a sense– the Maurer-Cartan differential of the structure sheaf. An appropriate differential (of order 2) on $\Omega^2(\mathcal{L})$ implies Bianchi's identity.

A considerable part of the chapter is devoted to the study of *flat connections* (i.e., connections of zero curvature). In particular, we show that this notion of flatness is equivalent to the relevant notions of *(complete) parallelism* and *integrability* of connections, and *∂-flatness* of a principal sheaf, under an appropriate Frobenius integrability condition. We also examine *flat principal sheaves*, namely, principal sheaves whose cocycles have coefficients in a constant sheaf, thus the transition sections are locally constant. Unlike the case of connections on ordinary smooth bundles, the last notion of flatness is not equivalent to flat connections and the other related notions mentioned above.

The case of flat connections on $\mathcal{GL}(n, \mathcal{A})$-principal sheaves, discussed in the final section, is an illuminating example clarifying many technicalities of the general theory.

8.1. Preliminaries

Throughout this chapter we fix a differential triad $(\mathcal{A}, d, \Omega)$ over a topological space $X \equiv (X, \mathfrak{T}_X)$.

In Section 2.5 we defined the p-th exterior power of Ω

$$\Omega^p := \underbrace{\Omega^1 \bigwedge_{\mathcal{A}} \cdots \bigwedge_{\mathcal{A}} \Omega^1}_{p-\text{factors}} \equiv \bigwedge^p \Omega^1; \qquad p \geq 0,$$

where $\Omega^0 = \mathcal{A}$ and $\bigwedge^1 \Omega = \Omega^1 = \Omega$, along with the differentials

$$d^p : \Omega^p \longrightarrow \Omega^{p+1}; \qquad p \geq 0,$$

where $d^0 := d$.

We also defined the exterior product

$$\wedge : \Omega^p \times \Omega^q \longrightarrow \Omega^p \bigwedge_{\mathcal{A}} \Omega^q \equiv \Omega^{p+q}.$$

Since the exterior power Ω^p may be confused with the fiber product of p factors all equal to Ω, in what follows, according to the concluding comment of Subsection 1.3.2, Ω^p *will exclusively denote the exterior power,* whereas *the fiber product will be denoted by* $\Omega^{(p)}$.

Given a Lie sheaf of groups $\mathcal{G} \equiv (\mathcal{G}, \rho, \mathcal{L}, \partial)$, analogously to (3.3.4), we define the ***$\mathcal{L}$-valued p-th exterior power*** of Ω

$$\boxed{\Omega^p(\mathcal{L}) := \Omega^p \otimes_{\mathcal{A}} \mathcal{L}} \tag{8.1.1}$$

and the exterior algebra

$$\Omega^{\bullet}(\mathcal{L}) \equiv \bigwedge(\Omega(\mathcal{L})) := \bigoplus_{p=0}^{\infty} \Omega^p(\mathcal{L}). \tag{8.1.2}$$

Referring also to the remarks of Subsection 1.3.6, we recall that the sheaf (8.1.1) is generated by the presheaf $U \longmapsto \bigwedge^p(\Omega^1(U)) \otimes \mathcal{L}(U)$.

We shall define an ***exterior product*** on $\Omega^{\bullet}(\mathcal{L})$, denoted by

$$\boldsymbol{\wedge} : \Omega^p(\mathcal{L}) \times_X \Omega^q(\mathcal{L}) \longrightarrow \Omega^{p+q}(\mathcal{L}), \tag{8.1.3}$$

which extends the exterior product (2.5.6). The reader might have noticed the typographical difference –made for the sake of distinction– between the wedge $\wedge$ of (2.5.6) and the (bigger and bolder) $\boldsymbol{\wedge}$ of (8.1.3).

To this end, for an open $U \subseteq X$, we define the (local) exterior product

$$\boldsymbol{\wedge}_U : \big(\bigwedge^p(\Omega^1(U)) \otimes_{\mathcal{A}(U)} \mathcal{L}(U)\big) \times \big(\bigwedge^q(\Omega^1(U)) \otimes_{\mathcal{A}(U)} \mathcal{L}(U)\big) \longrightarrow \bigwedge^{p+q}(\Omega^1(U)) \otimes_{\mathcal{A}(U)} \mathcal{L}(U),$$

by requiring that

$$(\omega \otimes u, \theta \otimes v) \longmapsto (\omega \otimes u)\boldsymbol{\wedge}_U(\theta \otimes v) := (\omega \wedge_U \theta) \otimes [u, v]. \tag{8.1.4}$$

The definition is extended to arbitrary tensors by $\mathcal{A}(U)$-linearity.

Being clear that the operators $\wedge_U$ $(U \in \mathfrak{T}_X)$ figuring in $\omega \wedge_U \theta$ of (8.1.4) generate the exterior product (2.5.6) on $\Omega^{\bullet}$, it is straightforward to show that the assignment $U \mapsto \boldsymbol{\wedge}_U$ is a presheaf morphism, whose sheafification is –by definition– the exterior product (8.1.3).

Since, classically, $\boldsymbol{\wedge}$ is denoted by [ ], in certain cases we follow the same trend, although the plethora of brackets used in this work may lead to some confusion. Thus we also set

$$a\boldsymbol{\wedge}b \equiv [a, b], \qquad (a, b) \in \Omega^p(\mathcal{L}) \times_X \Omega^q(\mathcal{L}). \tag{8.1.3'}$$

For details concerning the analogous case of differential forms with values in a Lie algebra, we refer, for instance, to Bleecker [10, p. 36], Naber [81, p. 234], Pham Mau Quan [101, p. 211].

By a routine application of (8.1.4), we establish the following fundamental properties:

$$(a + a')\bigwedge b = a\bigwedge b + a'\bigwedge b,$$

$$\alpha \cdot (a\bigwedge b) = (\alpha \cdot a)\bigwedge b = a\bigwedge(\alpha \cdot b),$$

for every $a, a', b \in \Omega^{\bullet}(\mathcal{L})_x = \big(\bigwedge(\Omega(\mathcal{L}))\big)_x \cong \bigwedge(\Omega(\mathcal{L}))_x$, $\alpha \in \mathcal{A}_x$, and $x \in X$;

$$a\bigwedge b = (-1)^{pq+1} b\bigwedge a$$

$$(*) \qquad (-1)^{pr}(a\bigwedge b)\bigwedge c + (-1)^{qp}(b\bigwedge c)\bigwedge a + (-1)^{rq}(c\bigwedge a)\bigwedge b = 0,$$

for every $a \in \Omega^p(\mathcal{L})_x$, $b \in \Omega^q(\mathcal{L})_x$, $c \in \Omega^r(\mathcal{L})_x$, and every $x \in X$.

As a consequence of the preceding, we see that $\bigwedge$ is not associative, and the product $a\bigwedge a$, $(a \in \Omega^{\bullet}(\mathcal{L}))$ is not necessarily identical to zero (in contrast to the case of the forms $a \in \Omega^{\bullet}$). For later reference we record two obvious equalities:

$$\text{(8.1.5a)} \qquad a\bigwedge b = b\bigwedge a; \qquad (a, b) \in \Omega^1(\mathcal{L}) \times_X \Omega^1(\mathcal{L}),$$

$$\text{(8.1.5b)} \qquad (a\bigwedge a)\bigwedge a = 0; \qquad a \in \Omega^1(\mathcal{L}).$$

As we have seen in (3.3.5) – (3.3.7), a representation $\rho : \mathcal{G} \to \mathcal{A}ut(\mathcal{L})$ induces a natural action of $\mathcal{G}$ on $\Omega(\mathcal{L}) \equiv \Omega^1(\mathcal{L})$. An analogous action of $\mathcal{G}$ on $\Omega^p(\mathcal{L})$, $p \geq 2$, is generated by the local actions (for all $U \in \mathfrak{T}_X$)

$$\mathcal{G}(U) \times \big(\bigwedge^p(\Omega^1(U)) \otimes_{\mathcal{A}(U)} \mathcal{L}(U)\big) \longrightarrow \bigwedge^p(\Omega^1(U)) \otimes_{\mathcal{A}(U)} \mathcal{L}(U),$$

given by

$$\text{(8.1.6)} \qquad (s, \omega \otimes u) \longmapsto \omega \otimes \rho(s)(u).$$

In conformity with the notations of Section 3.3, we still write

$$\text{(8.1.7)} \qquad \boxed{\rho(g).w}$$

in order to denote the result of the action of $g \in \mathcal{G}_x$ on $w \in (\Omega^n(\mathcal{L}))_x \cong (\Omega^n)_x \otimes_{\mathcal{A}_x} \mathcal{L}_x$, for any $x \in X$. For g and w as before, there are sections $s \in \mathcal{G}(U)$ and $t \in \bigwedge^p(\Omega^1(U)) \otimes_{\mathcal{A}(U)} \mathcal{L}(U)$ such that $g = \widetilde{s}(x) \equiv s(x)$ and $w = \widetilde{t}(x)$, for some $U \in \mathcal{N}(x)$. Therefore, as in (3.3.7′),

$$\text{(8.1.8)} \qquad \rho(g).w = \big((1 \otimes \rho(s))(t)\big)^{\sim}(x).$$

The previous action is also extended, in an obvious way, to an action on the exterior algebra $\Omega^{\bullet}(\mathcal{L})$ and is related with the exterior product $\bigwedge$ in the following manner:

8.1.1 Proposition. *The exterior product* $\bigwedge$ *is* $\mathcal{G}$*-equivariant in the sense that equality*

$$\rho(g).(a \bigwedge b) = (\rho(g).a) \bigwedge (\rho(g).b), \tag{8.1.9}$$

holds for every $g \in \mathcal{G}_x$, $a \in \Omega^p(\mathcal{L})_x$, $b \in \Omega^q(\mathcal{L})_x$, $x \in X$, *and for any* p, q.

Proof. As in (8.1.8), let

$$s \in \mathcal{G}(U),\ t \in \textstyle\bigwedge^p(\Omega^1(U)) \otimes_{\mathcal{A}(U)} \mathcal{L}(U),\ r \in \textstyle\bigwedge^q(\Omega^1(U)) \otimes_{\mathcal{A}(U)} \mathcal{L}(U)$$

be sections such that $g = s(x)$, $a = \widetilde{t}(x)$ and $b = \widetilde{r}(x)$, with $U \in \mathcal{N}(x)$. We further assume that $t = \omega \otimes u$, $r = \theta \otimes v$ (the general case of non-decomposable tensors is handled by linear extension). Then, by (8.1.3) and (8.1.4),

$$a \bigwedge b = \big((\omega \otimes u) \bigwedge_U (\theta \otimes v)\big)^{\sim}(x) = \big((\omega \wedge_U \theta) \otimes [u, v]\big)^{\sim}(x).$$

Hence, taking into account (8.1.8), we have that

$$\rho(g).(a \bigwedge b) = \Big(\big(1 \otimes \rho(s)\big)\big((\omega \wedge_U \theta) \otimes [u, v]\big)\Big)^{\sim}(x). \tag{8.1.10}$$

On the other hand,

$$\begin{aligned}
\big(1 \otimes \rho(s)\big)\big((\omega \wedge_U \theta) \otimes [u, v]\big) &= (\omega \wedge_U \theta) \otimes [\rho(s)(u), \rho(s)(v)] \\
&= \big(\omega \otimes \rho(s)(u)\big) \textstyle\bigwedge_U \big(\theta \otimes \rho(s)(v)\big) \\
&= \big(1 \otimes \rho(s)\big)(\omega \otimes u) \textstyle\bigwedge_U \big(1 \otimes \rho(s)\big)(\theta \otimes v) \\
&= \big(1 \otimes \rho(s)\big)(t) \textstyle\bigwedge_U \big(1 \otimes \rho(s)\big)(r).
\end{aligned}$$

Therefore, (8.1.10) transforms into

$$\begin{aligned}
\rho(g).(a \bigwedge b) &= \Big(\big(1 \otimes \rho(s)\big)(t) \textstyle\bigwedge_U \big(1 \otimes \rho(s)\big)(r)\Big)^{\sim}(x) \\
&= \big(\big(1 \otimes \rho(s)\big)(t)\big)^{\sim}(x) \textstyle\bigwedge \big(\big(1 \otimes \rho(s)\big)(r)\big)^{\sim}(x) \\
&= (\rho(g).a) \textstyle\bigwedge (\rho(g).b),
\end{aligned}$$

as stated. □

8.1.2 Examples. In anticipation of later applications, we focus our considerations on the following two basic cases:

(a) The exterior product of $\Omega^1(\mathcal{C}^\infty_X(\mathbb{G}))$

The module $\Omega^1(\mathcal{C}^\infty_X(\mathbb{G})) := \Omega^1 \otimes_{\mathcal{C}^\infty_X} \mathcal{C}^\infty_X(\mathbb{G})$, already encountered in many instances, has been defined in Example 3.3.6(a). We shall connect its exterior product in the sense of (8.1.3) with the usual exterior product of Lie algebra valued differential forms. For simplicity we restrict ourselves to 1-forms.

Given two ordinary forms $\omega, \theta \in \Lambda^1(U, \mathbb{G})$, their bracket $[\omega, \theta] \in \Lambda^2(U, \mathbb{G})$ (in other words, their exterior product with respect to the Lie algebra structure of $\mathbb{G}$), is defined by

$$[\omega, \theta]_x(u, v) := [\omega_x(u), \theta_x(v)] - [\omega_x(v), \theta_x(u)], \tag{8.1.11}$$

for every $x \in U$ and $u, v \in T_x X$ (the formula extends analogously to arbitrary forms; see, e.g., Bleecker [10, p. 35], Naber [81, p. 234]). Since

$$\omega = \sum_{i=1}^n \omega_i E_i^U, \quad \theta = \sum_{i=1}^n \theta_i E_i^U, \quad [E_i^U, E_j^U] = \sum_{k=1}^n c_{ij}^k \, E_k^U,$$

the definition of the ordinary exterior product of $\mathbb{R}$-valued forms leads to (see the notations of Example 3.3.6(a))

$$[\omega, \theta] = \sum_{k=1}^n \Big(\sum_{i,j=1}^n c_{ij}^k \, \omega_i \wedge \theta_j \Big) E_k^U. \tag{8.1.12}$$

If we define the 2-form analog of (3.3.13), namely

$$\underline{\lambda}_U^2 : \Lambda^2(U, \mathbb{G}) \xrightarrow{\ \simeq\ } \Lambda^2(U, \mathbb{R}) \otimes_{C^\infty(U,\mathbb{R})} C^\infty(U, \mathbb{G}),$$

by a formula similar to (3.3.13a), then (8.1.12) yields the element

$$\underline{\lambda}_U^2([\omega, \theta]) = \sum_{k=1}^n \Big(\sum_{i,j=1}^n c_{ij}^k \, \omega_i \wedge \theta_j \Big) \otimes E_k^U. \tag{8.1.13}$$

On the other hand, the definition of $\bigwedge_U$ and (3.3.13a) imply that

$$\begin{aligned} \underline{\lambda}_U^1(\omega) \textstyle\bigwedge_U \underline{\lambda}_U^1(\theta) &= \Big(\sum_{i=1}^n \omega_i \otimes E_i^U \Big) \textstyle\bigwedge_U \Big(\sum_{j=1}^n \theta_j \otimes E_j^U \Big) \\ &= \sum_{i,j=1}^n (\omega_i \wedge \theta_j) \otimes [E_i^U, E_j^U] \\ &= \sum_{k=1}^n \Big(\sum_{i,j=1}^n c_{ij}^k \, \omega_i \wedge \theta_j \Big) \otimes E_k^U. \end{aligned} \tag{8.1.14}$$

Note that, in the present case, the entities Ω^1 and $\wedge_U$ involved in the general formula (8.1.4) are replaced by $\Lambda^1(U,\mathbb{R})$ and $\wedge$, respectively. Thus, equalities (8.1.13) and (8.1.14) give

$$\underline{\lambda}^2_U([\omega,\theta]) = \underline{\lambda}^1_U(\omega)\bigwedge_U \underline{\lambda}^1_U(\theta); \qquad \omega,\theta \in \Lambda^1(U,\mathbb{G}), \tag{8.1.15}$$

for every open $U \subseteq X$. As a result, using (8.1.3$'$) and the inverse of $\underline{\lambda}^1$, namely

$$\underline{\mu}^1 : \Omega^1(\mathcal{C}^\infty_X(\mathbb{G})) = \Omega^1 \otimes_{\mathcal{C}^\infty_X} \mathcal{C}^\infty_X(\mathbb{G}) \xrightarrow{\simeq} \Omega^1_X(\mathbb{G})$$

(see (3.3.14)), we find the following (global) formula

$$a\bigwedge b \equiv [a,b] = \underline{\lambda}^2([\underline{\mu}^1(a),\underline{\mu}^1(b)]), \tag{8.1.16}$$

for every $(a,b) \in \Omega^1(\mathcal{C}^\infty_X(\mathbb{G})) \times_X \Omega^1(\mathcal{C}^\infty_X(\mathbb{G}))$.

Here $\underline{\lambda}^2$ is the sheafification of $(\underline{\lambda}^2_U)$, and the bracket, in the last term of (8.1.16), is the sheafification of the individual brackets of (8.1.12), for all open $U \subseteq X$. (Typically, we should have written $[\omega,\theta]_U$ and $\omega_i \wedge_U \theta_j$ in (8.1.12), but we omitted this detail as being easily understood).

Formula (8.1.16) establishes the relationship mentioned in the beginning of the example.

(b) The exterior product of $\Omega^1(\mathcal{M}_n(\mathcal{A}))$

For the module in title we refer to equalities (3.1.3) and (3.1.4), as well as to the general discussion of Section 3.3.

Thinking of the matrix sheaf $\mathcal{M}_n(\mathcal{A})$ as the Lie algebra sheaf of $\mathcal{GL}(n,\mathcal{A})$, according to (3.1.7) and its analog for 2nd order forms, we have the identifications

$$\mu^1 : \Omega^1(\mathcal{M}_n(\mathcal{A})) := \Omega^1 \otimes_{\mathcal{A}} \mathcal{M}_n(\mathcal{A}) \xrightarrow{\cong} \mathcal{M}_n(\Omega^1),$$
$$\mu^2 : \Omega^2(\mathcal{M}_n(\mathcal{A})) := \Omega^2 \otimes_{\mathcal{A}} \mathcal{M}_n(\mathcal{A}) \xrightarrow{\cong} \mathcal{M}_n(\Omega^2),$$

induced by the corresponding local isomorphisms (μ^1_U) and (μ^2_U). The inverse of μ^2 is denoted by λ^2.

We intend to connect the exterior product

$$\bigwedge : \Omega^1(\mathcal{M}_n(\mathcal{A})) \times_X \Omega^1(\mathcal{M}_n(\mathcal{A})) \longrightarrow \Omega^2(\mathcal{M}_n(\mathcal{A})) \tag{8.1.17}$$

with the *exterior product of matrices*

$$\wedge : \mathcal{M}_n(\Omega^1) \times_X \mathcal{M}_n(\Omega^1) \to \mathcal{M}_n(\Omega^2) \tag{8.1.18}$$

(same symbol as $\wedge : \Omega^1 \times_X \Omega^1 \to \Omega^2$), generated by

$$(\omega_{ij}) \wedge_U (\theta_{ij}) := \Big(\sum_{k=1}^{n} \omega_{ik} \wedge_U \theta_{kj} \Big), \tag{8.1.19}$$

if $(\omega_{ij}), (\theta_{ij}) \in M_n(\Omega^1(U)) \cong \mathcal{M}_n(\Omega^1)(U)$, for any open $U \subseteq X$. Recall that the products $(\wedge_U)$ on the right-hand side of (8.1.19) generate the original $\wedge : \Omega^1 \times_X \Omega^1 \to \Omega^2$.

Let $\omega, \theta \in \Omega^1(U) \otimes_{\mathcal{A}(U)} M_n(\mathcal{A}(U)) \cong \Omega^1(\mathcal{M}_n(\mathcal{A}))(U)$ be two arbitrary 1-forms (viz. sections). Since, by the notations following (3.1.6a), the forms ω, θ can be written as

$$\omega = \sum_{i,j=1}^{n} \omega_{ij} \otimes E_{ij}^U, \quad \theta = \sum_{i,j=1}^{n} \theta_{ij} \otimes E_{ij}^U,$$

taking into account the Lie algebra structure of $M_n(\mathcal{A}(U))$, and applying a direct calculation (as in the first example), we see that

$$\begin{aligned} \omega \bigwedge\nolimits_U \theta &= \Big(\sum_{i,j=1}^{n} \omega_{ij} \otimes E_{ij}^U \Big) \bigwedge\nolimits_U \Big(\sum_{i,j=1}^{n} \theta_{ij} \otimes E_{ij}^U \Big) \\ &= \sum_{i,j,k,l=1}^{n} (\omega_{ij} \wedge_U \theta_{kl}) \otimes [E_{ij}^U, E_{kl}^U] \\ &= \sum_{i,j=1}^{n} \Big(\sum_{k=1}^{n} (\omega_{ik} \wedge \theta_{kj} + \theta_{ik} \wedge \omega_{kj}) \Big) \otimes E_{ij}^U. \end{aligned} \tag{8.1.20}$$

Hence, (3.1.6b) and (8.1.19) transform (8.1.20) into

$$\mu_U^2(\omega \bigwedge\nolimits_U \theta) = \mu_U^1(\omega) \wedge_U \mu_U^1(\theta) + \mu_U^1(\theta) \wedge_U \mu_U^1(\omega), \tag{8.1.21}$$

for every $\omega, \theta \in \Omega^1(U) \otimes_{\mathcal{A}(U)} M_n(\mathcal{A}(U))$. Note that the definition of the product (8.1.19) does not allow a simplification of the second member of (8.1.21) as in the case of forms. Therefore, by sheafification,

$$\mu^2(a \bigwedge b) \equiv \mu^2([a,b]) = \mu^1(a) \wedge \mu^1(b) + \mu^1(b) \wedge \mu^1(a), \tag{8.1.22}$$

for every $(a,b) \in \Omega^1(\mathcal{M}_n(\mathcal{A})) \times_X \Omega^1(\mathcal{M}_n(\mathcal{A}))$, where the exterior product $\wedge$ on the right-hand side of (8.1.22) is the sheafification of the products $\wedge_U$ of (8.1.19), when U is varying in the topology of X.

In particular,

$$\mu^2_U(\omega \bigwedge_U \omega) = 2\, \mu^1_U(\omega) \wedge_U \mu^1_U(\omega), \tag{8.1.21$'$}$$

$$\mu^2(a \bigwedge a) \equiv \mu^2([a,a]) = 2\, \mu^1(a) \wedge \mu^1(a), \tag{8.1.22$'$}$$

for every $\omega \in \Omega^1(U) \otimes_{\mathcal{A}(U)} M_n(\mathcal{A}(U))$ and $a \in \Omega^1(\mathcal{M}_n(\mathcal{A}))$.

To prepare the definition of the curvature of a connection, given in the next section, we first need the following notion:

8.1.3 Definition. Let $(X, \mathcal{A})$ be an algebraized space and the differential triad $(\mathcal{A}, d, \Omega^1)$ over it. A ***precurvature datum*** on X is a quintuple

$$(\mathcal{A}, d, \Omega^1, d^1, \Omega^2), \tag{8.1.23}$$

where $d^1 : \Omega^1 \to \Omega^2$ is the 1st exterior derivation extending $d = d^0$ and satisfying (2.5.8), (2.5.9).

In Mallios [62, Vol. II, p. 188], (8.1.23) is called a "curvature" datum since it is sufficient for the definition of the curvature of $\mathcal{A}$-connections (on vector sheaves). However, to define the curvature of connections on principal sheaves, we need a different sequence of sheaves and morphisms, for which we reserve the term curvature datum (see Definition 8.1.4). Such a datum involves, unavoidably, the structure of the Lie sheaf of groups.

More precisely, given a Lie sheaf of groups $\mathcal{G} \equiv (\mathcal{G}, \rho, \mathcal{L}, \partial)$, we assume the existence of a $\mathbb{K}$-linear morphism

$$\boldsymbol{d}^1 : \Omega^1(\mathcal{L}) \longrightarrow \Omega^2(\mathcal{L}), \tag{8.1.24}$$

satisfying the following conditions:

$$(\boldsymbol{d}^1 \circ \partial)(g) = -\frac{1}{2}\, \partial(g) \bigwedge \partial(g), \tag{8.1.25}$$

$$\boldsymbol{d}^1(\rho(g).w) = \rho(g).\big(\boldsymbol{d}^1 w + \partial(g) \bigwedge w\big), \tag{8.1.26}$$

for every $g \in \mathcal{G}$, and every $(g, w) \in \mathcal{G} \times_X \Omega^1(\mathcal{L})$, respectively.

Condition (8.1.26) will be mainly used in its equivalent form

$$\boldsymbol{d}^1(\rho(g^{-1}).w) = \rho(g^{-1}).\boldsymbol{d}^1 w - \partial(g) \bigwedge \rho(g^{-1}).w, \tag{8.1.26$'$}$$

which is a direct consequence of Propositions 3.3.5 and 8.1.1. Both (8.1.26) and (8.1.26′) describe the behavior of $\boldsymbol{d}^1$ regarding the action of $\mathcal{G}$ on $\Omega^1(\mathcal{L})$. Therefore, $\boldsymbol{d}^1$ is not $\mathcal{G}$-equivariant.

> Conditions (8.1.25) and (8.1.26) are crucial for the development of a notion of curvature with the properties of its classical counterpart. Although both are inherent in the ordinary smooth context (see also Examples 8.1.6 below), they are never used in the study of the classical curvature because other approaches prevail. In particular, (8.1.26) and (8.1.26′) are rarely mentioned (and proved) in the literature.

With the previous notations, we introduce the next important $\mathbb{K}$-linear morphism

$$\mathcal{D} : \Omega^1(\mathcal{L}) \longrightarrow \Omega^2(\mathcal{L}), \tag{8.1.27}$$

defined by (see also (8.1.3′))

$$\mathcal{D}(w) := \boldsymbol{d}^1 w + \frac{1}{2}\, w \wedge w \equiv \boldsymbol{d}^1 w + \frac{1}{2}\, [w, w], \qquad w \in \Omega^1(\mathcal{L}). \tag{8.1.28}$$

The morphism $\mathcal{D}$ is called a ***Cartan (second) structure operator***. It depends, of course, on the construction of a morphism $\boldsymbol{d}^1$. Accordingly, condition (8.1.28) is called the ***Cartan (second) structure equation***.

8.1.4 Definition. A ***curvature datum*** is a pair $(\mathcal{G}, \mathcal{D})$, where $\mathcal{G}$ is a Lie sheaf of groups and $\mathcal{D}$ a Cartan structure operator.

8.1.5 Proposition. *The operator $\mathcal{D}$ has the following properties:*

$$\mathcal{D} \circ \partial = 0, \tag{8.1.29}$$

$$\mathcal{D}(w + w') = \mathcal{D}(w) + \mathcal{D}(w') + w \wedge w', \tag{8.1.30}$$

$$\mathcal{D}\big(\rho(g^{-1}).w + \partial(g)\big) = \rho(g^{-1}).\mathcal{D}(w), \tag{8.1.31}$$

for every $(w, w') \in \Omega^1(\mathcal{L}) \times_X \Omega^1(\mathcal{L})$ and $(g, w) \in \mathcal{G} \times_X \Omega^1(\mathcal{L})$.

Proof. The first property is an obvious consequence of (8.1.25). The second is a result of the additivity of $\boldsymbol{d}^1$ and $\wedge$, combined with equality (8.1.5a).

For the last property, (8.1.29) and (8.1.30) yield

$$\mathcal{D}\big(\rho(g^{-1}).w + \partial(g)\big) = \mathcal{D}(\rho(g^{-1}).w) + (\rho(g^{-1}).w) \wedge \partial(g).$$

Thus, (8.1.28) implies that

$$\begin{aligned}
\mathcal{D}\big(\rho(g^{-1}).w + \partial(g)\big) &= \boldsymbol{d}^1\big(\rho(g^{-1}).w\big) + \frac{1}{2}\,(\rho(g^{-1}).w)\wedge\rho(g^{-1}).w \\
&\quad + (\rho(g^{-1}).w)\wedge\partial(g) \\
\text{(see (8.1.26}'\text{))} \qquad &= \rho(g^{-1}).\boldsymbol{d}^1 w - \partial(g)\wedge\rho(g^{-1}).w \\
\text{(see (8.1.9))} \qquad &\quad + \frac{1}{2}\,\rho(g^{-1}).(w\wedge w) + (\rho(g^{-1}).w)\wedge\partial(g) \\
\text{(see (8.1.5a))} \qquad &= \rho(g^{-1}).\boldsymbol{d}^1 w + \frac{1}{2}\,\rho(g^{-1}).(w\wedge w) \\
&= \rho(g^{-1}).\mathcal{D}(w). \qquad \square
\end{aligned}$$

Equality (8.1.29) is also called the ***Maurer-Cartan equation***, with respect to $(\mathcal{G}, \mathcal{D})$. The reason for this terminology is explained in the note at the end of Example 8.1.6(a).

8.1.6 Examples. We continue the discussion of Examples 8.1.2 by describing their differential $\boldsymbol{d}^1$ and the corresponding curvature data.

(a) The curvature datum of $\mathcal{C}^\infty_X(G)$

We consider $\mathcal{C}^\infty_X(G)$ with the structure of the Lie sheaf of groups induced by an ordinary Lie group G, as in Example 3.3.6(a). Then we define the differential

$$\boldsymbol{d}^1 : \Omega^1(\mathcal{C}^\infty_X(\mathbb{G})) \longrightarrow \Omega^2(\mathcal{C}^\infty_X(\mathbb{G}))$$

to be the sheafification of the operators

$$\boldsymbol{d}^1_U : \Lambda^1(U,\mathbb{R}) \otimes_{C^\infty(U,\mathbb{R})} C^\infty(U,\mathbb{G}) \longrightarrow \Lambda^2(U,\mathbb{R}) \otimes_{C^\infty(U,\mathbb{R})} C^\infty(U,\mathbb{G}),$$

given by $\boldsymbol{d}^1_U := \underline{\lambda}^2_U \circ d^1 \circ \underline{\mu}^1_U$, as shown in the diagram

$$\begin{array}{ccc}
\Lambda^1(U,\mathbb{G}) & \xrightarrow{\quad d^1 \quad} & \Lambda^2(U,\mathbb{G}) \\
\uparrow{\scriptstyle \underline{\mu}^1_U} & & \downarrow{\scriptstyle \underline{\lambda}^2_U} \\
\Lambda^1(U,\mathbb{R}) \otimes_{C^\infty(U,\mathbb{R})} C^\infty(U,\mathbb{G}) & \xrightarrow{\;\boldsymbol{d}^1_U\;} & \Lambda^2(U,\mathbb{R}) \otimes_{C^\infty(U,\mathbb{R})} C^\infty(U,\mathbb{G})
\end{array}$$

Diagram 8.1

where d^1 is in fact the (restriction to U of the) ordinary differential of $\mathbb{G}$-valued 1-forms.

In particular, if we consider an 1-form of the type $\omega = \sum_{i=1}^n \omega_i \otimes E_i \in \Lambda^1(U,\mathbb{R}) \otimes_{C^\infty(U,\mathbb{R})} C^\infty(U,\mathbb{G})$, then it follows at once that

$$\boldsymbol{d}_U^1 \omega = \sum_{i=1}^n (d^1\omega_i) \otimes E_i \tag{8.1.32}$$

(see also the notations and analogous computations in Example 8.1.2(a), in conjunction with (3.3.13b) and the 2nd order analog of (3.3.13a)).

It is worth noticing that the identification $\Lambda^p(U,\mathbb{G}) \cong \Lambda^p(U,\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{G}$ implies that the ordinary differential (of order p) of $\mathbb{G}$-valued forms is identified with $d^p \otimes 1 : \Lambda^p(U,\mathbb{G}) \otimes_{\mathbb{R}} \mathbb{G} \to \Lambda^{p+1}(U,\mathbb{G}) \otimes_{\mathbb{R}} \mathbb{G}$ (see, e.g., Greub-Halperin-Vanstone [35, Vol. I, p. 150], Pham Mau Quan [101, p. 211]). Though (8.1.32) is quite close to the previous tensor product (for $p = 1$), here we cannot write $\boldsymbol{d}_U^1 = d^1 \otimes 1$, since now this tensor product should be taken over $C^\infty(U,\mathbb{R})$, whereas d^1 is *not* linear with respect to the latter algebra.

We show that $\boldsymbol{d}^1$ satisfies (8.1.25): For an open $U \subseteq X$ and any $g \in C^\infty(U,G)$, (3.3.17) implies that

$$\begin{aligned}
\boldsymbol{d}_U^1(\partial_U(g)) &= (\underline{\lambda}_U^2 \circ d^1 \circ \underline{\mu}_U^1)(\partial_U(g)) \\
&= (\underline{\lambda}_U^2 \circ d^1 \circ \underline{\mu}_U^1)\big(\underline{\lambda}_U^1(g^{-1}.dg)\big) \\
&= \underline{\lambda}_U^2\big(d^1(g^{-1}.dg)\big),
\end{aligned}$$

where $d = d^0$ is the usual differential of smooth functions. But, if α denotes the left Maurer-Cartan form of G, the ordinary Maurer-Cartan equation gives

$$d^1(g^{-1}.dg) + \frac{1}{2}\,[g^{-1}.dg, g^{-1}.dg] = g^*\big(d^1\alpha + \frac{1}{2}\,[\alpha,\alpha]\big) = 0,$$

with g^* denoting the ordinary pull-back of forms by g. Hence,

$$\begin{aligned}
\boldsymbol{d}_U^1(\partial_U(g)) &= \underline{\lambda}_U^2\big(d^1(g^{-1}.dg)\big) \\
&= -\frac{1}{2}\,\underline{\lambda}_U^2\big([g^{-1}.dg,\ g^{-1}.dg]\big) \\
\text{(see (3.3.17))} \qquad &= -\frac{1}{2}\,\underline{\lambda}_U^2\big(\big[\underline{\mu}_U^1(\partial_U(g)), \underline{\mu}_U^1(\partial_U(g))\big]\big) \\
\text{(see (8.1.15))} \qquad &= -\frac{1}{2}\,\partial_U(g) \bigwedge\nolimits_U \partial_U(g).
\end{aligned}$$

Varying U in the topology of X, we get (8.1.25).

We now verify (8.1.26′). Since in the present context it reads

$$\boldsymbol{d}^1\big(\mathcal{A}d(g^{-1}).w\big) = \mathcal{A}d(g^{-1}).\boldsymbol{d}^1 w - \partial(g)\bigwedge \mathcal{A}d(g^{-1}).w,$$

for every $(g,w) \in \mathcal{C}^\infty_X(G) \times_X \Omega^1(\mathcal{C}^\infty_X(\mathbb{G}))$, it suffices to verify its analog on the level of the generating presheaves and morphisms (see also the definition of $\mathcal{A}d$ in the same Example 3.3.6(a)); that is,

$$\begin{gathered}\boldsymbol{d}^1_U\left(\left(1\otimes \mathrm{Ad}(g^{-1})\right)(\omega\otimes f)\right) = \\ \big(1\otimes \mathrm{Ad}(g^{-1})\big)\big(\boldsymbol{d}^1_U(\omega\otimes f)\big) - \partial_U(g)\bigwedge\nolimits_U\big(1\otimes \mathrm{Ad}(g^{-1})\big)(\omega\otimes f),\end{gathered} \tag{8.1.33}$$

for every $(g,\omega\otimes f) \in C^\infty(U,G)\times\big(\Omega^1(U)\otimes_{C^\infty(U,\mathbb{R})} C^\infty(U,\mathbb{G})\big)$ and every open $U\subseteq X$.

Indeed, taking into account (3.3.15) and (3.3.16), as well as equality (△) on p. 108, namely

$$\underline{\mu}^1_U \circ \delta'_U = \delta_U \circ (1\times \underline{\mu}^1_U), \tag{8.1.34}$$

we have that

$$\begin{aligned}\boldsymbol{d}^1_U\left((1\otimes \mathrm{Ad}(g^{-1}))(\omega\otimes f)\right) &= (\underline{\lambda}^2_U\circ d^1\circ \underline{\mu}^1_U)\left(\delta'_U(g^{-1},\omega\otimes f)\right)\\ &= (\underline{\lambda}^2_U\circ d^1\circ\delta_U)\big(g^{-1},\underline{\mu}^1_U(\omega\otimes f)\big)\\ &= (\underline{\lambda}^2_U\circ d^1\circ\delta_U)(g^{-1},\omega f)\\ &= (\underline{\lambda}^2_U\circ d^1)\big(\mathrm{Ad}(g^{-1}).(\omega f)\big),\end{aligned} \tag{8.1.35}$$

where the meaning of ωf is explained in the equality following (3.3.13b).

However, by standard (though tedious) calculations on ordinary forms,

$$d^1\big(\mathrm{Ad}(g^{-1}).(\omega f)\big) = \mathrm{Ad}(g^{-1}).d^1(\omega f) - [g^{-1}.dg,\, \mathrm{Ad}(g^{-1}).(\omega f)]. \tag{8.1.36}$$

(A very detailed proof of this can be found, e.g., in Kreĭn-Yatskin [51, Chap. 3, Proposition 1.2], under an appropriate change of notations.) Therefore, applying (8.1.36), (3.3.17), the analog of (3.3.15) for 2-forms, (8.1.34) and its analog for 2-forms, as well as (3.3.13b), (3.3.16′) and (8.1.15), we transform (8.1.35) as follows:

$$\begin{aligned}&\boldsymbol{d}^1_U\big((1\otimes \mathrm{Ad}(g^{-1}))(\omega\otimes f)\big)\\ &\quad= \underline{\lambda}^2_U\big(\mathrm{Ad}(g^{-1}).d^1(\omega f)\big) - \underline{\lambda}^2_U\big([g^{-1}.dg,\, \mathrm{Ad}(g^{-1})(\omega f)]\big)\\ &\quad= \underline{\lambda}^2_U\big(\delta_U(g^{-1},d^1(\omega f))\big) - \underline{\lambda}^2_U\big([\underline{\mu}^1_U(\partial(g)),\delta_U(g^{-1},\omega f)]\big)\\ &\quad= \delta'_U\big(g^{-1},\,\underline{\lambda}^2_U(d^1(\omega f))\big) - \underline{\lambda}^2_U\big([\underline{\mu}^1_U(\partial(g)),\,\underline{\mu}^1_U(\delta'_U(g^{-1},1\otimes\omega f))]\big)\\ &\quad= \big(1\otimes \mathrm{Ad}(g^{-1})\big)\big(\boldsymbol{d}^1_U(\omega\otimes f)\big) - \partial(g)\bigwedge\nolimits_U\big(1\otimes \mathrm{Ad}(g^{-1})\big)(\omega\otimes f),\end{aligned}$$

which is precisely the desired equality (8.1.33).

Having defined $\boldsymbol{d}^1$, we obtain the corresponding operator $\mathcal{D}$ and the curvature datum $(\mathcal{C}^\infty_X, \mathcal{D})$. In summary:

> The Lie sheaf of groups $\mathcal{C}^\infty_X(G)$, obtained from a Lie group G (along with a smooth manifold X), is provided with a curvature datum derived from the usual differentials and the bracket (exterior product) of ordinary $\mathbb{G}$-valued forms. The curvature datum is essentially obtained from the precurvature datum $(\mathcal{C}^\infty_X, d, \Omega^1, d^1, \Omega^2)$, where $(\mathcal{C}^\infty_X, d, \Omega^1)$ is the differential triad of X (see Example 2.1.4(a)), Ω^2 the sheaf of germs of ($\mathbb{R}$-valued) differential 2-forms on X, and d^1 the sheafification of the ordinary differential of 1st order.

Note. Let us now explain the terminology applied to (8.1.29). For this purpose we observe that the operator $\mathcal{D}$ of the present example can also be defined by the sheafification of the local morphisms

$$\mathcal{D}_U : \Lambda^1(U,\mathbb{R}) \otimes_{C^\infty(U,\mathbb{R})} C^\infty(U,\mathbb{G}) \longrightarrow \Lambda^2(U,\mathbb{R}) \otimes_{C^\infty(U,\mathbb{R})} C^\infty(U,\mathbb{G})$$

given by

$$\mathcal{D}_U(\omega) := \boldsymbol{d}^1_U \omega + \frac{1}{2}\,\omega \bigwedge\nolimits_U \omega.$$

Therefore, working as in the verification of (8.1.25) by $\boldsymbol{d}^1$, we have that

$$\begin{aligned}
\mathcal{D}_U(\partial_U(g)) &= \boldsymbol{d}^1_U(\partial_U(g)) + \frac{1}{2}\,\partial_U(g) \bigwedge\nolimits_U \partial_U(g) \\
&= (\underline{\lambda}^2_U \circ d^1 \circ \underline{\mu}^1_U)(\underline{\lambda}^1_U(g^{-1}.dg)) \\
&\qquad + \frac{1}{2}\,\underline{\lambda}^1_U(g^{-1}.dg) \bigwedge\nolimits_U \underline{\lambda}^1_U(g^{-1}.dg) \\
&= \underline{\lambda}^2_U\big(d^1(g^{-1}.dg) + \frac{1}{2}\,[g^{-1}.dg,\, g^{-1}.dg]\big) \\
&= \underline{\lambda}^2_U\Big(g^*\big(d^1\alpha + \frac{1}{2}\,[\alpha,\alpha]\big)\Big) = 0.
\end{aligned}$$

This shows that $\mathcal{D} \circ \partial$ is ultimately related with the usual Maurer-Cartan equation of a Lie group and (as expected) verifies (8.1.29).

(b) The curvature datum of $\mathcal{GL}(n,\mathcal{A})$

We think of $\mathcal{GL}(n,\mathcal{A})$ as the Lie sheaf of groups described in Example 3.3.6(b).

In Example 8.1.2(b) we defined the identifications

(8.1.37) $\lambda^p : \mathcal{M}_n(\Omega^p) \xrightarrow{\simeq} \Omega^p \otimes_{\mathcal{A}} \mathcal{M}_n(\mathcal{A}) =: \Omega^p(\mathcal{M}_n(\mathcal{A}));\qquad (p = 1, 2),$

whose inverses are denoted by μ^p. They are generated by the corresponding local isomorphisms

$$\lambda^p_U : M_n(\Omega^p(U)) \to \Omega^p(U) \otimes_{\mathcal{A}(U)} M_n(\mathcal{A}(U))$$

and $\mu^p_U = (\lambda^p_U)^{-1}$, given respectively by

(8.1.37a) $$\lambda^p_U\big((\theta_{ij})\big) := \sum_{i,j=1}^{n} \theta_{ij} \otimes E^U_{ij},$$

(8.1.37b) $$\mu^p_U\big(\theta \otimes (a_{ij})\big) := (\theta \cdot a_{ij}) = (a_{ij} \cdot \theta).$$

We have already defined the matrix differential $d^0 : \mathcal{M}_n(\mathcal{A}) \to \mathcal{M}_n(\Omega)$ (see (3.1.11)) generated by the morphisms

(8.1.38) $$d^0_U : M_n(\mathcal{A}(U)) \longrightarrow M_n(\Omega^1(U)) : (a_{ij}) \mapsto (d^0 a_{ij}),$$

$(U \in \mathfrak{T}_X)$ where d^0 in the target is the differential of the given differential triad.

Hence, *if there exists* a precurvature datum $(\mathcal{A}, d = d^0, \Omega, d^1, \Omega^2)$, we can also define the differential (same symbol as before) $d^1 : \mathcal{M}_n(\Omega^1) \to \mathcal{M}_n(\Omega^2)$, generated, similarly, by the local morphisms

(8.1.39) $$d^1_U : M_n(\Omega^1(U)) \longrightarrow M_n(\Omega^2(U)) : (\omega_{ij}) \mapsto (d^1 \omega_{ij}).$$

The last matrix differentials extend to the morphisms

$$\boldsymbol{d}^1_U := \lambda^2_U \circ d^1_U \circ \mu^1_U : \Omega^1(U) \otimes_{\mathcal{A}(U)} M_n(\mathcal{A}(U)) \longrightarrow \Omega^2(U) \otimes_{\mathcal{A}(U)} M_n(\mathcal{A}(U)),$$

for all $U \in \mathfrak{T}_X$, which in turn generate the differential

$$\boldsymbol{d}^1 : \Omega^1(\mathcal{M}_n(\mathcal{A})) \longrightarrow \Omega^2(\mathcal{M}_n(\mathcal{A})).$$

To prove that the latter verifies (8.1.25), it suffices to work locally. Thus, for any open $U \subseteq X$ and $g \in \mathrm{Gl}(n, \mathcal{A}(U)) \cong \mathcal{GL}(n, \mathcal{A})(U)$, the above definition of $\boldsymbol{d}^1_U$, the local analog of (3.2.17), and the definition of $\widetilde{\partial}$ (see (3.2.9), (3.2.10)), yield

(8.1.40) $$(\boldsymbol{d}^1_U \circ \partial_U)(g) = (\lambda^2_U \circ d^1_U \circ \mu^1_U)(\lambda^1_U(\widetilde{\partial}(g))) = (\lambda^2_U \circ d^1_U)(g^{-1} \cdot d^0 g),$$

where $d^0 = d$ is now the matrix extension of the differential of the given differential triad (see also (3.1.10) and (3.1.11)).

Applying now (8.1.39), (2.5.8) and (2.5.9), we find that

$$d^1_U(g^{-1} \cdot d^0 g) = d^0(g^{-1}) \wedge_U d^0 g.$$

But an elementary calculation shows that

$$d^0(g \cdot g^{-1}) = 0 \quad \Longrightarrow \quad d^0(g^{-1}) = -g^{-1} \cdot (d^0 g) \cdot g^{-1};$$

hence,

$$\begin{aligned} d^1_U(g^{-1} \cdot d^0 g) &= -\big(g^{-1} \cdot (d^0 g) \cdot g^{-1}\big) \wedge_U d^0 g = \\ &= -(g^{-1} \cdot d^0 g) \wedge_U (g^{-1} \cdot d^0 g). \end{aligned}$$

As a result, using the last equality, along with (8.1.22′), we transform (8.1.40) into

$$\begin{aligned} (\boldsymbol{d}^1_U \circ \partial_U)(g) &= -\lambda^2_U\big((g^{-1} \cdot d^0 g) \wedge_U (g^{-1} \cdot d^0 g)\big) \\ &= -\lambda^2_U\big(\widetilde{\partial}_U(g) \wedge_U \partial_U(g)\big) \\ &= -\lambda^2_U\big(\mu^1_U(\partial_U(g)) \wedge_U \mu^1_U(\partial_U(g))\big) \\ &= -\frac{1}{2}\,(\lambda^2_U \circ \mu^2_U)\big(\partial_U(g) \bigwedge_U \partial_U(g)\big) \\ &= -\frac{1}{2}\,\partial_U(g) \bigwedge_U \partial_U(g), \end{aligned}$$

from which we get (8.1.25).

We shall now prove the analog of (8.1.26′). First observe that, for every $g \in \mathrm{Gl}(n, \mathcal{A}(U))$ and $\omega \in M_n(\Omega^1(U))$, a typical application of (2.5.8) as before, together with the exterior product of matrices (8.1.19), implies that

$$\begin{aligned} &d^1\left(\mathrm{Ad}_U(g^{-1})(\omega)\right) = d^1(g^{-1} \cdot \omega \cdot g) = \\ &\big(d^0(g^{-1}) \wedge_U \omega\big) \cdot g + g^{-1} \cdot (d^1\omega) \cdot g - g^{-1} \cdot (\omega \wedge_U d^0 g). \end{aligned} \tag{8.1.41}$$

The first summand of (8.1.41) is transformed into

$$\begin{aligned} \big(d^0(g^{-1}) \wedge_U \omega\big) \cdot g &= -\big((g^{-1} \cdot (d^0 g) \cdot g^{-1}) \wedge_U \omega\big) \cdot g \\ &= -\big((\widetilde{\partial}_U(g) \cdot g^{-1}) \wedge_U \omega\big) \cdot g \\ &= -\widetilde{\partial}_U(g) \wedge_U (g^{-1} \cdot \omega \cdot g) \\ &= -\widetilde{\partial}_U(g) \wedge_U \mathrm{Ad}_U(g^{-1})(\omega). \end{aligned} \tag{8.1.42}$$

Similarly, for the third summand we have that

$$(8.1.43)\qquad \begin{aligned} g^{-1}\cdot(\omega\wedge_U d^0 g) &= g^{-1}\cdot\big((\omega\cdot g)\wedge_U (g^{-1}\cdot d^0 g)\big)\\ &= (g^{-1}\cdot\omega\cdot g)\wedge_U \widetilde{\partial}_U(g)\\ &= \mathrm{Ad}_U(g^{-1})(\omega)\wedge_U \widetilde{\partial}_U(g). \end{aligned}$$

Consequently, substituting (8.1.42) and (8.1.43) in (8.1.41), we obtain

$$(8.1.44)\qquad d^1\big(\mathrm{Ad}_U(g^{-1})(\omega)\big) = \\ \mathrm{Ad}_U(g^{-1})(d^1\omega) - \widetilde{\partial}_U(g)\wedge_U \mathrm{Ad}_U(g^{-1})(\omega) - \mathrm{Ad}_U(g^{-1})(\omega)\wedge_U \widetilde{\partial}_U(g).$$

As earlier, let us notice that the last two terms cannot unified since the exterior product is the particular product of matrices (8.1.19).

Now, for the proof of (8.1.26′), we work locally by taking into account the action (3.2.13′). This means it suffices to show equality

$$(8.1.45)\qquad \boldsymbol{d}^1_U\,\big((1\otimes \mathrm{Ad}_U(g^{-1}))(\theta)\big) = \\ \big(1\otimes \mathrm{Ad}_U(g^{-1})\big)\big(\boldsymbol{d}^1_U\theta\big) - \partial_U(g){\textstyle\bigwedge}_U\,\big(1\otimes \mathrm{Ad}_U(g^{-1})\big)(\theta),$$

for every $g\in \mathrm{GL}(n,\mathcal{A}(U))$, $\theta\in\Omega^1(U)\otimes_{\mathcal{A}(U)} M_n(\mathcal{A}(U))$, and every open $U\subseteq X$. We notice that the action (3.2.13′) is generated by the local actions (3.2.14′), the latter being given (for arbitrary tensors) by

$$\delta'_{n,U}(g,\theta) = (1\otimes \mathrm{Ad}_U(g^{-1}))(\theta).$$

Indeed, for g and θ as before, the first member of (8.1.45) is transformed as follows:

$$\begin{aligned} \boldsymbol{d}^1_U\big((1\otimes \mathrm{Ad}_U(g^{-1}))(\theta)\big) &= (\lambda^2_U\circ d^1_U\circ\mu^1_U)\big((1\otimes \mathrm{Ad}_U(g^{-1}))(\theta)\big)\\ &= (\lambda^2_U\circ d^1_U\circ\mu^1_U)(\delta'_{n,U}(g^{-1},\theta))\\ \text{(see Diagram 3.2)}\quad &= (\lambda^2_U\circ d^1_U)\big(\delta_{n,U}(g^{-1},\mu^1_U(\theta))\big)\\ \text{(see (3.2.14))}\quad &= (\lambda^2_U\circ d^1_U)\big(\mathrm{Ad}_U(g^{-1})(\mu^1_U(\theta))\big). \end{aligned}$$

Thus, applying (8.1.44) for $\omega=\mu^1_U(\theta)$, we find that

$$\begin{aligned} \boldsymbol{d}^1_U\,\big((1\otimes \mathrm{Ad}_U(g^{-1}))(\theta)\big) &= \lambda^2_U\big(\mathrm{Ad}_U(g^{-1})(d^1_U(\mu^1_U(\theta)))\big)\\ &\quad - \lambda^2_U\big(\widetilde{\partial}_U(g){\textstyle\bigwedge}_U \mathrm{Ad}_U(g^{-1})(\mu^1_U(\theta))\big)\\ &\quad - \lambda^2_U\big(\mathrm{Ad}_U(g^{-1})(\mu^1_U(\theta)){\textstyle\bigwedge}_U\widetilde{\partial}_U(g)\big)\\ (\triangleright)\quad &:= \lambda^2_U(A) - \lambda^2_U(B) - \lambda^2_U(C), \end{aligned}$$

where the terms represented by A, B, C are clear.

But the analog of Diagram 3.2 for 2-forms implies that

$$\begin{aligned}\lambda_U^2(A) &= \lambda_U^2\big(\mathrm{Ad}_U(g^{-1})(d_U^1(\mu_U^1(\theta))\big)\\ &= (\lambda_U^2 \circ \delta_{n,U})\big(g^{-1},\, d_U^1(\mu_U^1(\theta))\big)\\ &= \big(\delta'_{n,U} \circ (1 \times \lambda_U^2)\big)\big(g^{-1},\, d_U^1(\mu_U^1(\theta))\big)\\ &= \delta'_{n,U}\big(g^{-1},\, (\lambda_U^2 \circ d_U^1 \circ \mu_U^1)(\theta)\big)\\ &= \big(1 \otimes \mathrm{Ad}_U(g^{-1}\big)(\boldsymbol{d}_U^1\theta)\big).\end{aligned}$$

On the other hand, using (3.2.17) and Diagram 3.2,

$$\begin{aligned}\widetilde{\partial}_U(g){\textstyle\bigwedge}_U \mathrm{Ad}_U(g^{-1})(\mu_U^1(\theta)) &= \mu_U^1(\partial_U(\theta)){\textstyle\bigwedge}_U \delta_{n,U}(g^{-1},\, \mu_U^1(\theta))\\ &= \mu_U^1(\partial_U(\theta)){\textstyle\bigwedge}_U (\mu_U^1 \circ \delta'_{n,U})(g^{-1}, \theta)\\ &= \mu_U^1(\partial_U(\theta)){\textstyle\bigwedge}_U \mu_U^1\big((1 \otimes \mathrm{Ad}_U(g^{-1}))(\theta)\big).\end{aligned}$$

Similarly,

$$\mathrm{Ad}_U(g^{-1})(\mu_U^1(\theta)){\textstyle\bigwedge}_U \widetilde{\partial}_U(g) = \mu_U^1\big((1 \otimes \mathrm{Ad}_U(g^{-1}))(\theta)\big){\textstyle\bigwedge}_U \mu_U^1(\partial_U(g)).$$

Therefore, applying (8.1.21), we see that

$$-\lambda_U^2(B) - \lambda_U^2(C) = -\partial_U(g){\textstyle\bigwedge}_U \big(1 \otimes \mathrm{Ad}_U(g^{-1})\big)(\theta).$$

As a result, if we substitute the expressions of $\lambda_U^2(A)$ and $-\lambda_U^2(B) - \lambda_U^2(C)$ in ($\triangleright$), we obtain (8.1.45), as required.

Taking into account (3.2.15′), the sheafification of (8.1.45) leads to

$$\boldsymbol{d}^1\big(\mathcal{A}d(g^{-1}).w\big) = \mathcal{A}d(g^{-1}).\boldsymbol{d}^1 w - \partial(g){\textstyle\bigwedge}\mathcal{A}d(g^{-1}).w,$$

for every $(g, w) \in \mathcal{GL}(n, \mathcal{A}) \times_X \Omega^1(\mathcal{M}_n(\mathcal{A}))$, which is formula (8.1.26′) in the context of the present example.

The previous $\boldsymbol{d}^1$ determines an operator $\mathcal{D}$ and the corresponding curvature datum $(\mathcal{GL}(n, \mathcal{A}), \mathcal{D})$. In conclusion:

> The Lie sheaf of groups $\mathcal{GL}(n, \mathcal{A})$ has a natural curvature datum provided that the original differential triad $(\mathcal{A}, d, \Omega)$ extends to a pre-curvature datum $(\mathcal{A}, d, \Omega^1, d^1, \Omega^2)$.

Note. As a complement to the comments following Definition 8.1.2, let us remark that the conclusions of both of the previous examples show that the curvature data of them have been naturally constructed from an appropriate precurvature datum $(\mathcal{A}, d, \Omega^1, d^1, \Omega^2)$.

Thus, although the curvature of a connection on an arbitrary $\mathcal{G}$-principal sheaf is generally determined by a curvature datum (see the next section), in the particular case of connections on the $\mathcal{C}^\infty_X(G)$-principal sheaf of germs of sections of a smooth principal bundle (see Example 4.1.9(a)), and that of connections on a $\mathcal{GL}(n, \mathcal{A})$-principal sheaf, the curvature is ultimately constructed from the aforementioned precurvature datum.

For the above reason, and *only within the context of the aforementioned particular cases*, $(\mathcal{A}, d, \Omega^1, d^1, \Omega^2)$ may legitimately be called a curvature datum, as in Mallios [62, Vol. II, p. 188].

8.2. The curvature in general

Let $\mathcal{P} \equiv (\mathcal{P}, \mathcal{G}, X, \pi)$ be a given principal sheaf admitting connections. We also assume the existence of a curvature datum $(\mathcal{G}, \mathcal{D})$. With this assumption in mind, we come to the following fundamental notion.

8.2.1 Definition. The ***curvature*** of a connection D on $\mathcal{P}$ is the morphism (of sheaves of sets) $R \equiv R^D : \mathcal{P} \longrightarrow \Omega^2(\mathcal{L})$ defined by $R := \mathcal{D} \circ D$, as pictured in the next diagram.

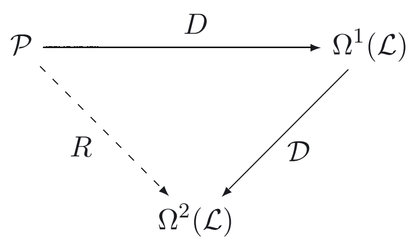

Diagram 8.2

8.2.2 Proposition. *The curvature R of a connection D on the principal sheaf $(\mathcal{P}, \mathcal{G}, X, \pi)$ is $\mathcal{G}$-equivariant with respect to the actions of $\mathcal{G}$ on $\mathcal{P}$ and $\Omega^2(\mathcal{L})$; that is,*

$$R(p.g) = \rho(g^{-1}).R(p),$$

for every $(p, g) \in \mathcal{P} \times_X \mathcal{G}$.

Proof. In virtue of (6.1.1) and (8.1.31), we have

$$\begin{aligned} R(p\cdot g) &= \mathcal{D}\left(\rho(g^{-1}).D(p) + \partial(g)\right) \\ &= \rho(g^{-1}).\mathcal{D}(D(p)) = \rho(g^{-1}).R(p). \end{aligned}$$ □

In the terminology of Definition 5.3.8, the previous result shows that R is a *tensorial* morphism. Hence, if we denote by

$$\mathcal{H}om_{\mathcal{G}}(\mathcal{P}, \Omega^2(\mathcal{L})) \tag{8.2.1}$$

the sheaf of germs of tensorial morphisms of $\mathcal{P}$ into $\Omega^2(\mathcal{L})$ (see (5.3.19)), Proposition 8.2.2 implies the following:

8.2.3 Corollary. *Up to an isomorphism, R can be considered as a global section of the sheaf of germs of tensorial morphisms (8.2.1); that is,*

$$R \in \mathrm{Hom}_{\mathcal{G}}(\mathcal{P}, \Omega^2(\mathcal{L})) \cong \mathcal{H}om_{\mathcal{G}}(\mathcal{P}, \Omega^2(\mathcal{L}))(X).$$

Proof. Recall that (8.2.1) is generated by the complete presheaf of tensorial morphisms

$$U \longmapsto \mathrm{Hom}_{\mathcal{G}|_U}(\mathcal{P}|_U, \Omega^2(\mathcal{L})|_U).$$ □

As usual, we denote the morphism of sections induced by R by the same symbol. Thus, evaluating R at the natural sections (s_α) of $\mathcal{P}$ (with respect to a fixed local frame $\mathcal{U}$ of $\mathcal{P}$), we obtain the local sections

$$\Omega_\alpha := R^D(s_\alpha) \equiv R(s_\alpha) \in \Omega^2(\mathcal{L})(U_\alpha), \qquad \alpha \in I. \tag{8.2.2}$$

Inspired by the classical case of connections on principal bundles, we give the following definition.

8.2.4 Definition. The local sections (Ω_α) are called the ***local curvature forms*** of the curvature $R \equiv R^D$, with respect to a local frame $\mathcal{U}$ of $\mathcal{P}$.

8.2.5 Proposition. *Let R be the curvature of a connection D with local connection forms (ω_α). Then the local curvature forms satisfy:*

i) The ***local Cartan (second) structure equations***

$$\Omega_\alpha = \boldsymbol{d}^1\omega_\alpha + \frac{1}{2}\,\omega_\alpha \wedge \omega_\alpha \equiv \boldsymbol{d}^1\omega_\alpha + \frac{1}{2}\,[\omega_\alpha, \omega_\alpha]; \qquad \alpha \in I. \tag{8.2.3}$$

ii) The compatibility condition

$$\Omega_\beta = \rho(g_{\alpha\beta}^{-1}).\Omega_\alpha, \tag{8.2.4}$$

on every $U_{\alpha\beta} \neq \emptyset$.

Clearly, $\boldsymbol{d}^1$ and $\bigwedge$ in (8.2.3) are now the induced operators on the corresponding local sections.

Proof. For any $x \in U_\alpha$, we check that

$$\begin{aligned}\Omega_\alpha(x) &= R(s_\alpha)(x) = R(s_\alpha(x)) = \mathcal{D}\big(D(s_\alpha(x))\big)\\ &= \mathcal{D}\big(D(s_\alpha)(x)\big) = \mathcal{D}(\omega_\alpha(x)) = \mathcal{D}(\omega_\alpha)(x)\\ &= (\boldsymbol{d}^1\omega_\alpha + \frac{1}{2}\,\omega_\alpha{\bigwedge}\omega_\alpha)(x),\end{aligned}$$

from which (8.2.3) follows.

Finally, equality (4.3.3) and Proposition 8.2.2 imply that

$$\Omega_\beta = R(s_\beta) = R(s_\alpha \cdot g_{\alpha\beta}) = \rho(g_{\alpha\beta}^{-1}).R(s_\alpha) = \rho(g_{\alpha\beta}^{-1}).\Omega_\alpha,$$

which is the desired compatibility condition. □

8.2.6 Theorem. *The curvature R of a connection $D \equiv (\omega_\alpha)$ is completely determined by the 0-cochain of its local curvature forms $(\Omega_\alpha) \in C^0(\mathcal{U}, \Omega^2(\mathcal{L}))$.*

Proof. For an arbitrary open $U \subseteq X$, we define the mapping

$$R'_U : \mathcal{P}(U) \longrightarrow \Omega^2(\mathcal{L})(U)$$

as follows: for any section $s \in \mathcal{P}(U)$, we set

$$R'_U(s)|_{U\cap U_\alpha} := \rho(g_\alpha^{-1}).\Omega_\alpha|_{U\cap U_\alpha}, \tag{8.2.5}$$

where $g_\alpha \in \mathcal{G}(U \cap U_\alpha)$ is uniquely determined by the equality $s|_{U\cap U_\alpha} = s_\alpha|_{U\cap U_\alpha} \cdot g_\alpha$. Since $g_\alpha = g_{\alpha\beta} \cdot g_\beta$ over $U \cap U_{\alpha\beta}$, (8.2.4) implies that

$$\rho(g_\beta^{-1}).\Omega_\beta = \rho(g_\alpha^{-1} \cdot g_{\alpha\beta}).\big(\rho(g_{\alpha\beta}^{-1}).\Omega_\alpha\big) = \rho(g_\alpha^{-1}).\Omega_\alpha,$$

where, for simplicity, we have omitted the notation of restrictions. This proves that R'_U is well defined, taking, of course, into account that

$$U = \bigcup_{\alpha\in I}(U \cap U_\alpha).$$

Therefore, the family (R'_U), with U running in the topology of X, is a presheaf morphism generating a morphism of sheaves $R' : \mathcal{P} \to \Omega^2(\mathcal{L})$.

By the definition of (Ω_α) and R', we have that $R(s_\alpha) = \Omega_\alpha = R'(s_\alpha)$. Hence, for any $s \in \mathcal{P}(U)$ and $\alpha \in I$, Proposition 8.2.2 and equality (8.2.5) imply that

$$\begin{aligned} R(s)|_{U\cap U_\alpha} &= R(s|_{U\cap U_\alpha}) = R(s_\alpha|_{U\cap U_\alpha} \cdot g_\alpha) \\ &= \rho(g_\alpha^{-1}).R(s_\alpha)|_{U\cap U_\alpha} = R'(s)|_{U\cap U_\alpha}, \end{aligned}$$

from which we conclude that $R(s) = R'(s)$, for every $s \in \mathcal{P}(U)$ and every open $U \subseteq X$. Therefore, $R = R'$ and the statement is proved. □

We shall show that R induces another global object, related with the adjoint sheaf $\rho(\mathcal{P})$. Here we identify $\rho(\mathcal{P})$ with $\mathcal{P} \times_X^{\mathcal{G}} \mathcal{L}$, whose coordinates are $\widetilde{\Phi}_\alpha : \rho(\mathcal{P})|_{U_\alpha} \to \mathcal{L}|_{U_\alpha}$. The change of coordinates, computed on local sections, gives

$$(\widetilde{\Phi}_\alpha \circ \widetilde{\Phi}_\beta^{-1})(\ell) = \rho(g_{\alpha\beta}^{-1})(\ell), \qquad \ell \in \mathcal{L}(U_{\alpha\beta})$$

(see Subsection 5.4(d) and its concluding comments).

We consider the sheaf (actually $\mathcal{A}$-module) $\Omega^2(\rho(\mathcal{P})) = \Omega^2 \otimes_{\mathcal{A}} \rho(\mathcal{P})$. Working as in the discussion preceding the proof of Corollary 6.3.4, we see that the previous sheaf is locally of type $\Omega^2(\mathcal{L})$ by means of the local coordinates

$$1\otimes\widetilde{\Phi}_\alpha : \Omega^2(\rho(\mathcal{P}))|_{U_\alpha} = \Omega^2|_{U_\alpha}\otimes_{\mathcal{A}|_{U_\alpha}}\rho(\mathcal{P})|_{U_\alpha} \to \Omega^2|_{U_\alpha}\otimes_{\mathcal{A}|_{U_\alpha}}\mathcal{L}|_{U_\alpha} = \Omega^2(\mathcal{L})|_{U_\alpha},$$

$(\alpha \in I)$, where 1 is the identity of Ω^2 restricted to U_α. Their change

$$(1 \otimes \widetilde{\Phi}_\alpha) \circ (1 \otimes \widetilde{\Phi}_\beta^{-1}) = 1 \otimes (\widetilde{\Phi}_\alpha \circ \widetilde{\Phi}_\beta^{-1}),$$

over $U_{\alpha\beta}$, is given (section-wise) by the following analog of (6.3.11)

$$\big(1 \otimes (\widetilde{\Phi}_\alpha \circ \widetilde{\Phi}_\beta^{-1})\big)(\theta) = \rho(g_{\alpha\beta}).\theta, \qquad \theta \in \Omega^2(\mathcal{L})(U). \tag{8.2.6}$$

8.2.7 Proposition. *The curvature R determines a global section of the $\mathcal{A}$-module $\Omega^2(\rho(\mathcal{P}))$.*

Proof. Setting $\Theta_\alpha := (1 \otimes \widetilde{\Phi}_\alpha^{-1})(\Omega_\alpha)$, (8.2.4) and (8.2.6) imply

$$\begin{gathered} \Theta_\beta = (1 \otimes \widetilde{\Phi}_\beta^{-1})(\Omega_\beta) = (1 \otimes \widetilde{\Phi}_\beta^{-1})\big(\rho(g_{\alpha\beta}^{-1}).\Omega_\alpha\big) = \\ \Big((1 \otimes \tilde{\Phi}_\beta^{-1}) \circ \big(1 \otimes (\tilde{\Phi}_\beta \circ \tilde{\Phi}_\alpha^{-1})\big)\Big)(\Omega_\alpha) = (1 \otimes \tilde{\Phi}_\alpha^{-1})(\Omega_\alpha) = \Theta_\alpha, \end{gathered}$$

over $U_{\alpha\beta} \neq \emptyset$. Therefore, gluing the Θ_α's we get a global section (in classical terms: a global 2-form) $\Theta \in \Omega^2(\mathcal{P})(X)$, as stated. □

8.3. The Bianchi identity

In order to derive the abstract analog of the classical Bianchi identity, we need to assume that the curvature datum $(\mathcal{G}, \mathcal{D})$ can be extended to a triplet, called henceforth ***Bianchi datum***,

$$(\mathcal{G}, \mathcal{D}, \boldsymbol{d}^2), \tag{8.3.1}$$

where $\boldsymbol{d}^2 : \Omega^2(\mathcal{L}) \to \Omega^3(\mathcal{L})$ is a $\mathbb{K}$-linear morphism satisfying the following conditions:

$$\boldsymbol{d}^2 \circ \boldsymbol{d}^1 = 0, \tag{8.3.2}$$

$$\boldsymbol{d}^2(a \bigwedge b) = (\boldsymbol{d}^1 a) \bigwedge b - a \bigwedge \boldsymbol{d}^1 b, \tag{8.3.3}$$

for every $(a, b) \in \Omega^1(\mathcal{L}) \times_X \Omega^1(\mathcal{L})$.

Note. Working as in Example 8.1.6(a), we easily see that the 2nd order exterior differential of ordinary (Lie algebra) $\mathbb{G}$-valued forms determines an operator $\boldsymbol{d}^2 : \Omega^2(\mathcal{C}_X^\infty(\mathbb{G})) \to \Omega^3(\mathcal{C}_X^\infty(\mathbb{G}))$ satisfying the above properties.

Similarly, an operator $\boldsymbol{d}^2 : \Omega^2(\mathcal{M}_n(\mathcal{A})) \to \Omega^2(\mathcal{M}_n(\mathcal{A}))$ is constructed within the context of Example 8.1.6(b), if we are given a $\mathbb{K}$-linear morphism $d^2 : \Omega^2 \longrightarrow \Omega^3$ satisfying the conditions

$$d^2 \circ d^1 = 0,$$
$$d^2(a \wedge b) = (d^1 a) \wedge b - a \wedge d^1 b,$$

for every $(a, b) \in \Omega^1 \times_X \Omega^1$.

Thus, in both cases, the Bianchi datum is essentially determined by the collection $(\mathcal{A}, d, \Omega, d^1, \Omega^2, d^2, \Omega^3)$ extending the (pre)curvature datum $(\mathcal{A}, d, \Omega, d^1, \Omega^2)$. For this reason, in these two cases, one may call the former collection itself a Bianchi datum (see also the final note of Section 8.1 and Mallios [62, Vol. II, p. 220]).

The 2nd exterior differential $\boldsymbol{d}^2$ induces the $\mathbb{K}$-linear morphism

$$\boldsymbol{d}_H^2 : \mathrm{Hom}(\mathcal{P}, \Omega^2(\mathcal{L})) \longrightarrow \mathrm{Hom}(\mathcal{P}, \Omega^3(\mathcal{L})), \tag{8.3.4}$$

defined by

$$\boldsymbol{d}_H^2(f) := \boldsymbol{d}^2 \circ f, \tag{8.3.5}$$

for every $f \in \mathrm{Hom}(\mathcal{P}, \Omega^2(\mathcal{L}))$. For convenience, we also set

$$\boldsymbol{d}_H^2 \equiv \boldsymbol{d}^2. \tag{8.3.6}$$

The distinction between $\boldsymbol{d}^2 \equiv \boldsymbol{d}_H^2$ and $\boldsymbol{d}^2 : \Omega^1(\mathcal{L}) \to \Omega^2(\mathcal{L})$ is understood either by the context or by mentioning their domains.

In the same way, the exterior product $\bigwedge$ (see (8.1.3)) induces the $\mathbb{K}$-linear morphism

$$\bigwedge_H : \mathrm{Hom}(\mathcal{P}, \Omega^p(\mathcal{L})) \times \mathrm{Hom}(\mathcal{P}, \Omega^q(\mathcal{L})) \longrightarrow \mathrm{Hom}(\mathcal{P}, \Omega^{p+q}(\mathcal{L})), \tag{8.3.7}$$

determined by

$$(f \bigwedge_H g)(p) := f(p) \bigwedge g(p), \tag{8.3.8}$$

for every (f, g) in the domain of (8.3.7) and $p \in \mathcal{P}$. Again, for the sake of simplicity, we set

$$\bigwedge_H \equiv \bigwedge. \tag{8.3.9}$$

Since, by Corollary 8.2.3, $R \in \mathrm{Hom}_{\mathcal{G}}(\mathcal{P}, \Omega^2(\mathcal{L})) \subseteq \mathrm{Hom}(\mathcal{P}, \Omega^2(\mathcal{L}))$, we prove the analog of the classical Bianchi identity.

8.3.1 Theorem. *Let D be a connection on $\mathcal{P}$ with curvature R. If there exists a Bianchi datum, then the* ***Bianchi (second) identity***

$$\boldsymbol{d}^2 R = R \bigwedge D \equiv [R, D] \tag{8.3.10}$$

holds true.

Clearly, the bracket appearing in (8.3.10) extends to $\mathrm{Hom}(\mathcal{P}, \Omega^2(\mathcal{L}))$ the bracket of (8.1.3′).

Proof. For every $p \in \mathcal{P}$, (8.3.6) and (8.3.9) imply that

$$\begin{aligned}
(\boldsymbol{d}^2 R)(p) &\equiv (\boldsymbol{d}_H^2 R)(p) := \boldsymbol{d}^2(R(p)) \\
&= \boldsymbol{d}^2\Big(\boldsymbol{d}^1(D(p)) + \frac{1}{2}\, D(p) \bigwedge D(p)\Big) \\
\text{(see (8.3.2))} \quad &= \frac{1}{2}\, \boldsymbol{d}^2\big(D(p) \bigwedge D(p)\big) \\
\text{(see (8.3.3))} \quad &= \boldsymbol{d}^1\,(D(p)) \bigwedge D(p) \\
\text{(see (8.1.28))} \quad &= \mathcal{D}(D(p)) \bigwedge D(p) - \frac{1}{2}\,(D(p) \bigwedge D(p)) \bigwedge D(p) \\
\text{(see (8.1.5b))} \quad &= R(p) \bigwedge D(p) := (R \bigwedge_H D)(p) \\
&\equiv (R \bigwedge D)(p),
\end{aligned}$$

from which the Bianchi identity follows. □

8.3.2 Remark. Following the custom of the classical literature, we may omit the order of the differential and write identity (8.3.10) as

$$\boldsymbol{d}R = [R, D]. \tag{8.3.10$'$}$$

We shall give a variant of the Bianchi identity, which reminds us of another familiar version of its classical counterpart. To this end, given a fixed connection D, we introduce the ***covariant exterior differential***

$$\boldsymbol{D} \equiv \boldsymbol{D}^D : \mathrm{Hom}(\mathcal{P}, \Omega^2(\mathcal{L})) \longrightarrow \mathrm{Hom}(\mathcal{P}, \Omega^3(\mathcal{L})), \tag{8.3.11}$$

defined by

$$\boldsymbol{D}(f) = \boldsymbol{d}^2_H(f) + D \bigwedge_H f; \qquad f \in \mathrm{Hom}(\mathcal{P}, \Omega^2(\mathcal{L})),$$

or, in virtue of (8.3.5) and (8.3.8),

$$(\boldsymbol{D}(f))\,(p) = \boldsymbol{d}^2(f(p)) + D(p) \bigwedge f(p), \qquad p \in \mathcal{P}. \tag{8.3.12}$$

Therefore, the preceding definition and the properties of $\bigwedge$ (see the beginning of Section 8.1) prove at once:

8.3.3 Corollary. *The Bianchi (second) identity takes the form* $\boldsymbol{D}(R) = 0$.

Furthermore, from Theorem 8.3.1 we obtain:

8.3.4 Corollary. *The* ***local Bianchi (second) identities***

$$\boldsymbol{d}^2\Omega_\alpha = \Omega_\alpha \bigwedge \omega_\alpha = (\boldsymbol{d}^1\omega_\alpha) \bigwedge \omega_\alpha; \qquad \alpha \in I, \tag{8.3.13}$$

are valid. Equivalently, by (8.1.3′) and (8.3.10′),

$$\boldsymbol{d}^2\Omega_\alpha = [\Omega_\alpha, \omega_\alpha] = [\boldsymbol{d}^1\omega_\alpha, \omega_\alpha], \qquad \alpha \in I. \tag{8.3.13$'$}$$

In the previous statement, all the operators are now the ones induced on the corresponding modules of sections over U_α.

Proof. Based on (8.2.2), (8.3.10), and the interplay between sheaf morphisms and the induced morphisms of sections, we have:

$$\begin{aligned}
\boldsymbol{d}^2(\Omega_\alpha)(x) &= \boldsymbol{d}^2(\Omega_\alpha(x)) = \boldsymbol{d}^2(R(s_\alpha(x))) \\
&= R(s_\alpha(x)) \bigwedge D(s_\alpha(x)) \\
&= \Omega_\alpha(x) \bigwedge \omega_\alpha(x) \\
&= (\Omega_\alpha \bigwedge \omega_\alpha)(x),
\end{aligned}$$

for every $x \in U_\alpha$. This proves the first equality of (8.3.13).

For the second, we differentiate the local structure equation (8.2.3):

$$\begin{aligned} \boldsymbol{d}^2(\Omega_\alpha) &= \boldsymbol{d}^2(\boldsymbol{d}^1\omega_\alpha + \frac{1}{2}\,\omega_\alpha \bigwedge \omega_\alpha) \\ \text{(see (8.3.2))} \qquad &= \frac{1}{2}\,\boldsymbol{d}^2(\omega_\alpha \bigwedge \omega_\alpha) \\ \text{(see (8.3.3))} \qquad &= (\boldsymbol{d}^1\omega_\alpha)\bigwedge \omega_\alpha, \end{aligned}$$

which completes the proof. □

Conversely, we shall prove:

8.3.5 Theorem. *The local Bianchi identities (8.3.13) imply the Bianchi identity (8.3.10).*

For the proof we need the following auxiliary result.

8.3.6 Lemma. *If $D \equiv (\omega_\alpha)$ is a connection with curvature $R \equiv (\Omega_\alpha)$, then*

$$\text{(8.3.14)} \qquad \boldsymbol{d}^2\big(\rho(g_\alpha^{-1}).\Omega_\alpha\big) = \rho(g_\alpha^{-1}).\boldsymbol{d}^2\Omega_\alpha - \partial(g_\alpha)\bigwedge \rho(g_\alpha^{-1}).\Omega_\alpha,$$

for every $g_\alpha \in \mathcal{G}(U_\alpha)$.

Proof. By the structure equation (8.2.3) and Proposition 8.1.1,

$$\text{(8.3.15)} \qquad \begin{aligned} \boldsymbol{d}^2\big(\rho(g_\alpha^{-1}).\Omega_\alpha\big) &= \boldsymbol{d}^2\big(\rho(g_\alpha^{-1}).(\boldsymbol{d}^1\omega_\alpha + \frac{1}{2}\,\omega_\alpha \bigwedge \omega_\alpha)\big) \\ &= \boldsymbol{d}^2\big(\rho(g_\alpha^{-1}).\boldsymbol{d}^1\omega_\alpha\big) \\ &\quad + \frac{1}{2}\,\boldsymbol{d}^2\big(\rho(g_\alpha^{-1}).\omega_\alpha \bigwedge \rho(g_\alpha^{-1}).\omega_\alpha\big). \end{aligned}$$

Also, by (8.1.26′),

$$\rho(g_\alpha^{-1}).\boldsymbol{d}^1\omega_\alpha = \boldsymbol{d}^1\big(\rho(g_\alpha^{-1}).\omega_\alpha\big) + \partial(g_\alpha)\bigwedge \rho(g_\alpha^{-1}).\omega_\alpha.$$

Thus, by the preceding equality and (8.3.2), (8.3.3), we successively transform the first summand in the last term of (8.3.15) to

$$\begin{aligned} \boldsymbol{d}^2\big(\rho(g_\alpha^{-1}).\boldsymbol{d}^1\omega_\alpha\big) &= \boldsymbol{d}^2\big(\partial(g_\alpha)\bigwedge \rho(g_\alpha^{-1}).\omega_\alpha\big) \\ &= \boldsymbol{d}^1(\partial(g_\alpha))\bigwedge \rho(g_\alpha^{-1}).\omega_\alpha - \partial(g_\alpha)\bigwedge \boldsymbol{d}^1\big(\rho(g_\alpha^{-1}).\omega_\alpha\big), \end{aligned}$$

or, by applying (8.1.25),

$$\begin{aligned}\boldsymbol{d}^2\big(\rho(g_\alpha^{-1}).\boldsymbol{d}^1\omega_\alpha\big) = &-\frac{1}{2}\big(\partial(g_\alpha)\wedge\partial(g_\alpha)\big)\wedge\rho(g_\alpha^{-1}).\omega_\alpha \\ &- \partial(g_\alpha)\wedge\boldsymbol{d}^1\big(\rho(g_\alpha^{-1}).\omega_\alpha\big).\end{aligned} \tag{8.3.16}$$

Since, in virtue of (8.1.26′), the second summand on the right-hand side of (8.3.16) becomes

$$\partial(g_\alpha)\wedge\boldsymbol{d}^1\big(\rho(g_\alpha^{-1}).\omega_\alpha\big) = \partial(g_\alpha)\wedge\rho(g_\alpha^{-1}).\boldsymbol{d}^1\omega_\alpha - \partial(g_\alpha)\wedge\big(\partial(g_\alpha)\wedge\rho(g_\alpha^{-1}).\omega_\alpha\big),$$

equality (8.3.16) turns into

$$\begin{aligned}\boldsymbol{d}^2\big(\rho(g_\alpha^{-1}).\boldsymbol{d}^1\omega_\alpha\big) = &-\frac{1}{2}\big(\partial(g_\alpha)\wedge\partial(g_\alpha)\big)\wedge\rho(g_\alpha^{-1}).\omega_\alpha \\ &- \partial(g_\alpha)\wedge\rho(g_\alpha^{-1}).\boldsymbol{d}^1\omega_\alpha \\ &+ \partial(g_\alpha)\wedge\big(\partial(g_\alpha)\wedge\rho(g_\alpha^{-1}).\omega_\alpha\big).\end{aligned} \tag{8.3.17}$$

On the other hand, applying (8.3.3) and (8.1.26′) to the second summand of the last term of (8.3.15), we get

$$\begin{aligned}\frac{1}{2}\boldsymbol{d}^2\big(\rho(g_\alpha^{-1}).\omega_\alpha\wedge\rho(g_\alpha^{-1}).\omega_\alpha\big) &= \boldsymbol{d}^1\big(\rho(g_\alpha^{-1}).\omega_\alpha\big)\wedge\rho(g_\alpha^{-1}).\omega_\alpha \\ &= \rho(g_\alpha^{-1}).(\boldsymbol{d}^1\omega_\alpha)\wedge\rho(g_\alpha^{-1}).\omega_\alpha \\ &\quad - \big(\partial(g_\alpha)\wedge\rho(g_\alpha^{-1}).\omega_\alpha\big)\wedge\rho(g_\alpha^{-1}).\omega_\alpha.\end{aligned} \tag{8.3.18}$$

Therefore, by (8.3.17) and (8.3.18), (8.3.15) takes the form

$$\begin{aligned}\boldsymbol{d}^2\big(\rho(g_\alpha^{-1}).\Omega_\alpha\big) = &-\frac{1}{2}\big(\partial(g_\alpha)\wedge\partial(g_\alpha)\big)\wedge\rho(g_\alpha^{-1}).\omega_\alpha \\ &- \partial(g_\alpha)\wedge\rho(g_\alpha^{-1}).\boldsymbol{d}^1\omega_\alpha \\ &+ \partial(g_\alpha)\wedge\big(\partial(g_\alpha)\wedge\rho(g_\alpha^{-1}).\omega_\alpha\big) \\ &+ \rho(g_\alpha^{-1}).(\boldsymbol{d}^1\omega_\alpha)\wedge\rho(g_\alpha^{-1}).\omega_\alpha \\ &- \big(\partial(g_\alpha)\wedge\rho(g_\alpha^{-1}).\omega_\alpha\big)\wedge\rho(g_\alpha^{-1}).\omega_\alpha.\end{aligned} \tag{8.3.19}$$

Now we work out the right-hand side of (8.3.14). First, using similar arguments, together with (8.1.9), we check that

$$\begin{aligned}\rho(g_\alpha^{-1}).\boldsymbol{d}^2\Omega_\alpha &= \rho(g_\alpha^{-1}).\boldsymbol{d}^2\big(\boldsymbol{d}^1\omega_\alpha + \frac{1}{2}\,\omega_\alpha\bigwedge\omega_\alpha\big)\\ &= \rho(g_\alpha^{-1}).\big((\boldsymbol{d}^1\omega_\alpha)\bigwedge\omega_\alpha\big)\\ &= \rho(g_\alpha^{-1}).(\boldsymbol{d}^1\omega_\alpha)\bigwedge\rho(g_\alpha^{-1}).\omega_\alpha.\end{aligned}\tag{8.3.20}$$

Analogously,

$$\begin{aligned}\partial(g_\alpha)\bigwedge\rho(g_\alpha^{-1}).\Omega_\alpha = \partial(g_\alpha)\bigwedge\rho(g_\alpha^{-1}).(\boldsymbol{d}^1\omega_\alpha)\\ +\frac{1}{2}\,\partial(g_\alpha)\bigwedge\big(\rho(g_\alpha^{-1}).\omega_\alpha\bigwedge\rho(g_\alpha^{-1}).\omega_\alpha\big).\end{aligned}\tag{8.3.21}$$

Combining (8.3.20) and (8.3.21), we see that the right-hand side of (8.3.14) turns into

$$\begin{aligned}\rho(g_\alpha^{-1}).\boldsymbol{d}^2\Omega_\alpha - \partial(g_\alpha)\bigwedge\rho(g_\alpha^{-1}).\Omega_\alpha = \rho(g_\alpha^{-1}).(\boldsymbol{d}^1\omega_\alpha)\bigwedge\rho(g_\alpha^{-1}).\omega_\alpha\\ - \partial(g_\alpha)\bigwedge\rho(g_\alpha^{-1}).(\boldsymbol{d}^1\omega_\alpha)\\ -\frac{1}{2}\,\partial(g_\alpha)\bigwedge\big(\rho(g_\alpha^{-1}).\omega_\alpha\bigwedge\rho(g_\alpha^{-1}).\omega_\alpha\big).\end{aligned}\tag{8.3.22}$$

If we compare (8.3.19) with (8.3.22), we conclude that (8.3.14) is satisfied if and only if

$$\begin{aligned}-\frac{1}{2}\,\big(\partial(g_\alpha)\bigwedge\partial(g_\alpha)\big)\bigwedge\rho(g_\alpha^{-1}).\omega_\alpha + \partial(g_\alpha)\bigwedge\big(\partial(g_\alpha)\bigwedge\rho(g_\alpha^{-1}).\omega_\alpha\big)\\ -\big(\partial(g_\alpha)\bigwedge\rho(g_\alpha^{-1}).\omega_\alpha\big)\bigwedge\rho(g_\alpha^{-1}).\omega_\alpha = -\frac{1}{2}\,\partial(g_\alpha)\bigwedge\big(\rho(g_\alpha^{-1}).\omega_\alpha\bigwedge\rho(g_\alpha^{-1}).\omega_\alpha\big).\end{aligned}$$

Setting $a = \partial(g_\alpha)$ and $b = \rho(g_\alpha^{-1}).\omega_\alpha$, after a few elementary calculations we see that the last equality amounts to

$$(a\bigwedge a)\bigwedge b + (b\bigwedge b)\bigwedge a = -2(a\bigwedge b)\bigwedge a - 2(a\bigwedge b)\bigwedge b.$$

But this is always true, as it follows at once by applying the section analog of the identity ($*$) of the exterior product $\bigwedge$, firstly to the triplet (a, a, b) and secondly to (b, b, a). □

Proof of Theorem 8.3.5. Let $p \in \mathcal{P}$ be an arbitrary element and assume that $\pi(p) = x \in U_\alpha$. Then there is a (unique) $g_\alpha(x) \in \mathcal{G}_x$ such that $p = s_\alpha(x) \cdot g_\alpha(x)$, thus we have the equalities:

$$R(p) = R(s_\alpha(x) \cdot g_\alpha(x)) = \rho(g_\alpha(x)^{-1}).\Omega_\alpha(x),$$

$$D(p) = D(s_\alpha(x) \cdot g_\alpha(x)) = \rho(g_\alpha(x)^{-1}).\omega_\alpha + \partial(g_\alpha(x)).$$

Applying (8.3.14) and the local Bianchi identities, evaluated at x, we find that

$$\begin{aligned}(\boldsymbol{d}^2 R)(p) &= \boldsymbol{d}^2(R(p)) = \boldsymbol{d}^2(\rho(g_\alpha(x)^{-1}).\Omega_\alpha(x))\\ &= \rho(g_\alpha(x)^{-1}).\boldsymbol{d}^2(\Omega_\alpha(x)) - \partial(g_\alpha(x))\bigwedge\rho(g_\alpha(x)^{-1}).\Omega_\alpha(x)\\ &= \rho(g_\alpha(x)^{-1}).(\Omega_\alpha(x)\bigwedge\omega_\alpha(x)) - \partial(g_\alpha(x))\bigwedge\rho(g_\alpha(x)^{-1}).\Omega_\alpha(x)\\ &= \rho(g_\alpha(x)^{-1}).\Omega_\alpha(x)\bigwedge\rho(g_\alpha(x)^{-1}).\omega_\alpha(x)\\ &\qquad + \rho(g_\alpha(x)^{-1}).\Omega_\alpha(x)\bigwedge\partial(g_\alpha(x))\\ &= \rho(g_\alpha(x)^{-1}).\Omega_\alpha(x)\bigwedge\big(\rho(g_\alpha(x)^{-1}).\omega_\alpha + \partial(g_\alpha(x))\big)\\ &= R(p)\bigwedge D(p) = (R\bigwedge D)(p),\end{aligned}$$

which proves the assertion of the theorem. □

The results of this section, combined together, also lead to:

8.3.7 Corollary. *The Bianchi (second) identity (8.3.10) (or its variant in Corollary 8.3.3) is equivalent to the local Bianchi identities (8.3.13).*

8.4. The sheaf of curvatures

For the sake of completeness we shall describe the curvature as a global section of the sheaf of curvatures, in analogy to the sheaf of connections $\mathcal{C}(\mathcal{P})$, whose global sections correspond to the connections of $\mathcal{P}$ (see Theorem 6.2.4).

Let $\mathcal{P}$ be a principal sheaf with local frame $\mathcal{U}$ and natural sections (s_α). If $D \equiv (\omega_\alpha)$ is a connection on $\mathcal{P}$, then the corresponding section $S \in \mathcal{C}(\mathcal{P})(X)$ satisfies equality

$$S(x) = [(s_\alpha(x), D(s_\alpha(x)))] = [(s_\alpha(x), \omega_\alpha(x))]; \qquad x \in U_\alpha, \tag{8.4.1}$$

as a result of (6.6.14) discussed in the proof of Lemma 6.6.4 (see also the second part of the proof of Theorem 5.3.9 ("ontoness")). By the same token, we have that the definition of the equivalence relation involved yields

$$S(x) = [(s_\alpha(x), \omega_\alpha(x))] = [(s_\beta(x), \omega_\beta(x))], \qquad x \in U_{\alpha\beta}.$$

8.4.1 Definition. The ***sheaf of curvatures*** of a principal sheaf $\mathcal{P}$ is the sheaf

$$\mathcal{R}(\mathcal{P}) := \mathcal{P} \times_X^{\mathcal{G}} \Omega^2(\mathcal{L})$$

defined by the following equivalence relation:

$$(p, w) \sim (q, w') \Longleftrightarrow \exists| \; g \in \mathcal{G} : (q, w') = (p, w).g := (p.g, \rho(g^{-1}).w).$$

The previous terminology is justified by the following direct consequence of Theorem 5.3.9:

8.4.2 Proposition. *Global sections T of $\mathcal{R}(\mathcal{P})$ correspond bijectively to morphisms $R \in Hom_{\mathcal{G}}(\mathcal{P}, \Omega^2(\mathcal{L}))$, so that*

$$T(x) = [(s_\alpha(x), R(s_\alpha(x)))], \qquad x \in U_\alpha. \tag{8.4.2}$$

In particular, if R is the curvature of a connection and T the corresponding section of $\mathcal{R}(\mathcal{P})$, then

$$T(x) = [(s_\alpha(x), \Omega_\alpha(x))], \qquad x \in U_\alpha. \tag{8.4.2'}$$

To connect the sections S and T (corresponding, respectively, to a connection and its curvature), we define the morphism $\boldsymbol{\mathcal{D}} : \mathcal{C}(\mathcal{P}) \to \mathcal{R}(\mathcal{P})$ by

$$\boldsymbol{\mathcal{D}}([(p, w)]) := [(p, \mathcal{D}(w))]; \qquad [(p, w)] \in \mathcal{C}(\mathcal{P}), \tag{8.4.3}$$

where $\mathcal{D}$ is the Cartan structure operator (8.1.27).

8.4.3 Theorem. *Let D be a connection on $\mathcal{P}$, corresponding to the global section $S \in \mathcal{C}(\mathcal{P})(X)$ of the sheaf of connections. Then the global section $T \in \mathcal{R}(\mathcal{P})(X)$, corresponding to R^D, satisfies equality*

$$T = \boldsymbol{\mathcal{D}}(S), \tag{8.4.4}$$

if $\boldsymbol{\mathcal{D}} : \mathcal{C}(\mathcal{P})(X) \to \mathcal{R}(\mathcal{P})(X)$ is the induced morphism of global sections.

Conversely, a section $T \in \mathcal{R}(\mathcal{P})(X)$ satisfying (8.4.4) determines (in a unique way) the curvature R^D of the connection D corresponding to S.

Proof. Let T be the section determined by R^D. Then (8.4.1) – (8.4.3) imply that

$$\begin{aligned} T(x) &= \big[\big(s_\alpha(x), R^D(x)\big)\big] = \big[\big(s_\alpha(x), (\mathcal{D} \circ D)(s_\alpha(x))\big)\big] \\ &= \boldsymbol{\mathcal{D}}\big([(s_\alpha(x), D(s_\alpha(x)))]\big) = \boldsymbol{\mathcal{D}}(S(x)) = \boldsymbol{\mathcal{D}}(S)(x), \end{aligned}$$

for every $x \in U_\alpha$. Hence, $T|_{U_\alpha} = \boldsymbol{\mathcal{D}}(S)|_{U_\alpha}$, for every $U_\alpha \in \mathcal{U}$, and the first part of the statement is proved.

For the converse part, let us denote by R' the morphism corresponding to a section T. Then, in virtue of our assumptions, we obtain

$$\big[\big(s_\alpha(x), R'(s_\alpha(x))\big)\big] = T(x) = \boldsymbol{\mathcal{D}}(S(x)) = \\ \boldsymbol{\mathcal{D}}\big([(s_\alpha(x), D(s_\alpha(x))]\big) = \big[\big(s_\alpha(x), R^D(s_\alpha(x))\big)\big];$$

that is, $R'(s_\alpha) = R^D(s_\alpha)$, for all the natural sections (s_α) of $\mathcal{P}$. Since R' and R^D are both tensorial, arguing as in the last part of the proof of Theorem 8.2.6, we conclude that $R' = R^D$. □

8.5. The curvature of various connections

After some typical examples, we describe the curvature of related and associated connections, the curvature of the pull-back connection, as well as the curvature of $\mathcal{A}$-connections on vector sheaves.

8.5.1. Some typical examples

(a) Thinking of the Maurer-Cartan differential ∂ as a connection (see Example 6.1.2(a)), the Maurer-Cartan equation (8.1.29) implies that

$$R^\partial = \mathcal{D} \circ \partial = 0.$$

(b) Similarly, the curvature of each canonical local connection D_α (see Example 6.1.2(b)) is given by

$$R^{D_\alpha} = \mathcal{D} \circ D_\alpha = \mathcal{D} \circ \partial \circ \phi_\alpha = 0.$$

The previous connections are also typical examples of *flat connections*, treated in detail in Section 8.6.

(c) Let (P, G, X, π_P) be a principal bundle and $(\mathcal{P}, \mathcal{G}, X, \pi)$ the sheaf of germs of smooth sections of P. As we have already seen in Example 6.2(a), a connection on P, say ω, amounting to the family of its local connection forms $\omega_\alpha \in \Lambda^1(U_\alpha, \mathbb{G})$, $\alpha \in I$, corresponds bijectively to a connection D on $\mathcal{P}$ with local connection forms (viz. sections) $\underline{\omega}_\alpha \in \Omega(\mathcal{C}^\infty_X(\mathbb{G}))(U_\alpha)$. The curvature of the principal bundle connection ω is fully determined by the local curvature forms

$$\Omega_\alpha = d^1\omega_\alpha + \frac{1}{2}[\omega_\alpha, \omega_\alpha] \in \Lambda^2(U_\alpha, \mathbb{G}); \qquad \alpha \in I,$$

satisfying the compatibility condition $\Omega_\beta = \mathrm{Ad}(g_{\alpha\beta}^{-1}).\Omega_\alpha$ on $U_{\alpha\beta}$.

To find the curvature R^D we start with the local connection forms $(\underline{\omega}_\alpha)$ and then we define the curvature forms $(\underline{\Omega}_\alpha)$ by the structure equations (8.2.3), i.e.,

$$\underline{\Omega}_\alpha = \boldsymbol{d}^1 \underline{\omega}_\alpha + \frac{1}{2}\, \underline{\omega}_\alpha \bigwedge \underline{\omega}_\alpha,$$

where $\bigwedge$ is the exterior product (8.1.16) of Example 8.1.2(a) and $\boldsymbol{d}^1$ the differential defined in Example 8.1.6(a). Therefore, by Theorem 8.2.6, we determine the curvature R^D.

Another way to obtain R^D is the following: By the 2-form analogs of (3.3.13) and (6.2.2), we have the isomorphisms

$$\begin{aligned}\Lambda^2(U_\alpha, \mathbb{G}) \xrightarrow{\underline{\lambda}^2_{U_\alpha}} &\ \Lambda^2(U_\alpha, \mathbb{R}) \otimes_{C^\infty(U_\alpha,\mathbb{R})} C^\infty(U_\alpha, \mathbb{G}) \\ &\cong \Omega^2(U_\alpha) \otimes_{\mathcal{A}(U_\alpha)} \mathcal{L}(U_\alpha) \\ &\cong (\Omega^2 \otimes_{\mathcal{A}} \mathcal{L})(U_\alpha),\end{aligned}$$

where $\mathcal{L} = \mathcal{C}^\infty_X(\mathbb{G})$ and $\mathcal{A} = \mathcal{C}^\infty_X((R)$. We set $\underline{\Theta}_\alpha := \underline{\lambda}^2_{U_\alpha}(\Omega_\alpha)$. Then, applying (8.1.15) and the definition of $\boldsymbol{d}^1$, we check that

$$\begin{aligned}\underline{\Theta}_\alpha := \underline{\lambda}^2_{U_\alpha}(\Omega_\alpha) &= \underline{\lambda}^2_{U_\alpha}\big(d^1\omega_\alpha + \frac{1}{2}\,[\omega_\alpha, \omega_\alpha]\big) \\ &= (\underline{\lambda}^2_{U_\alpha} \circ d^1 \circ \underline{\mu}^1_{U_\alpha})\big(\underline{\lambda}^1_{U_\alpha}(\omega_\alpha)\big) + \frac{1}{2}\, \underline{\lambda}^1_{U_\alpha}(\omega_\alpha) \bigwedge_{U_\alpha} \underline{\lambda}^1_{U_\alpha}(\omega_\alpha) \\ &= \boldsymbol{d}^1_{U_\alpha} \underline{\omega}_\alpha + \frac{1}{2}\, \underline{\omega}_\alpha \bigwedge_{U_\alpha} \underline{\omega}_\alpha.\end{aligned}$$

Therefore, by the above identification and (1.2.17), the form $\Omega'_\alpha = \widetilde{\underline{\Theta}_\alpha} \in \Omega^2(\mathcal{L})(U_\alpha)$ satisfies

$$\begin{aligned}\Omega'_\alpha = \widetilde{\underline{\Theta}_\alpha} &= \big(\boldsymbol{d}^1_{U_\alpha} \underline{\omega}_\alpha + \frac{1}{2}\, \underline{\omega}_\alpha \bigwedge_{U_\alpha} \underline{\omega}_\alpha\big)^{\sim} \\ &\equiv \boldsymbol{d}^1 \underline{\omega}_\alpha + \frac{1}{2}\, \underline{\omega}_\alpha \bigwedge \underline{\omega}_\alpha \\ &= \underline{\Omega}_\alpha;\end{aligned}$$

that is, we obtain the 2-forms $(\underline{\Omega}_\alpha)$ which determine R^D as in the previous approach.

For the sake of completeness, let us notice that the compatibility condition $\Omega_\beta = \mathrm{Ad}(g_{\alpha\beta}^{-1}).\Omega_\alpha$ implies the analogous condition $\underline{\Omega}_\beta = \mathcal{A}d(\widetilde{g}_{\alpha\beta}^{-1}).\underline{\Omega}_\alpha$. This is easily proved by working as in the proof of (6.2.1′).

8.5.2. The curvature of related connections

Following the notations and terminology of Definition 6.4.1, we consider two $(f, \phi, \overline{\phi}, id_X)$-*related* connections D and D' on the principal sheaves $(\mathcal{P}, \mathcal{G}, X, \pi)$ and $(\mathcal{P}', \mathcal{G}', X, \pi')$, respectively.

We assume that $\Omega^1(\mathcal{L})$ and $\Omega^1(\mathcal{L}')$ are endowed with 1st order differentials $\boldsymbol{d}^1$ (same symbol for both) such that

$$(1_{\Omega^2} \otimes \overline{\phi}) \circ \boldsymbol{d}^1 = \boldsymbol{d}^1 \circ (1_{\Omega^1} \otimes \overline{\phi}). \tag{8.5.1}$$

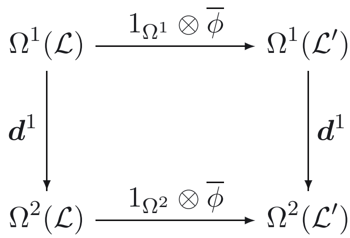

Diagram 8.3

Then the curvature data $(\mathcal{G}, \mathcal{D})$ and $(\mathcal{G}', \mathcal{D}')$ determine the curvatures $R \equiv R^D$ and $R' \equiv R^{D'}$. We claim that

$$R' \circ f = (1_{\Omega^2} \otimes \overline{\phi}) \circ R. \tag{8.5.2}$$

The proof is a consequence of the commutativity of the sub-diagrams of the next diagram illustrating the present situation.

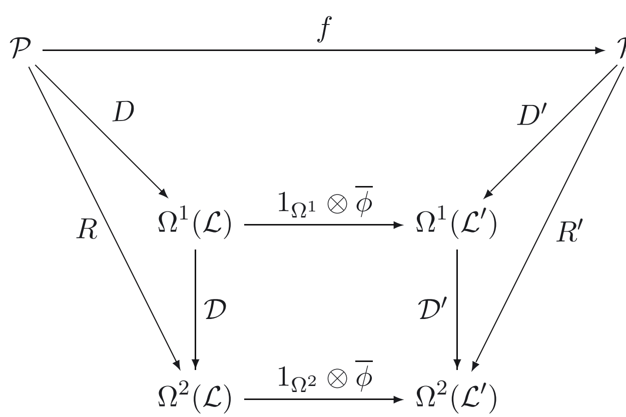

Diagram 8.4

Indeed, the upper (trapezoid) sub-diagram is commutative by (6.4.1), while the left and right triangles are commutative by the definition of the curvature.

It remains to show the commutativity of the lower (square) sub-diagram. In accordance with (8.5.1), it is sufficient to prove the general equality

$$(1_{\Omega^1} \otimes \overline{\phi})(a) \bigwedge (1_{\Omega^1} \otimes \overline{\phi})(b) = (1_{\Omega^1} \otimes \overline{\phi})(a \bigwedge b), \tag{8.5.3}$$

for every $(a, b) \in \Omega^1(\mathcal{L}) \times_X \Omega^1(\mathcal{L})$. We still use the same symbol for the exterior products defined with respect to $\mathcal{L}$ and $\mathcal{L}'$.

For the proof of (8.5.3) we work locally, by taking decomposable tensors. More precisely, for any open $U \subseteq X$, we consider the local exterior products $\bigwedge_U$ (generating $\bigwedge$), given by (8.1.4). Then, for any

$$\omega \otimes u,\ \theta \otimes v \in \Omega^1(U) \otimes_{\mathcal{A}(U)} \mathcal{L}(U),$$

we check that

$$(1_{\Omega^1} \otimes \overline{\phi})(\omega \otimes u) \bigwedge_U (1_{\Omega^1} \otimes \overline{\phi})(\theta \otimes v) = (\omega \otimes \overline{\phi}(u)) \bigwedge_U (\theta \otimes \overline{\phi}(v)) =$$
$$(\omega \wedge \theta) \otimes [\,\overline{\phi}(u), \overline{\phi}(v)] = (\omega \wedge \theta) \otimes \overline{\phi}([u, v]) =$$
$$(1_{\Omega^2} \otimes \overline{\phi})\big((\omega \wedge \theta) \otimes [u, v]\big) = (1_{\Omega^2} \otimes \overline{\phi})\big((\omega \otimes u) \bigwedge_U (\theta \otimes v)\big).$$

Hence, by linear extension, we obtain (8.5.3) and, consequently, (8.5.2).

For the sake of completeness let us find the relationship between the local curvature forms (Ω_α) and (Ω'_α) of R and R', respectively, defined over local frames with a common open covering $\mathcal{U} = (U_\alpha)$ of X. We denote by (s_α) and (s'_α) the natural sections of $\mathcal{P}$ and $\mathcal{P}'$.

For every s_α, we have that

$$(R' \circ f)(s_\alpha) = R'(f(s_\alpha)) = R'(s'_\alpha \cdot h_\alpha) =$$
$$\rho'(h_\alpha^{-1}).R'(s'_\alpha) = \rho'(h_\alpha^{-1}).\Omega'_\alpha,$$

as well as

$$\big((1_{\Omega^2} \otimes \overline{\phi}) \circ R\big)\,(s_\alpha) = (1_{\Omega^2} \otimes \overline{\phi})(\Omega_\alpha),$$

where $h_\alpha \in \mathcal{G}'(U_\alpha)$ with $f(s_\alpha) = s'_\alpha \cdot h_\alpha$ (see also Theorem 4.4.1). Thus (8.5.2) implies

$$\rho'(h_\alpha^{-1}).\Omega'_\alpha = (1_{\Omega^2} \otimes \overline{\phi})(\Omega_\alpha), \qquad \alpha \in I. \tag{8.5.4}$$

Another way to obtain the previous equality is to use the local structure equation (8.2.3) and equality (6.4.3) of Theorem 6.4.2. In fact, together with (8.5.1), (8.5.3), the equivariance of $\bigwedge$ (see (8.1.9)), conditions (8.1.25), (8.1.26′), and the structure equation (8.1.31), we see that

$$\begin{aligned}
(1_{\Omega^2} \otimes \overline{\phi})(\Omega_\alpha) &= (1_{\Omega^2} \otimes \overline{\phi})(\boldsymbol{d}^1\omega_\alpha + \frac{1}{2}\,\omega_\alpha{\textstyle\bigwedge}\omega_\alpha) \\
&= \boldsymbol{d}^1\left((1_{\Omega^1} \otimes \overline{\phi})(\omega_\alpha)\right) + \frac{1}{2}\,(1_{\Omega^1} \otimes \overline{\phi})(\omega_\alpha){\textstyle\bigwedge}(1_{\Omega^1} \otimes \overline{\phi})(\omega_\alpha) \\
&= \boldsymbol{d}^1\left(\rho'(h_\alpha^{-1}).\omega'_\alpha + \partial'(h_\alpha)\right) \\
&\quad + \frac{1}{2}\left(\rho'(h_\alpha^{-1}).\omega'_\alpha + \partial'(h_\alpha)\right) {\textstyle\bigwedge} \left(\rho'(h_\alpha^{-1}).\omega'_\alpha + \partial'(h_\alpha)\right) \\
&= \rho'(h_\alpha^{-1}).\boldsymbol{d}^1\omega'_\alpha + \boldsymbol{d}^1(\partial'(h_\alpha)) \\
&\quad + \frac{1}{2}\,\rho'(h_\alpha^{-1}).(\omega'_\alpha{\textstyle\bigwedge}\omega'_\alpha) + \frac{1}{2}\,\partial'(h_\alpha){\textstyle\bigwedge}\partial'(h_\alpha) \\
&= \rho'(h_\alpha^{-1}).\left(\boldsymbol{d}^1\omega'_\alpha + \frac{1}{2}\,\omega'_\alpha{\textstyle\bigwedge}\omega'_\alpha\right) \\
&= \rho'(h_\alpha^{-1}).\Omega'_\alpha.
\end{aligned}$$

We can also check that the conditions (8.5.2) and (8.5.4) are *equivalent.* We have already seen that (8.5.2) implies (8.5.4). The converse is obtained by applying the 2-form analog of Lemma 6.4.3: For any $p \in \mathcal{P}$ with $\pi(p) = x \in U_\alpha$, there is a unique $g_\alpha \in \mathcal{G}_x$ with $p = s_\alpha(x) \cdot g_\alpha$; hence,

$$f(p) = f(s_\alpha(x) \cdot g_\alpha) = f(s_\alpha(x)) \cdot \phi(g_\alpha) = s'_\alpha(x) \cdot h_\alpha(x) \cdot \phi(g_\alpha),$$

from which it follows that

$$\begin{aligned}
(R' \circ f)(p) &= R'\left(s'_\alpha(x) \cdot h_\alpha(x) \cdot \phi(g_\alpha)\right) \\
&= \rho'(\phi(g_\alpha)^{-1}).\left(\rho'(h_\alpha^{-1}(x)).\Omega'_\alpha(x)\right) \\
\text{(see (8.5.4))} \qquad &= \rho'(\phi(g_\alpha)^{-1}).\left((1_{\Omega^2} \otimes \overline{\phi})(\Omega_\alpha(x))\right) \\
&= (1_{\Omega^2} \otimes \overline{\phi})\left(\rho(g_\alpha^{-1}).\Omega_\alpha(x)\right) \\
&= (1_{\Omega^2} \otimes \overline{\phi})(R(p)) \\
&= \left((1_{\Omega^2} \otimes \overline{\phi}) \circ R\right)(p),
\end{aligned}$$

thus proving the assertion.

In particular, if we consider two $\mathcal{G}$-principal sheaves and a $\mathcal{G}$-isomorphism $f \equiv (f, id_{\mathcal{G}}, id_{\mathcal{L}}, id_X)$ between them, equalities (8.5.2) and (8.5.4) reduce to

$$R' \circ f = R, \tag{8.5.2$'$}$$

$$\rho(h_\alpha^{-1}).\Omega'_\alpha = \Omega_\alpha, \tag{8.5.4$'$}$$

respectively, for every $\alpha \in I$.

8.5.3. The curvature of associated connections

We apply the results of the preceding subsection to the associated connections discussed in Section 7.3.

(a) First we work out the case of the connection $D_{\phi(\mathcal{P})}$ on $\phi(\mathcal{P})$, induced by a morphism of Lie sheaves of groups $(\phi, \overline{\phi})$. We assume that the principal sheaf $\mathcal{P}$ is endowed with a connection $D \equiv D_{\mathcal{P}}$ with curvature $R \equiv R^D$. If (8.5.1) is satisfied, then, in virtue of Proposition 7.3.2 and equality (8.5.2), we obtain

$$R^{D_{\phi(\mathcal{P})}} \circ \varepsilon = (1_{\Omega^2} \otimes \overline{\phi}) \circ R. \tag{8.5.5}$$

On the other hand, if (for simplicity) we denote by (Ω'_α) the local curvature forms of $R^{D_{\phi(\mathcal{P})}}$, equality (8.5.4) reduces to

$$\Omega'_\alpha = (1_{\Omega^2} \otimes \overline{\phi})(\Omega_\alpha); \qquad \alpha \in I, \tag{8.5.6}$$

since (5.4.6) implies that $h_\alpha = \mathbf{1}_{\mathcal{H}}|_{U_\alpha}$. Note that, as in the general case of the preceding subsection, (8.5.6) is also derived from (7.3.2) and (8.2.3).

(b) The second case refers to a representation $(\varphi, \overline{\varphi})$ of the form (7.3.3). Such a representation induces a vector sheaf $\mathcal{E}_\varphi$ associated with a given $\mathcal{G}$-principal sheaf $\mathcal{P}$, a morphism

$$(F_{\mathcal{P}}, \varphi, \overline{\varphi}, id_X) : (\mathcal{P}, \mathcal{G}, X, \pi) \longrightarrow (\mathcal{P}(\mathcal{E}_\varphi), \mathcal{GL}(n, \mathcal{A}), X, \pi'),$$

and a connection $D_{\mathcal{P}(\mathcal{E}_\varphi)}$ which is $(F_{\mathcal{P}}, \varphi, \overline{\varphi}, id_X)$-related with a given connection $D \equiv D_{\mathcal{P}}$ on $\mathcal{P}$. Then (8.5.1) and Proposition 7.3.3 yield

$$R^{D_{\mathcal{P}(\mathcal{E}_\varphi)}} \circ F_{\mathcal{P}} = (1_{\Omega^2} \otimes \overline{\varphi}) \circ R. \tag{8.5.7}$$

The respective formula for the local curvature forms is given again by (8.5.6), as a result of (5.5.9). It is also obtained from (7.3.5) and (8.2.3).

(c) The final case is concerned with the connections D_φ and $D_{\mathcal{P}(\mathcal{E}_\varphi)}$ of Proposition 7.3.4. Then

$$R^{D_\varphi} = R^{D_{\mathcal{P}(\mathcal{E}_\varphi)}} \circ \theta.$$

Moreover, if $R^{D_\varphi} \equiv (\Omega'_\alpha)$ and $R^{D_{\mathcal{P}(\mathcal{E}_\varphi)}} \equiv (\Omega_\alpha)$, (7.3.7) shows that

$$\Omega'_\alpha = \Omega_\alpha, \qquad \alpha \in I.$$

8.5.4. The curvature of the pull-back connection

Let $\mathcal{P} \equiv (\mathcal{P}, \mathcal{G}, X, \pi)$ be a principal sheaf equipped with a connection D. We assume that $(\mathcal{G}, \mathcal{D})$ is a curvature datum, thus the curvature $R \equiv R^D$ of D is defined.

By Proposition 6.5.1, a continuous map $f : Y \to X$ induces the connection $D^* := \boldsymbol{\tau} \circ f^*(D)$ on the pull-back principal sheaf $f^*(\mathcal{P})$. For a reason which will immediately become clear, we set $\boldsymbol{\tau}_1 := \boldsymbol{\tau}$. We recall that $\boldsymbol{\tau}_1$ is the isomorphism of $f^*(\mathcal{A})$-modules (see Lemma 3.5.1)

$$\boldsymbol{\tau}_1 : f^*(\Omega^1(\mathcal{L})) \xrightarrow{\simeq} f^*(\Omega^1)(f^*(\mathcal{L})). \tag{8.5.8}$$

Analogously, we define the 2-form $f^*(\mathcal{A})$-isomorphism

$$\boldsymbol{\tau}_2 : f^*(\Omega^2(\mathcal{L})) \xrightarrow{\simeq} f^*(\Omega^2)(f^*(\mathcal{L})), \tag{8.5.9}$$

thus we may define the exterior differential

$$\boldsymbol{d}^1_* := \boldsymbol{\tau}_2 \circ f^*(\boldsymbol{d}^1) \circ \boldsymbol{\tau}_1^{-1}, \tag{8.5.10}$$

also shown in the next diagram.

$$\begin{array}{ccc}
f^*(\Omega^1(\mathcal{L})) & \xrightarrow{\;f^*(\boldsymbol{d}^1)\;} & f^*(\Omega^2(\mathcal{L})) \\
\boldsymbol{\tau}_1 \downarrow & & \downarrow \boldsymbol{\tau}_2 \\
f^*(\Omega^1)(f^*(\mathcal{L})) & \dashrightarrow_{\boldsymbol{d}^1_*} & f^*(\Omega^2)(f^*(\mathcal{L}))
\end{array}$$

Diagram 8.5

Similarly, applying the obvious identification

$$f^*\big(\Omega^1(\mathcal{L}) \times_X \Omega^1(\mathcal{L})\big) \cong f^*(\Omega^1(\mathcal{L})) \times_X f^*(\Omega^1(\mathcal{L})), \tag{8.5.11}$$

we define the exterior product

$$\bigwedge\nolimits^* := \boldsymbol{\tau}_2 \circ f^*(\bigwedge) \circ (\boldsymbol{\tau}_1 \times \boldsymbol{\tau}_1)^{-1}, \tag{8.5.12}$$

where

$$f^*(\bigwedge) : \big((y,a),(y,b)\big) \longmapsto (y, a \bigwedge b).$$

Equality (8.5.12) is illustrated in the following diagram.

$$\begin{array}{ccc}
f^*(\Omega^1(\mathcal{L})) \times_X f^*(\Omega^1(\mathcal{L})) & \xrightarrow{\quad f^*(\bigwedge) \quad} & f^*(\Omega^2(\mathcal{L})) \\
\Big\downarrow {\scriptstyle \boldsymbol{\tau}_1 \times \boldsymbol{\tau}_1} & & \Big\downarrow {\scriptstyle \boldsymbol{\tau}_2} \\
f^*(\Omega^1)(f^*(\mathcal{L})) \times_X f^*(\Omega^1)(f^*(\mathcal{L})) & \overset{\bigwedge^*}{\dashrightarrow} & f^*(\Omega^2)(f^*(\mathcal{L}))
\end{array}$$

Diagram 8.6

We shall prove that $\boldsymbol{d}^1_*$ verifies the analogs of (8.1.25) and (8.1.26). Before the proof of (8.1.25) we note that equalities (1.4.5) and (8.5.12), in conjunction with (8.5.11), imply

$$\boldsymbol{\tau}_2(y, \theta \bigwedge \omega) = \boldsymbol{\tau}_1(y,\theta) \bigwedge\nolimits^* \boldsymbol{\tau}_1(y,\omega), \tag{8.5.13}$$

for every $(y,\theta),(y,\omega) \in f^*(\Omega^1(\mathcal{L}))$. Therefore, for every $(y,g) \in f^*(\mathcal{G})_y \cong \{y\} \times \mathcal{G}_{f(y)}$, (3.5.7) implies that

$$\begin{aligned}
\big(\boldsymbol{d}^1_* \circ \partial^*\big)(y,g) &= \boldsymbol{d}^1_*\big(\boldsymbol{\tau}_1(y, \partial(g)\big) \\
&= \big(\boldsymbol{\tau}_2 \circ f^*(\boldsymbol{d}^1)\big)(y, \partial(g)) \\
&= \boldsymbol{\tau}_2\big(y, \boldsymbol{d}^1(\partial(g))\big) \\
\text{(see (8.1.25))} \qquad &= -\frac{1}{2}\, \boldsymbol{\tau}_2\big(y, \partial(g) \bigwedge \partial(g)\big) \\
\text{(see (8.5.13))} \qquad &= -\frac{1}{2}\, \boldsymbol{\tau}_1(y, \partial(g)) \bigwedge\nolimits^* \boldsymbol{\tau}_1(y, \partial(g)) \\
&= -\frac{1}{2}\, \partial^*(y,g) \bigwedge\nolimits^* \partial^*(y,g),
\end{aligned}$$

which is the analog of (8.1.25) in our context.

For the proof of the analog of (8.1.26), we recall that the action Δ^* of $f^*(\mathcal{G})$ on $f^*(\Omega^1)(f^*(\mathcal{L}))$ satisfies

$$\Delta^*\big((y,g),\boldsymbol{\tau}_1(y,w)\big) = \boldsymbol{\tau}_1\big(y,\Delta(g,\omega)\big),$$

for every $(y,g) \in f^*(\mathcal{G})_y$ and $(y,w) \in f^*(\Omega^1(\mathcal{L}))_y$ (see Lemma 3.5.3), from which, together with (3.3.7), it follows that

$$\rho^*(y,g).\boldsymbol{\tau}_1(y,w) = \boldsymbol{\tau}_1(y,\rho(g).w), \tag{8.5.14}$$

for (y,g) and (y,w) as before.

The 2-form analog of (8.5.14) is

$$\rho^*(y,g).\boldsymbol{\tau}_2(y,\theta) = \boldsymbol{\tau}_2(y,\rho(g).\theta), \tag{8.5.15}$$

for every $(y,g) \in f^*(\mathcal{G})_y$ and $(y,\theta) \in f^*(\Omega^2(\mathcal{L}))_y$, resulting from the action of $\mathcal{G}$ on $\Omega^2(\mathcal{L})$ and that of $f^*(\mathcal{G})$ on $f^*(\Omega^2(\mathcal{L}))$.

Now, let any $(y,g) \in f^*(\mathcal{G})_y$ and $u \in \big(f^*(\Omega^1)(f^*(\mathcal{L}))\big)_y$. There exists a (unique) $(y,w) \in f^*(\Omega^1(\mathcal{L}))_y$ such that $\boldsymbol{\tau}_1(y,w) = u$. Therefore, (8.5.14), (8.1.26) and (8.5.15) yield

$$\begin{aligned}
\boldsymbol{d}^1_*\big(\rho^*(y,g).u\big) &= \big(\boldsymbol{\tau}_2 \circ f^*(\boldsymbol{d}^1) \circ \boldsymbol{\tau}_1^{-1}\big)\big(\rho^*(y,g).\boldsymbol{\tau}_1(y,w)\big)\\
&= \big(\boldsymbol{\tau}_2 \circ f^*(\boldsymbol{d}^1) \circ \boldsymbol{\tau}_1^{-1}\big)\big(\boldsymbol{\tau}_1(y,\rho(g).w)\big)\\
&= \big(\boldsymbol{\tau}_2 \circ f^*(\boldsymbol{d}^1)\big)(y,\rho(g).w)\\
&= \boldsymbol{\tau}_2\big(y,\boldsymbol{d}^1(\rho(g).w)\big)\\
&= \boldsymbol{\tau}_2\big(y,\rho(g).(\boldsymbol{d}^1 w + \partial(g)\textstyle\bigwedge w)\big)\\
&= \rho^*(y,g).\boldsymbol{\tau}_2\big(y,\boldsymbol{d}^1 w + \partial(g)\textstyle\bigwedge w\big).
\end{aligned} \tag{8.5.16}$$

However (see also (8.5.13)),

$$\boldsymbol{d}^1_* u = \big(\boldsymbol{\tau}_2 \circ f^*(\boldsymbol{d}^1) \circ \boldsymbol{\tau}_1^{-1}\big)(\boldsymbol{\tau}_1(y,w)) = \boldsymbol{\tau}_2(y,\boldsymbol{d}^1 w), \tag{8.5.17}$$

$$\partial^*(y,g)\textstyle\bigwedge^* u = \boldsymbol{\tau}_1(y,\partial(g))\textstyle\bigwedge^*\boldsymbol{\tau}_1(y,w) = \boldsymbol{\tau}_2\big(y,\partial(g)\textstyle\bigwedge w\big). \tag{8.5.18}$$

Thus, substituting (8.5.17) and (8.5.18) in (8.5.16), we obtain

$$\begin{aligned}
\boldsymbol{d}^1_*\big(\rho^*(y,g).u\big) &= \rho^*(y,g).\boldsymbol{\tau}_2\big(y,\boldsymbol{d}^1 w + \partial(g)\textstyle\bigwedge w\big)\\
&= \rho^*(y,g).\big(\boldsymbol{\tau}_2(y,\boldsymbol{d}^1 w) + \boldsymbol{\tau}_2(y,\partial(g)\textstyle\bigwedge w)\big)\\
&= \rho^*(y,g).\big(\boldsymbol{d}^1_* u + \partial^*(y,g)\textstyle\bigwedge^* u\big),
\end{aligned}$$

which is the analog of (8.5.26) for $\boldsymbol{d}^1_*$.

Now, using $\bigwedge^*$ and $\boldsymbol{d}^1_*$, we define the structure operator $\mathcal{D}^*$ by

$$\mathcal{D}^*(u) := \boldsymbol{d}^1_* u + \frac{1}{2}\, u \bigwedge^* u,$$

for every $u \in f^*(\Omega^1)(f^*(\mathcal{L}))$. As one expects,

$$\mathcal{D}^*(u) = \big(\boldsymbol{\tau}_2 \circ f^*(\mathcal{D}) \circ \boldsymbol{\tau}_1^{-1}\big)(u). \tag{8.5.19}$$

Indeed, for every u as before (: $u = \boldsymbol{\tau}_1(y, w)$), we have that

$$\begin{aligned}
\mathcal{D}^*(u) &= \boldsymbol{d}^1_*\big(\boldsymbol{\tau}_1(y, w)\big) + \frac{1}{2}\, \boldsymbol{\tau}_1(y, w) \textstyle\bigwedge^* \boldsymbol{\tau}_1(y, w) \\
&= \big(\boldsymbol{\tau}_2 \circ f^*(\boldsymbol{d}^1)\big)(y, w) + \frac{1}{2}\, \big(\boldsymbol{\tau}_2 \circ f^*(\textstyle\bigwedge)\big)\big((y, w), (y, w)\big) \\
&= \boldsymbol{\tau}_2(y, \boldsymbol{d}^1 w) + \frac{1}{2}\, \boldsymbol{\tau}_2(y, w \textstyle\bigwedge w) \\
&= \boldsymbol{\tau}_2\big(y, \boldsymbol{d}^1 w + \frac{1}{2}\, w \textstyle\bigwedge w\big) \\
&= \boldsymbol{\tau}_2\big(y, \mathcal{D}(w)\big) = \big(\boldsymbol{\tau}_2 \circ f^*(\mathcal{D})\big)(y, w) \\
&= \big(\boldsymbol{\tau}_2 \circ f^*(\mathcal{D}) \circ \boldsymbol{\tau}_1^{-1}\big)(u);
\end{aligned}$$

that is, we obtain (8.5.19).

With the previous constructions we get a curvature datum $(f^*(\mathcal{G}), \mathcal{D}^*)$ which determines the curvature $R^* := R^{D^*}$ of D^*. We now see that

$$R^* = \boldsymbol{\tau}_2 \circ f^*(R), \tag{8.5.20}$$

since, by (8.5.19) and the definition of the pull-back connection D^*,

$$\begin{aligned}
R^* = \mathcal{D}^* \circ D^* &= \big(\boldsymbol{\tau}_2 \circ f^*(\mathcal{D}) \circ \tau_1^{-1}\big) \circ \big(\boldsymbol{\tau}_1 \circ f^*(D)\big) \\
&= \boldsymbol{\tau}_2 \circ f^*(\mathcal{D} \circ D) = \boldsymbol{\tau}_2 \circ f^*(R).
\end{aligned}$$

In a more informative way, showing all the morphisms involved so far, one can equivalently write that

$$R^{D^*} = R^{\boldsymbol{\tau}_1 \circ f^*(D)} = \boldsymbol{\tau}_2 \circ f^*(R^D), \tag{8.5.20$'$}$$

for every connection D on $\mathcal{P}$.

Omitting the isomorphism $\boldsymbol{\tau}_2$, (8.5.20) reduces to

$$(8.5.20'') \qquad R^*(y,p) \equiv f^*(R)(y,p) = (y, R(p)), \qquad (y,p) \in Y \times_X \mathcal{P}$$

(compare with (6.5.2)).

For the sake of completeness, let us find the relationship between the local curvature forms (Ω_α) of R, over a local frame $\mathcal{U} = (U_\alpha)$ of $\mathcal{P}$, and the local curvature forms (Ω^*_α) of R^*, over the local frame $\mathcal{V} = \{f^{-1}(U_\alpha) \mid U_\alpha \in \mathcal{U}\}$ of $f^*(\mathcal{P})$. Let us denote by

$$f^*_{U_\alpha} : \Omega^2(\mathcal{L})(U_\alpha) \longrightarrow f^*\left(\Omega^2(\mathcal{L})\right)(f^{-1}(U_\alpha))$$

the adjunction map over U_α, given by $f^*_{U_\alpha}(\theta)(y) := \big(y, \theta(f(y))\big)$, for every $\theta \in \Omega^2(\mathcal{L})(U_\alpha)$ and $y \in f^{-1}(U_\alpha)$. Then equality (4.1.11) –relating the natural sections (s_α) of $\mathcal{P}$ with those of $f^*(\mathcal{P})$– implies that

$$\begin{aligned} \Omega^*_\alpha(y) &= R^*(s^*_\alpha)(y) = R^*(s^*_\alpha(y)) = (\boldsymbol{\tau}_2 \circ f^*(R))\big(y, s_\alpha(f(y))\big) \\ &= \boldsymbol{\tau}_2\big(y, R(s_\alpha(f(y)))\big) = \boldsymbol{\tau}_2\big(y, \Omega_\alpha(f(y))\big) = \big(\boldsymbol{\tau}_2 \circ f^*_{U_\alpha}(\Omega_\alpha)\big)(y), \end{aligned}$$

for every $y \in f^{-1}(U_\alpha)$; that is,

$$(8.5.21) \qquad \Omega^*_\alpha = \boldsymbol{\tau}_2 \circ f^*_{U_\alpha}(\Omega_\alpha); \qquad \alpha \in I,$$

or, omitting again $\boldsymbol{\tau}_2$,

$$(8.5.21') \qquad \Omega^*_\alpha(y) \equiv \big(y, \Omega_\alpha(f(y))\big), \qquad y \in f^{-1}(U_\alpha).$$

We note that the same formula can be obtained from the structure equation of D^*, which now has the form $\Omega^*_\alpha = \boldsymbol{d}^1_* \omega^*_\alpha + \frac{1}{2}\, \omega^*_\alpha \bigwedge^* \omega^*_\alpha$, where ω^*_α is given by (6.5.3). This is a matter of routine checking requiring no further elaboration.

8.5.5. The curvature of an $\mathcal{A}$-connection

In the first place we define the curvature of an $\mathcal{A}$-connection ∇ on a vector sheaf $\mathcal{E} \equiv (\mathcal{E}, \pi, X)$ in a direct way, as expounded in Mallios [62, Chap. VIII]. Later on, it will be connected with the curvature of the corresponding connection on the sheaf of frames $\mathcal{P}(\mathcal{E})$ of $\mathcal{E}$.

For our purpose, we assume the existence of a (pre)curvature datum $(\mathcal{A}, d = d^0, \Omega^1, d^1, \Omega^2)$ and set $\Omega^i(\mathcal{E}) := \mathcal{E} \otimes_{\mathcal{A}} \Omega^i \cong \Omega^i \otimes_{\mathcal{A}} \mathcal{E}$, $i = 1, 2.$

Then the ***1st prolongation*** of an $\mathcal{A}$-connection $\nabla : \mathcal{E} \longrightarrow \Omega^1(\mathcal{E})$ is the $\mathbb{K}$-*morphism*

$$\nabla^1 : \Omega^1(\mathcal{E}) \longrightarrow \Omega^2(\mathcal{E}),$$

satisfying the property

$$\nabla^1(e \otimes w) = e \otimes d^1 w - w \wedge \nabla e = e \otimes d^1 w + (\nabla e) \wedge w,$$

for every $(e, w) \in \mathcal{E} \times_X \Omega^1$.

It is easily proved that

$$\nabla^1(a \cdot (e \otimes w)) = a \cdot \nabla^1(e \otimes w) + (da) \wedge (e \otimes w),$$
$$\nabla^1(a \cdot \nabla e) = a \cdot \nabla^1(\nabla e) - (\nabla e) \wedge da,$$

for every $(e, w) \in \mathcal{E} \times_X \Omega^1$ and $a \in \mathcal{A}$ on stalks over the same base point (for more details we refer to Mallios op. cit., p. 190, under an appropriate change of notations).

The ***curvature*** of an $\mathcal{A}$-connection ∇ is defined to be the morphism

$$R^\nabla := \nabla^1 \circ \nabla,$$

also shown in the diagram:

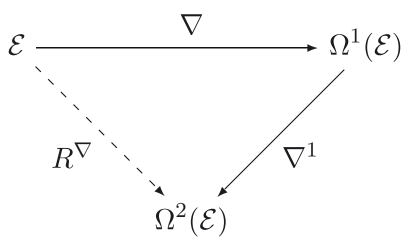

Diagram 8.7

A straightforward calculation shows that R^∇ is an $\mathcal{A}$-morphism; that is,

$$R^\nabla \in \mathrm{Hom}_{\mathcal{A}}(\mathcal{E}, \Omega^2(\mathcal{E})) \cong \mathcal{H}om_{\mathcal{A}}(\mathcal{E}, \Omega^2(\mathcal{E}))(X).$$

If $\mathcal{U} = (U_\alpha)$ is a local frame of $\mathcal{E}$, we have already proved that ∇ is completely determined by its local connection matrices

$$\omega^\alpha := \left(\omega^\alpha_{ij}\right) \in M_n(\Omega(U_\alpha)); \qquad \alpha \in I,$$

(see equality (7.1.4) and Theorem 7.1.4). Similarly, if we evaluate R^{∇} (viz. the induced morphism on sections) at the elements of the natural basis e^{α}, then, in virtue of (7.1.3) and the preceding properties of ∇^1, we find that

$$\begin{aligned} R^{\nabla}(e_j^{\alpha}) &= \nabla^1(\nabla(e_j^{\alpha})) = \nabla^1\Big(\sum_{i=1}^{n} e_i^{\alpha}\otimes\omega_{ij}^{\alpha}\Big) \\ &= \sum_{i=1}^{n}\big(e_i^{\alpha}\otimes d^1\omega_{ij}^{\alpha} - \omega_{ij}^{\alpha}\wedge\nabla(e_i^{\alpha})\big) \\ &= \sum_{i=1}^{n}\big(e_i^{\alpha}\otimes d^1\omega_{ij}^{\alpha}\big) + \sum_{i=1}^{n}\Big(\sum_{k=1}^{n}\big(e_k^{\alpha}\otimes\omega_{ki}^{\alpha}\big)\wedge\omega_{ij}^{\alpha}\Big), \end{aligned}$$

or, by a suitable change of indices,

$$R^{\nabla}(e_j^{\alpha}) = \sum_{i=1}^{n} e_i^{\alpha}\otimes\Big(d^1\omega_{ij}^{\alpha} + \sum_{k=1}^{n}\omega_{ik}^{\alpha}\wedge\omega_{kj}^{\alpha}\Big).$$

Setting

$$R_{ij}^{\alpha} := d^1\omega_{ij}^{\alpha} + \sum_{k=1}^{n}\omega_{ik}^{\alpha}\wedge\omega_{kj}^{\alpha}, \tag{8.5.22}$$

we obtain the $n\times n$ matrix

$$R^{\alpha} = \big(R_{ij}^{\alpha}\big) \in M_n(\Omega^2(U_{\alpha})) \cong \mathcal{M}_n(\Omega^2)(U_{\alpha}), \tag{8.5.23}$$

called the ***curvature matrix of*** ∇ over U_{α}.

Taking into account the definition of the differential d^1 of matrices (see (8.1.39)) and the definition of the product $\wedge$ of matrices (see (8.1.18) and (8.1.19)), we check that (8.5.22) leads to the matrix equalities

$$R^{\alpha} = d^1\omega^{\alpha} + \omega^{\alpha}\wedge\omega^{\alpha}; \qquad \alpha\in I, \tag{8.5.24}$$

known as the local ***Cartan (second) structure equations of*** ∇.

Working as in the proof of Lemma 7.1.2, we obtain the compatibility condition

$$R^{\beta} = \mathrm{Ad}(\psi_{\alpha\beta}^{-1})\,(R^{\alpha}) = \psi_{\alpha\beta}^{-1}\cdot R^{\alpha}\cdot\psi_{\alpha\beta}, \tag{8.5.25}$$

over $U_{\alpha\beta}\neq\emptyset$. Here the transition sections $\psi_{\alpha\beta}^{-1}\in\mathcal{GL}(n,\mathcal{A})(U_{\alpha})$ are viewed as elements of $\mathrm{GL}(n,\mathcal{A}(U_{\alpha}))$, in virtue of (3.2.7).

In analogy to Theorem 8.2.6, we verify that R^∇ is completely determined by the local matrices (8.5.23). This is a consequence of (8.5.25) and the following calculations (see a similar argumentation in the proof of Lemma 7.1.3): If $s \in \mathcal{E}(U)$ is any section of $\mathcal{E}$ over an arbitrary open $U \subseteq X$, then the restriction $s|_{U\cap U_\alpha}$ can be written in the form

$$s|_{U\cap U_\alpha} = \sum_{i=1}^{n} s_i^\alpha \cdot e_i^\alpha|_{U\cap U_\alpha},$$

with coefficients $s_i^\alpha \in \mathcal{A}(U \cap U_\alpha)$. Therefore, omitting –for simplicity– the restrictions, we have that

$$\begin{aligned} R^\nabla(s) &= R^\nabla\Big(\sum_{i=1}^{n} s_i^\alpha \cdot e_i^\alpha\Big) = \sum_{i=1}^{n} s_i^\alpha \cdot R^\nabla(e_i^\alpha) \\ &= \sum_{i=1}^{n} s_i^\alpha \Big(\sum_{k=1}^{n} e_k^\alpha \otimes \big(d^1\omega_{ki}^\alpha + \sum_{l=1}^{n} \omega_{kl}^\alpha \wedge \omega_{li}^\alpha\big)\Big) \\ &= \sum_{k=1}^{n} e_k^\alpha \otimes \Big(\sum_{i=1}^{n} s_i^\alpha \cdot \big(d^1\omega_{ki}^\alpha + \sum_{l=1}^{n} \omega_{kl}^\alpha \wedge \omega_{li}^\alpha\big)\Big) \\ &= \sum_{k=1}^{n} e_k^\alpha \otimes \Big(\sum_{i=1}^{n} s_i^\alpha \cdot R_{ki}^\alpha\Big) \\ &= \sum_{k=1}^{n} e_k^\alpha \otimes (s \cdot R^\alpha), \end{aligned}$$

where the second factor in the last tensor product denotes the (matrix) multiplication of $s \equiv (s_1, \ldots, s_n)$ by R^α.

Now let us interpret the foregoing in terms of the sheaf of frames $\mathcal{P}(\mathcal{E})$. As we already know (see Theorem 7.1.6), an $\mathcal{A}$-connection $\nabla \equiv (\omega^\alpha)$ on $\mathcal{E}$ corresponds bijectively to a connection $D \equiv (\omega_\alpha)$ on $\mathcal{P}(\mathcal{E})$. In particular (see also (3.1.8) and (3.1.9)),

$$\omega_\alpha = \lambda^1_{U_\alpha}(\omega^\alpha) \in \Omega^1(\mathcal{M}_n(\mathcal{A}))(U_\alpha) = \\ (\Omega^1 \otimes_{\mathcal{A}} \mathcal{M}_n(\mathcal{A}))(U_\alpha) \cong \Omega^1(U_\alpha) \otimes_{\mathcal{A}(U_\alpha)} \mathcal{M}_n(\mathcal{A})(U_\alpha)$$

for all $\alpha \in I$. Also, each local curvature form over U_α,

$$\Omega_\alpha \in \Omega^2(\mathcal{M}_n(\mathcal{A}))(U_\alpha) = \\ (\Omega^2 \otimes_{\mathcal{A}} \mathcal{M}_n(\mathcal{A}))(U_\alpha) \cong \Omega^2(U_\alpha) \otimes_{\mathcal{A}(U_\alpha)} \mathcal{M}_n(\mathcal{A})(U_\alpha),$$

is determined by the structure equation (8.2.3), where now $\boldsymbol{d}^1$ and $\bigwedge$ are defined in Example 8.1.6(b). Thus, taking into account (1.2.17),

$$\Omega_\alpha = \boldsymbol{d}^1\omega_\alpha + \frac{1}{2}\,\omega_\alpha\bigwedge\omega_\alpha \equiv \big(\boldsymbol{d}^1_{U_\alpha}\omega_\alpha + \frac{1}{2}\,\omega_\alpha\bigwedge_{U_\alpha}\omega_\alpha\big)^{\sim}.$$

Applying the definitions of $\boldsymbol{d}^1$, $\bigwedge$, equality (8.1.21′) and the structure equation (8.5.24), we have that

$$\begin{aligned}\boldsymbol{d}^1_{U_\alpha}\omega_\alpha + \frac{1}{2}\,\omega_\alpha\bigwedge_{U_\alpha}\omega_\alpha &= (\lambda^2_{U_\alpha}\circ d^1\circ \mu^1_{U_\alpha})(\lambda^1_{U_\alpha}(\omega^\alpha)) \\ &\quad + \frac{1}{2}\,\lambda^1_{U_\alpha}(\omega^\alpha)\wedge_{U_\alpha}\lambda^1_{U_\alpha}(\omega^\alpha) \\ &= \lambda^2_{U_\alpha}(d^1\omega^\alpha + \omega^\alpha\wedge\omega^\alpha) \\ &= \lambda^2_{U_\alpha}(R^\alpha).\end{aligned}$$

Therefore, the 2-form equivalent of (3.1.8) and (3.1.9) imply that

$$\Omega_\alpha \equiv \big(\lambda^2_{U_\alpha}(R^\alpha)\big)^{\sim} \equiv \lambda^2_{U_\alpha}(R^\alpha),$$

in other words, *up to an isomorphism*, $\Omega_\alpha = R^\alpha$.

8.5.6. Some particular cases

(a) Assume that the structure group of $\mathcal{P}$ is abelian. If $D \equiv (\omega_\alpha)$ is a connection on $\mathcal{P}$ with curvature $R^D \equiv (\Omega_\alpha)$, then (by Definition 3.3.4) the compatibility condition (8.2.4) reduces to

$$\Omega_\beta = \Omega_\alpha \quad \text{on} \quad U_{\alpha\beta} \neq \emptyset.$$

As a consequence, R^D is equivalent to a global section, say $\boldsymbol{\Omega} \in \Omega^2(\mathcal{L})(X)$.

(b) An interesting example of a principal sheaf with an abelian structure sheaf is $(\mathcal{P}, \mathcal{A}^{\bullet}, X, \pi)$. In this case $\mathcal{L} = \mathcal{A}$ and $\Omega^1(\mathcal{A}) = \Omega^1 \otimes_{\mathcal{A}} \mathcal{A} \cong \Omega^1$. Hence,

$$\omega\bigwedge\omega = 0, \qquad \omega \in \Omega^1(\mathcal{A}) \cong \Omega^1.$$

Moreover, we take $\boldsymbol{d}^1 = d^1$. Thus $\mathcal{D} = d^1$ and the curvature of a connection D is now given by $R^D = d^1 \circ D$. As in the previous example, $\Omega_\beta = \Omega_\alpha$ on $U_{\alpha\beta}$ and R^D is equivalent to a global section $\boldsymbol{\Omega} \in \Omega^2(X)$.

(c) A combination of the previous case (b) with the general discussion of Subsection 8.5.5 (especially its concluding comments) allows us to establish an analogous interplay between the curvature of a Maxwell field $(\mathcal{E}, \nabla)$ (see definition 7.2.7) and the curvature of the corresponding connection D on $(\mathcal{P}(\mathcal{E}), \mathcal{A}^{\cdot}, X, \pi)$.

8.6. Flat connections

In this section we deal with connections of zero curvature. Integrable connections, complete parallelism, and other relevant notions are related with flat connections. Their equivalence, partially established here, will be completed in the next section.

Let $\mathcal{P} \equiv (\mathcal{P}, \mathcal{G}, X, \pi)$ be a principal sheaf equipped with a curvature datum; hence, the curvature of connections on $\mathcal{P}$ can be defined.

8.6.1 Definition. A connection D on $\mathcal{P}$ is said to be ***flat*** if $R \equiv R^D = 0$.

As a consequence of Theorem 8.2.6, we obtain the following immediate *local* criterion of flatness.

8.6.2 Proposition. *A connection D is flat if and only if the local curvature forms of R vanish, i.e.,*

$$\Omega_\alpha = 0, \qquad \alpha \in I.$$

Likewise, an $\mathcal{A}$-connection ∇ on a vector sheaf $\mathcal{E}$ is said to be ***flat*** if $R^\nabla = 0$. Hence, Proposition 8.6.2, combined with the results of Subsection 8.5.5, also implies:

8.6.3 Proposition. *Let ∇ be an $\mathcal{A}$-connection with curvature R^∇. Then the following conditions are equivalent:*

i) ∇ is flat.

ii) The curvature matrices R^α $(\alpha \in I)$ vanish.

iii) The corresponding connection D on $\mathcal{P}(\mathcal{E})$ is flat.

Two elementary examples of flat connections are provided by the Maurer-Cartan differential ∂ and the canonical local (or Maurer-Cartan) connections (D_α), as alluded to in the comments following Example 8.5.1(b).

We shall relate flat connections with the notions of complete parallelism and integrability, which are also significant for the geometry of the principal sheaf. They are defined relative to a connection D, but, unlike the flatness, they are independent of the curvature R^D. First we give the following:

8.6.4 Definition. We say that a connection D on $\mathcal{P}$ induces a ***complete parallelism*** if there exists a local frame $\mathcal{U}$ of $\mathcal{P}$, whose corresponding natural sections (s_α) are ***parallel*** or ***horizontal***; that is,

$$D(s_\alpha) = 0, \qquad \alpha \in I.$$

In this case, it is customary to call the cochain of sections $(s_\alpha) \in C^0(U_\alpha, \mathcal{P})$ a ***horizontal frame*** (of sections).

The second relevant notion is given in the next definition.

8.6.5 Definition. A connection D on $\mathcal{P}$ is called ***integrable*** if there is a local frame $\mathcal{U} \equiv ((U_\alpha), (\phi_\alpha))$ of $\mathcal{P}$, over which D coincides with the canonical local connections (6.1.2); that is,

$$D|_{\mathcal{P}_\alpha} = D_\alpha := \partial \circ \phi_\alpha, \tag{8.6.1}$$

where $\mathcal{P}_\alpha := \mathcal{P}|_{U_\alpha}$.

We show that there is no essential difference between complete parallelism and integrability.

8.6.6 Proposition. *Let D be a connection on $\mathcal{P}$. Then the following conditions are equivalent:*

i) D induces a complete parallelism.

ii) D is integrable.

iii) The local connection forms (ω_α) of D annihilate.

Proof. First observe that the natural local sections (s_α) satisfy

$$D_\alpha(s_\alpha) = \partial(\mathbf{1}|_{U_\alpha}) = 0; \qquad \alpha \in I, \tag{8.6.2}$$

as a result of (6.1.2), (4.1.7′), and Proposition 3.3.5.

Now assume that D induces a complete parallelism. Then $D(s_\alpha) = 0 = D_\alpha(s_\alpha)$. Since, for any $p \in \mathcal{P}|_{U_\alpha}$ with $\pi(p) = x$, there exists a $g \in \mathcal{G}_x$ such that $p = s_\alpha(x) \cdot g$, we see that

$$D(p) = \rho(g^{-1}).D(s_\alpha) + \partial(g) = \partial(g) = D_\alpha(p),$$

which implies that $D|_{\mathcal{P}_\alpha} = D_\alpha$. Hence $i) \Rightarrow ii)$.

Conversely, assume that D is integrable. Then, by (8.6.2),

$$D(s_\alpha) = D|_{\mathcal{P}_\alpha}(s_\alpha) = D_\alpha(s_\alpha) = 0; \qquad \alpha \in I,$$

which shows that (s_α) is a horizontal frame; thus $ii) \Rightarrow i)$.

Finally, $\omega_\alpha := D(s_\alpha) = 0$ if and only if D induces a complete parallelism. This terminates the proof. □

8.6.7 Corollary. *All the conditions of Proposition 8.6.6 are equivalent to*

$$D_\alpha = D_\beta \quad on \quad \mathcal{P}|_{U_{\alpha\beta}},$$

for every $\alpha, \beta \in I$ *with* $U_{\alpha\beta} \neq \emptyset$. *In other words,* (D_α) *is a 0-cocycle, or* $(D_\alpha) \in Z^0\big(\mathcal{U}, \mathcal{H}om(\mathcal{P}, \Omega^1(\mathcal{L}))\big)$.

Proof. If $\mathcal{P}$ has an integrable connection D, then, by definition, $D_\alpha = D = D_\beta$ on $\mathcal{P}|_{U_{\alpha\beta}}$. Conversely, if the equality of the statement holds, then, by gluing together the D_α's, we define an integrable connection D. □

8.6.8 Corollary. *Any one of the equivalent conditions of Proposition 8.6.6 and Corollary 8.6.7 implies that* D *is a flat connection.*

Proof. Since, in all cases, $\omega_\alpha = 0$, the structure equation (8.2.3) implies that $\Omega_\alpha = 0$, for every $\alpha \in I$. Thus, by Proposition 8.6.2, D is flat. □

The next notion of flatness, ultimately related with flat connections, depends only on ∂ and the local structure of $\mathcal{P}$, and not on any connection. More precisely, we have:

8.6.9 Definition. A principal sheaf $\mathcal{P}$ is said to be ***∂-flat*** if there is a local frame $\mathcal{U}$ with corresponding transition sections $g_{\alpha\beta} \in \mathcal{G}(U_{\alpha\beta})$ such that

$$\partial(g_{\alpha\beta}) = 0; \qquad \alpha, \beta \in I.$$

8.6.10 Proposition. $\mathcal{P}$ *is* ∂*-flat if and only if it has an integrable connection.*

Proof. Let $\mathcal{U}$ be a local frame of $\mathcal{P}$. Any $p \in \mathcal{P}|_{U_{\alpha\beta}}$, with $\pi(p) = x$, is written as $p = s_\beta(x) \cdot g$, for a (unique) $g \in \mathcal{G}_x$. Thus, taking into account (4.3.3) and (8.6.2), we obtain

$$(D_\alpha - D_\beta)(p) = \rho(g^{-1}).D_\alpha(s_\beta(x)) = \rho(g^{-1}).\partial(g_{\alpha\beta}(x)). \tag{8.6.3}$$

If $\mathcal{P}$ is ∂-flat, (8.6.3) implies that $D_\alpha = D_\beta$ on $\mathcal{P}|_{U_{\alpha\beta}}$, for all $\alpha, \beta \in I$. Therefore, by Corollary 8.6.7, the 0-cocycle (D_α) determines an integrable connection D.

Conversely, assume that $\mathcal{P}$ admits an integrable connection. Again, by Corollary 8.6.7, we have that $D_\alpha = D_\beta$ on $\mathcal{P}|_{U_{\alpha\beta}}$. Hence, (8.6.3) implies that $\partial(g_{\alpha\beta}) = 0$; that is, $\mathcal{P}$ is ∂-flat. □

The preceding result and Corollary 8.6.8 also yield:

8.6.11 Corollary. *Every ∂-flat principal sheaf admits a flat connection.*

The relationship between the flatness of connections and the relative notions studied so far is summarized in the next statement.

8.6.12 Theorem. *Let $\mathcal{P}$ be a principal sheaf and consider the following conditions:*

(F. 1) *$\mathcal{P}$ admits a flat connection.*
(F. 2) *$\mathcal{P}$ admits a connection inducing a complete parallelism.*
(F. 3) *$\mathcal{P}$ admits an integrable connection.*
(F. 4) *$\mathcal{P}$ is ∂-flat.*
(F. 5) *$\mathcal{P}$ admits a connection with vanishing local connection forms.*

Then (F. 2) – (F. 5) are equivalent and all of them imply (F. 1).

The statement is depicted as follows:

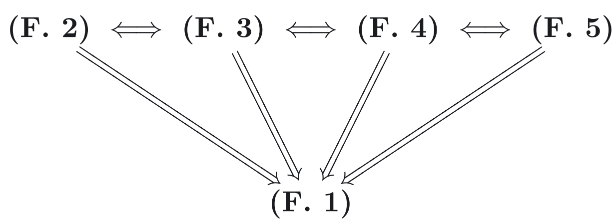

Diagram 8.8

The equivalence of (F. 1) with the other conditions will be discussed in Section 8.7.

Let now $\mathcal{P}$ and $\mathcal{P}'$ be two $\mathcal{G}$-principal sheaves over the same base X, and let $f \equiv (f, id_{\mathcal{G}}, id_{\mathcal{L}}, id_X)$ be a $\mathcal{G}$-(iso)morphism between them. Assume that D and D' are connections on $\mathcal{P}$ and $\mathcal{P}'$, respectively, with curvatures R and R'. If the connections are f-conjugate (see Definition 6.4.1), then equality (8.5.2′) implies that:

> *D is flat if and only if the same is true for D'. Hence, conjugation preserves flatness (of connections).*

A similar assertion is not necessarily true for integrability, parallelism and ∂-flatness. However, under suitable conditions, a given non integrable connection can be transformed to an integrable one, as in the following:

8.6.13 Proposition. *Let $\mathcal{P} \equiv (\mathcal{P}, \mathcal{G}, X, \pi)$ be a principal sheaf with a local frame $\mathcal{U} \equiv ((U_\alpha), (\phi_\alpha))$. Let D be a connection on $\mathcal{P}$ whose connection forms (ω_α) satisfy the condition*

$$\omega_\alpha = \partial(h_\alpha), \tag{8.6.4}$$

for a 0-cochain $h_\alpha \in C^0(\mathcal{U}, \mathcal{G})$. Then there exists a principal sheaf $\mathcal{P}' \equiv (\mathcal{P}', \mathcal{G}, X, \pi')$, an integrable connection D' on $\mathcal{P}'$ (with respect to a local frame over the same open covering (U_α) of X), and a $\mathcal{G}$-isomorphism of $\mathcal{P}$ onto $\mathcal{P}'$ such that D and D' are f-conjugate.

Conversely, let $\mathcal{P}$ be equipped with an arbitrary connection $D \equiv (\omega_\alpha)$. If there is a pair $(\mathcal{P}', D')$ such that $\mathcal{P}$ and $\mathcal{P}'$ are f-isomorphic and D and D' are f-conjugate, then (ω_α) necessarily satisfy (8.6.4).

Proof. The assumption (8.6.4), along with (3.3.8), turns the compatibility condition (6.1.5) into

$$\partial(h_\beta) = \rho(g_{\alpha\beta}^{-1}).\partial(h_\alpha) + \partial(g_{\alpha\beta}) = \partial(h_\alpha \cdot g_{\alpha\beta});$$

hence, in virtue of Proposition 3.3.5, $\partial(h_\alpha \cdot g_{\alpha\beta} \cdot h_\beta^{-1}) = 0$. Setting

$$g'_{\alpha\beta} := h_\alpha \cdot g_{\alpha\beta} \cdot h_\beta^{-1}, \tag{8.6.5}$$

we obtain a cocycle $(g'_{\alpha\beta}) \in Z^1(\mathcal{U}, \mathcal{G})$ which determines a $\mathcal{G}$-principal sheaf $\mathcal{P}'$ (Theorem 4.5.1).

Since $\partial(g'_{\alpha\beta}) = 0$, for all $\alpha, \beta \in I$, $\mathcal{P}'$ is ∂-flat. By Propositions 8.6.6 and 8.6.10, $\mathcal{P}'$ admits an integrable connection, say D', with corresponding local connection forms $\omega'_\alpha = 0$, $\alpha \in I$. Therefore, (8.6.4) amounts to (6.4.5) which, together with (8.6.5), proves that D and D' are f-conjugate (see Theorem 6.4.5). Here f is the isomorphism determined by the 0-cochain (h_α), in virtue of Theorem 4.4.2.

The proof of the converse follows a similar argumentation. □

8.6.14 Remarks. 1) A connection D satisfying (8.6.4) is called ***trivial***. Clearly, any integrable connection $D \equiv (\omega_\alpha)$ is trivial, since $\omega_\alpha = 0$ is also written as $\omega_\alpha = \partial(\mathbf{1}|_{U_\alpha})$.

2) Using Proposition 8.1.5, we see that a trivial connection is necessarily flat, since

$$\Omega_\alpha = \mathcal{D}(\omega_\alpha) = (\mathcal{D} \circ \partial)(h_\alpha) = 0.$$

3) Condition (8.6.4) means that equation $\partial x = \omega_\alpha$ has a global solution $h_\alpha \in \mathcal{G}(U_\alpha)$ (cf. also Definition 8.7.2). Hence, (8.6.4) is a strong *integrability* condition implying that *a trivial connection is flat.*

Conversely, as we shall see in the next section, the flatness of an *arbitrary* connection, together with a Frobenius condition, implies the local integrability of the aforementioned equation and, in turn, the equivalence of all the conditions of Theorem 8.6.12.

Using Definition 6.7.1 and notation (6.7.4), referring to the equivalence of pairs of the form $(\mathcal{P}, D)$, Proposition 8.6.13 implies the following:

8.6.15 Corollary. *Let D be a trivial connection on $\mathcal{P}$. Then, for every $(\mathcal{P}', D') \in [(\mathcal{P}, D)]$, the connection D' is trivial.*

Proof. Since D is trivial, in virtue of the direct part of Proposition 8.6.13, there is a pair $(\bar{\mathcal{P}}, \bar{D})$ such that $(\bar{\mathcal{P}}, \bar{D}) \sim (\mathcal{P}, D)$, with $\bar{D}$ integrable. Then, for any $(\mathcal{P}', D') \in [(\mathcal{P}, D)]$, we have that $(\mathcal{P}', D') \sim (\mathcal{P}, D) \sim (\bar{\mathcal{P}}, \bar{D})$. Therefore, by the converse part of the same proposition, we get the result. □

The proof of Proposition 8.6.13 suggests that, under an appropriate change of the local frame, D itself is integrable, with respect to the new frame. As a matter of fact, we have:

8.6.16 Proposition. *If D is a trivial connection on $\mathcal{P}$, then D is integrable; hence, $\mathcal{P}$ is ∂-flat.*

Proof. As in the proof of Proposition 8.6.13, we consider the isomorphism $f : \mathcal{P} \to \mathcal{P}'$ defined by an appropriate 0-cochain $(h_\alpha) \in C^0(\mathcal{U}, \mathcal{G})$, if $\mathcal{U} \equiv ((U_\alpha), (\phi_\alpha))$ is a local frame of $\mathcal{P}$. Setting $\sigma_\alpha := s_\alpha \cdot h_\alpha^{-1}$ and $\psi_\alpha := \phi'_\alpha \circ f$, $\alpha \in I$, where (ϕ'_α) are the local coordinates of $\mathcal{P}'$ over $\mathcal{U}$, we see that each $\psi_\alpha : \mathcal{P}|_{U_\alpha} \to \mathcal{G}|_{U_\alpha}$ is a $\mathcal{G}$-equivariant coordinate mapping.

Moreover, by (4.1.7′) and (4.4.6),

$$\psi_\alpha^{-1}(\mathbf{1}|_{U_\alpha}) = f^{-1}(s'_\alpha) = s_\alpha \cdot h_\alpha^{-1} = \sigma_\alpha,$$

which means that $\mathcal{U}' \equiv ((U_\alpha), (\psi_\alpha))$ is a local frame of $\mathcal{P}$ with corresponding natural sections (σ_α). Hence, in virtue of our assumption and Proposition 3.3.5,

$$D(\sigma_\alpha) = \rho(h_\alpha).D(s_\alpha) + \partial(h_\alpha^{-1}) = \rho(h_\alpha).\omega_\alpha - \rho(h_\alpha).\partial(h_\alpha) = 0,$$

i.e., D induces a complete parallelism. Therefore, D is integrable and $\mathcal{P}$ is ∂-flat, with respect to $\mathcal{U}'$. □

We close with one more notion of flatness, inspired by an analogous situation encountered in the context of ordinary fiber bundles. Towards this end, in the remainder of this section we assume that

$$\mathcal{G} \text{ contains the constant sheaf } G_X \equiv X \times G, \tag{8.6.6}$$

where G is a given *group*. We denote by $\varepsilon : G_X \hookrightarrow \mathcal{G}$ the natural inclusion, also induced by the inclusions $\varepsilon_U : G \cong G_X(U) \hookrightarrow \mathcal{G}(U)$, for every open $U \subseteq X$.

8.6.17 Definition. A $\mathcal{G}$-principal sheaf $\mathcal{P}$, with $\mathcal{G}$ as in (8.6.6), is called ***flat*** if it has a cocycle $(g_{\alpha\beta}) \in Z^1(\mathcal{U}, G_X)$.

To be more precise, in the preceding definition we should have written $(g_{\alpha\beta}) \in Z^1(\mathcal{U}, \varepsilon(G_X))$. However, for the sake of simplicity, we omit an explicit mention of ε, whenever its use is obviously understood.

In a different manner (see, e.g., Gunning [38, p. 96], Mallios [62, p. 370]), one says that

> $\mathcal{P} \equiv [\mathcal{P}] \in H^1(X, \mathcal{G})$ is flat (or ***admits a flat representative***) if and only if $\mathcal{P} \in \varepsilon^*(H^1(X, G_X))$,

where ε^* is the induced morphism of cohomology groups.

From the previous definitions, we see that the transition sections are *locally constant*. Thus, we have a situation analogous to the classical flat principal bundles, both within the topological and smooth context (in this respect see also Dupont [24], Kamber-Tondeur [46]). We note that in the smooth case, by a *flat bundle* we usually mean a bundle equipped with a *flat connection*, a fact which amounts to the existence of constant transition functions. However, in view of the non equivalence of the various notions of flatness given in our abstract setting, we are obliged to adhere to the distinctive terminology applied to each notion.

8.6.18 Proposition. *If* $G_X \equiv \varepsilon(G_X) \subseteq \ker(\partial)$, *then any flat* $\mathcal{G}$*-principal sheaf satisfies all the conditions (F. 1) – (F. 5) of Theorem 8.6.12.*

Conversely, if $\ker(\partial) \subseteq G_X$, *then a* $\mathcal{G}$*-principal sheaf, satisfying any one of the (equivalent) conditions (F. 2) – (F. 5) of the same theorem, is flat.*

Proof. By our assumptions, $\partial(g_{\alpha\beta}) = 0$, thus $\mathcal{P}$ is ∂-flat, i.e., we obtain Condition (F. 4) of Theorem 8.6.12. This proves the direct part of the statement.

Conversely, the ∂-flatness of $\mathcal{P}$ implies that

$$g_{\alpha\beta} \in \ker(\partial)(U_{\alpha\beta}) \subseteq (G_X)(U_{\alpha\beta}) \equiv \varepsilon(G_X)(U_{\alpha\beta}),$$

which completes the proof. □

8.6.19 Corollary. *If the sequence of sheaves*

$$1 \longrightarrow G_X \stackrel{\varepsilon}{\hookrightarrow} \mathcal{G} \stackrel{\partial}{\longrightarrow} \Omega(\mathcal{L}) \tag{8.6.7}$$

is exact, then the flatness of $\mathcal{P}$ *(in the sense of Definition 8.6.18) is equivalent with all the conditions (F. 2) – (F. 5) of Theorem 8.6.12. Thus we get the following diagram of implications completing Diagram 8.8:*

$$\begin{array}{c} [\,\mathcal{P}\ \textbf{is flat}\,] \\ \Updownarrow \\ [\,\textbf{(F. 2)} \Longleftrightarrow \textbf{(F. 3)} \Longleftrightarrow \textbf{(F. 4)} \Longleftrightarrow \textbf{(F. 5)}\,] \\ \Downarrow \\ \textbf{(F. 1)} \end{array}$$

DIAGRAM 8.9

The discussion on flat sheaves will be completed in Section 8.8, where we examine *flat* $\mathcal{GL}(n, \mathcal{A})$*-principal sheaves.* They provide an interesting example, illuminating the general considerations of the present section.

8.7. A Frobenius condition

In this section we intend to establish the equivalence of all the conditions of Theorem 8.6.12.

Reverting to the standard case of smooth bundles and connections, one may recall that the aforementioned equivalence is always valid there, as a consequence of the fundamental integrability theorem of Frobenius for equations with total (logarithmic) differential.

More precisely, let X be a smooth manifold and G a Lie group with Lie algebra $\mathbb{G}$. If θ is a $\mathbb{G}$-valued differential 1-form on X, i.e., $\theta \in \Lambda^1(X, \mathbb{G})$, then locally $\theta = f^{-1} \cdot df$, for a smooth map $f : U \to G$, $U \in \mathfrak{T}_X$, if and only if θ is *integrable*; that is, the (integrability) condition $d\theta + \frac{1}{2}\,\theta \wedge \theta = 0$ is satisfied.

In other words, the integrability condition ensures the existence of (local) solutions of the equation $x^{-1} \cdot dx = \theta$. In particular, if X is simply connected, then there are global solutions. For details we refer to Bourbaki [15, p. 179], Kreĭn-Yatskin [51, p. 57], Kriegl-Michor [52, p. 427], Oniščik [86, p. 76].

Using the terminology and notations of Example 3.3.6(a), the previous classical result is stated in the following form:

Frobenius theorem (restated). *Let $\theta \in \Omega^1(\mathcal{C}^\infty_X(\mathbb{G}))$. Then the equation $\partial x = \theta$ admits (local) solutions if and only if $\mathcal{D}(\theta) = 0$.*

It would be too much, of course, to expect an immediate extension of this result within the present framework. To achieve our purpose, we axiomatize the ordinary (smooth) situation. Apart from the classical case, this point of view is further supported by other examples given in Section 8.8.

In this respect, we consider the sequence of sheaves of sets (in fact a complex, in virtue of the Maurer-Cartan equation (8.1.29) and Proposition 3.3.5)

$$1 \longrightarrow \ker \partial \overset{i}{\hookrightarrow} \mathcal{G} \overset{\partial}{\longrightarrow} \Omega^1(\mathcal{L}) \overset{\mathcal{D}}{\longrightarrow} \Omega^2(\mathcal{L}), \tag{8.7.1}$$

and, throughout the present section, we *assume* that

(8.7.2) the (abstract) ***Frobenius condition*** holds true, namely *the sequence (8.7.1) is exact.*

Clearly (8.7.2) reverses the inclusion $\operatorname{im} \partial \subset \ker \mathcal{D}$.

8.7.1 Lemma. *Let $\theta \in (\ker \mathcal{D})(U)$ be a 1-form (viz. section) over any open $U \subseteq X$. Then, for each $x \in U$, there exists an open neighborhood $V \subseteq U$ of x and a section $g \in \mathcal{G}(V)$ such that*

$$\partial(g) = \theta|_V. \tag{8.7.3}$$

Proof. Condition (8.7.2) implies that

$$\theta(x) \in (\operatorname{im} \partial)_x = \operatorname{im}\big(\partial_x : \mathcal{G}_x \to \Omega^1(\mathcal{L})_x\big).$$

Hence, there is a local section $h \in \mathcal{G}(W)$, with $W \in \mathcal{N}(x)$, so that

$$\theta(x) = \partial_x(h(x)) = \partial(h(x)) = \partial(h)(x).$$

As a result, there exists an open V such that $x \in V \subseteq U \cap W$ and $\partial(h)|_V = \partial(h|_V) = \theta|_V$. The desired g is precisely $h|_V$. □

8.7.2 Definition. Following the standard terminology, a section g satisfying (8.7.3) is called a ***(local) solution*** of the equation $\partial x = \theta$ or, even, a solution of the equation (8.7.3).

For the sake of completeness we also prove:

8.7.3 Corollary. *Let $\theta \in (\ker \mathcal{D})(U)$ and let (x_o, g_o) be a pair with $x_o \in U$ and $g_o \in \mathcal{G}_{x_o}$. Then the following conditions are equivalent:*

i) There exists a solution $g \in \mathcal{G}(V)$ of (8.7.3) such that $g(x_o) = g_o$.

ii) For every solution $\bar{g} \in \mathcal{G}(\bar{V})$, $\bar{V} \in \mathcal{N}(x_o)$, it follows that

$$g_o \cdot \bar{g}(x_o)^{-1} \in \ker \partial_x.$$

Proof. Let $\bar{g} \in \mathcal{G}(\bar{V})$, $\bar{V} \in \mathcal{N}(x_o)$, be any solution and $g \in \mathcal{G}(V)$ one with $g(x_o) = g_o$. Since $\partial(\bar{g}(x_o)) = \theta(x_o) = \partial(g_o)$, Proposition 3.3.5 implies condition *ii)*.

Conversely, assuming *ii)*, we can find a local section $s \in (\ker \partial)(W)$ with $s(x_o) = g_o \cdot \bar{g}(x_o)^{-1}$ and $W \in \mathcal{N}(x_o)$. Setting $V := \bar{V} \cap W$ and $g := s \cdot \bar{g}|_V$, we check that $g(x_o) = g_o$ and

$$\partial(g) = \partial(s \cdot \bar{g}|_V) = \rho\big((\bar{g}|_V)^{-1}\big).\partial(s) + \partial(\bar{g}|_V) = \theta|_V;$$

that is, g is a solution satisfying condition *i)*. □

Note. Obviously, in virtue of the standard properties of the local sections of a sheaf, two solutions of (8.7.3) with the same initial condition coincide on an open subset of their common domain.

We now come to the main result of this section, which completes Theorem 8.6.12.

8.7.4 Theorem. *Under the Frobenius condition (8.7.2), all the conditions of Theorem 8.6.12 are equivalent.*

Proof. Let D be a connection with curvature $R \equiv R^D = 0$. We denote by (ω_α) the local connection forms and by (Ω_α) the local curvature forms, with respect to a local frame $\mathcal{U}$ and the corresponding natural sections (s_α).

If all the ω_α's vanish identically, then the result is trivially verified, since $D(s_\alpha) = \omega_\alpha = 0$, for every $\alpha \in I$, implies that (s_α) is a horizontal frame; hence, D induces a complete parallelism.

Now assume that not all the local connection forms vanish identically. We shall construct a new local frame $\mathcal{V}$ whose natural sections induce a complete parallelism. Indeed, for an arbitrary $x \in X$, there is some $U_\alpha \in \mathcal{U}$ with $x \in U_\alpha$. By the assumption, $\mathcal{D}(\omega_\alpha) = \Omega_\alpha = 0$; that is, $\omega_\alpha \in (\ker \mathcal{D})(U_\alpha)$ and (by Lemma 8.7.1) there exists an open neighborhood $V_x \subseteq U_\alpha$ of x and a section $g_x \in \mathcal{G}(V_x)$ such that $\partial(g_x) = \omega_\alpha|_{V_x}$. Running x in the entire X, we obtain an open refinement $\mathcal{V} = \{V_j \subseteq X \mid j \in J\}$ of $\mathcal{U}$ and a 0-cochain $(g_j) \in C^0(\mathcal{V}, \mathcal{G})$ satisfying

$$\partial(g_j) = \omega_{\tau(j)}|_{V_j}, \tag{8.7.4}$$

where $\tau : J \to I$ is a refining map with $V_j \subseteq U_{\tau(j)}$, for every $j \in J$.

The morphisms $\psi_j := \phi_{\tau(j)}|_{\mathcal{P}_j} \cdot (g_j \circ \pi|_{\mathcal{P}_j})$, with $\mathcal{P}_j := \mathcal{P}|_{V_j}$, determine a family of local coordinates with respect to $\mathcal{V}$. In fact, a simple calculation shows that $\psi_j^{-1} = \big(\phi_{\tau(j)}|_{\mathcal{P}_j}\big)^{-1} \cdot (g_j^{-1} \circ \pi_{\mathcal{G}}|_{\mathcal{G}_j})$, where $\pi_{\mathcal{G}}$ is the projection of $\mathcal{G}$, $\mathcal{G}_j := \mathcal{G}|_{V_j}$, and g_j^{-1} is the inverse section of g_j (see (1.1.4)). Hence, the new natural sections $(\sigma_j)_{j\in J}$ of $\mathcal{P}$, with respect to $\mathcal{V}$, are given by

$$\sigma_j(x) := (\psi_j^{-1} \circ \mathbf{1}|_{V_j})(x) = \phi_{\tau(j)}^{-1}(e_x) \cdot g_j^{-1}(x) = s_{\tau(j)}(x) \cdot g_j^{-1}(x),$$

for every $x \in V_j$, i.e., $\sigma_j = s_{\tau(j)}|_{V_j} \cdot g_j^{-1}$. Therefore, (8.7.4) and Proposition 3.3.5 imply that

$$\begin{aligned} D(\sigma_j) &= D(s_{\tau(j)}|_{V_j} \cdot g_j^{-1}) = \rho(g_j).D(s_{\tau(j)}|_{V_j}) + \partial(g_j^{-1}) \\ &= \rho(g_j).\omega_{\tau(j)}|_{V_j} - \rho(g_j).\partial(g_j) = 0, \end{aligned}$$

which closes the proof. □

In particular, Corollary 8.6.19 is completed as follows:

8.7.5 Corollary. *Let $\mathcal{P}$ be any $\mathcal{G}$-principal sheaf admitting a flat connection with $\mathcal{G}$ containing a constant sheaf of groups G_X. If the sequence (8.6.7) is exact, then the Frobenius condition implies that $\mathcal{P}$ is flat. Hence, all the conditions shown in Diagram 8.9 are equivalent.*

The Frobenius condition (8.7.2) leads to an interesting exact sequence of cohomology groups. First, observe that the same condition results in the (short) exact sequence of sheaves

$$0 \longrightarrow \ker \partial \overset{i}{\hookrightarrow} \mathcal{G} \overset{\partial}{\longrightarrow} \ker \mathcal{D} \longrightarrow 0. \tag{8.7.5}$$

Thus, also motivated by Asada [2], [3] and Oniščik [86] (all dealing with certain aspects of *non-abelian cohomology* related with connections), we derive the cohomology sequence

$$0 \longrightarrow H^0(X, \ker\partial) \xrightarrow{i^*} H^0(X, \mathcal{G}) \xrightarrow{\partial^*} H^0(X, \ker\mathcal{D}) \xrightarrow{\delta} \longrightarrow H^1(X, \ker\partial) \xrightarrow{i^*} H^1(X, \mathcal{G}), \tag{8.7.6}$$

where the morphism δ is defined as follows: Let $\Theta \in (\ker\mathcal{D})(X)$ be a given (global) section. As in the proof of Theorem 8.7.4, we can find local solutions $h_U \in \mathcal{G}(U)$ of $\partial x = \Theta$, i.e., $\partial(h_U) = \Theta|_U$, where U is running some open covering $\mathcal{U}$ of X. We define $\delta(\Theta)$ to be the 1-cochain given by

$$\delta(\Theta)_{UV} := h_U \cdot h_V^{-1}; \qquad U, V \in \mathcal{U}.$$

It is now clear that $\delta(\Theta)$ is in fact a 1-cocycle, thus δ is well defined.

Note. In virtue of Definition 8.6.9, $H^1(X, \ker\partial)$ represents the equivalence classes of ∂-flat $\mathcal{G}$-principal sheaves. Moreover, if (8.7.2) is in force, then the same cohomology set represents classes of $\mathcal{G}$-principal sheaves admitting flat connections.

8.7.6 Theorem. *Under the Frobenius condition, the sequence (8.7.6) is exact. Moreover, $\delta(\Theta) = \delta(\bar{\Theta})$ if and only if Θ and $\bar{\Theta}$ are gauge equivalent; that is, there exists $g \in H^0(X, \mathcal{G}) \cong \mathcal{G}(X)$ such that*

$$\Theta = \rho(g^{-1}).\bar{\Theta} + \partial(g). \tag{8.7.7}$$

Proof. The exactness of (8.7.6) is immediately verified (see also Subsection 1.6.4). For the direct part of the second assertion, we may assume that both equations with "coefficients" Θ and $\bar{\Theta}$ (see Definition 8.7.2) admit the respective local solutions (h_U) and $(\bar{h}_U)$ over the same open covering $\mathcal{U}$ of X, otherwise we take a common refinement of the respective coverings. Therefore, if $\delta(\Theta) = \delta(\bar{\Theta})$, then the 1-cocycles $(\delta(\Theta)_{UV})$ and $(\delta(\bar{\Theta})_{UV})$ are cohomologous, i.e., there exists a 0-cochain $(\lambda_U) \in C^0(\mathcal{U}, \ker\partial)$ satisfying

$$h_U \cdot h_V^{-1} = \lambda_U \cdot (\bar{h}_U \cdot \bar{h}_V^{-1}) \cdot \lambda_V^{-1}, \tag{8.7.8}$$

on every $U \cap V \neq \emptyset$. Setting $g_U := \bar{h}_U^{-1} \cdot \lambda_U^{-1} \cdot h_U$, (8.7.8) implies that $g_U = g_V$ on $U \cap V$, thus we obtain a global section $g \in \mathcal{G}(X)$ with $g|_U := g_U$. Hence, taking into account that $\partial(\lambda_U) = 0$, we have that

$$\begin{aligned}\Theta|_U = \partial(h_U) &= \partial(\lambda_U \cdot \bar{h}_U \cdot g_U) \\ &= \rho(g_U^{-1}).\partial(\bar{h}_U) + \partial(g_U) \\ &= \big(\rho(g^{-1}).\bar{\Theta} + \partial(g)\big)|_U,\end{aligned}$$

for every $U \in \mathcal{U}$. This proves (8.7.7).

Conversely, assume that $\Theta, \bar{\Theta} \in (\ker \partial)(X)$ are gauge equivalent by means of a $g \in \mathcal{G}(X)$. Then, using an appropriate common open covering and working as before (in a reverse way), we see that $\partial(\bar{h}_U \cdot g|_U) = \partial(h_U)$ or, by Proposition 3.3.5, $\partial(\bar{h}_U \cdot g|_U \cdot h_U^{-1}) = 0$. Setting

$$\lambda_U^{-1} := \bar{h}_U \cdot g|_U \cdot h_U^{-1}, \tag{8.7.9}$$

we have that $\partial(\lambda_U) = 0$, for every $U \in \mathcal{U}$, thus obtaining a 0-cochain $(\lambda_U) \in C^0(\mathcal{U}, \ker \partial)$. Comparing (8.7.9) for U and V, we get (8.7.8), thus $\delta(\Theta) = \delta(\bar{\Theta})$ which concludes the proof. □

We close with the following immediate result:

8.7.7 Corollary. *Assume that $\mathcal{G}$ contains the constant sheaf of groups G_X. If, in addition to the Frobenius condition, the sequence (8.6.7) is exact, then the exact sequence (8.7.6) takes the form*

$$0 \longrightarrow H^0(X, G_X) \xrightarrow{\varepsilon^*} H^0(X, \mathcal{G}) \xrightarrow{\partial^*} H^0(X, \ker \mathcal{D}) \xrightarrow{\delta} \longrightarrow H^1(X, G_X) \xrightarrow{\varepsilon*} H^1(X, \mathcal{G}).$$

8.8. $\mathcal{GL}(n, \mathcal{A})$-principal sheaves and flatness

The case of the sheaves in the title merits a particular treatment. On the one hand, taking $\mathcal{GL}(n, \mathcal{A})$ as the structure sheaf allows one to reduce the number of the axioms imposed at various stages of our approach. More specifically:

1) As we have already seen in Example 3.3.6(b), in conjunction with Proposition 3.2.1, $\mathcal{GL}(n, \mathcal{A})$ is a Lie sheaf of groups in a natural way: Its Maurer-Cartan differential is derived directly from the differential of the original differential triad $(\mathcal{A}, d \equiv d^0, \Omega \equiv \Omega^1)$ and automatically satisfies the fundamental property (3.3.8) of Definition 3.3.2, with respect to the adjoint representation of $\mathcal{GL}(n, \mathcal{A})$ on $\mathcal{M}_n(\mathcal{A})$.

2) If the differential triad extends to a precurvature datum (see Definition 8.1.2), then $\mathcal{GL}(n, \mathcal{A})$ is provided with a convenient curvature datum, already constructed in Example 8.1.6(b).

On the other hand, the same group sheaf is the structure group of the sheaf of frames $\mathcal{P}(\mathcal{E})$ of a vector sheaf $\mathcal{E}$ of rank n. Thus, according to Subsection 8.5.5, the results pertaining to the curvature of a $\mathcal{GL}(n, \mathcal{A})$-principal sheaf imply analogous results concerning the curvature of $\mathcal{A}$-connections. Consequently, taking into account Theorem 7.1.6, we see that a considerable part of the geometry of vector sheaves derives from the general theory of $\mathcal{GL}(n,\mathcal{A})$-principal sheaves.

Furthermore, flat $\mathcal{GL}(n,\mathcal{A})$-principal sheaves (see Definition 8.6.17) are naturally defined by considering the constant sheaf

$$\mathcal{GL}(n,\mathbb{K}) := X \times GL(n,\mathbb{K}),$$

($\mathbb{K}=\mathbb{R},\mathbb{C}$), and the morphism

$$\varepsilon : \mathcal{GL}(n,\mathbb{K}) \hookrightarrow \mathcal{GL}(n,\mathcal{A}),$$

induced by the obvious imbeddings of presheaves

$$\mathcal{GL}(n,\mathbb{K})(U) \cong GL(n,\mathbb{K}) \stackrel{\varepsilon}{\hookrightarrow} GL(n,\mathcal{A}(U)) \cong \mathcal{GL}(n,\mathcal{A})(U),$$

for every open $U \subseteq X$. Therefore, in virtue of the discussion following Definition 8.6.17, we conclude that

$$\mathcal{P} \equiv [\mathcal{P}] \in H^1(X,\mathcal{GL}(n,\mathcal{A})) \text{ is } \textit{flat} \text{ if it lies in } \operatorname{im}\varepsilon^*.$$

In analogy to (the first part of) Proposition 8.6.18, this conclusion leads to:

8.8.1 Proposition. *A flat $\mathcal{GL}(n,\mathcal{A})$-principal sheaf satisfies all the conditions of Theorem 8.6.12.*

Proof. Let $\mathcal{P}$ be any flat $\mathcal{GL}(n,\mathcal{A})$-principal sheaf. Each transition section $g_{\alpha\beta} \in \mathrm{GL}(n,\mathcal{A}(U_{\alpha\beta}))$ of the cocycle of $\mathcal{P}$ can be identified, by means of ε, with a matrix $(g_{ij}^{\alpha\beta}) \in \mathrm{GL}(n,\mathbb{K})$ $(i,j=1,\dots,n)$. Since d vanishes on constants (see Proposition 2.1.3), the Maurer-Cartan differential (see (3.2.10)) yields

$$\tilde{\partial}(g_{\alpha\beta}) \equiv \tilde{\partial}_{U_{\alpha\beta}}\big((g_{ij}^{\alpha\beta})\big) = \big(g_{ij}^{\alpha\beta}\big)^{-1}\cdot d\big((g_{ij}^{\alpha\beta})\big) = \big(g_{ij}^{\alpha\beta}\big)^{-1}\cdot\big(dg_{ij}^{\alpha\beta}\big) = 0, \tag{8.8.1}$$

as a consequence of (1.2.17), (3.2.9),(8.1.38) and $d=d^0$. Thus $\mathcal{P}$ is $\tilde{\partial}$-flat. Theorem 8.6.12 now completes the proof. □

The converse of the previous statement goes as follows (compare with the converse part of Proposition 8.6.18 in conjunction with the sequence (8.6.7)).

8.8.2 Proposition. *Let $\mathcal{P}$ be a $\mathcal{GL}(n, \mathcal{A})$-principal sheaf satisfying any one of the (equivalent) conditions (F. 2) – (F. 5) of Theorem 8.6.12. If the sequence*

$$0 \longrightarrow \mathbb{K} \equiv \mathbb{K}_X \overset{i}{\hookrightarrow} \mathcal{A} \overset{d}{\longrightarrow} \Omega^1 \tag{8.8.2}$$

is exact, then $\mathcal{P}$ is flat.

Proof. By the $\widetilde{\partial}$-flatness of $\mathcal{P}$ and equality (8.8.1), $\big(d\, g^{ij}_{\alpha\beta}\big) = 0$; that is, $d\, g^{ij}_{\alpha\beta} = 0$. Hence, for each $x \in U_{\alpha\beta}$, the exactness of (8.8.2) and the identification $(\mathbb{K}_X)_x \equiv i((\mathbb{K}_X)_x) \subseteq \mathcal{A}_x$, imply that $g^{ij}_{\alpha\beta}(x) = i([\, s\,]_x) \equiv [\, s\,]_x$, for a locally constant section s of $\mathbb{K}$. Consequently, there exists an open neighborhood $V^{ij}_x \subseteq U_{\alpha\beta}$ of x such that $g^{\alpha\beta}_{ij}|_{V^{ij}_x} = i(c_{ij}) \equiv c_{ij} \in \mathbb{K}$ (constant). Therefore, on

$$V_x := \bigcap_{i,j=1}^{n} V^{ij}_x ,$$

we have that

$$g_{\alpha\beta}|_{V_x} = \Big(g^{\alpha\beta}_{ij}|_{V_x}\Big) = \varepsilon\big((c_{ij})\big),$$

which proves that each $g_{\alpha\beta}$ is a locally constant section. □

A consequence of the above two results is:

8.8.3 Corollary. *With the assumptions of Proposition 8.8.2, Diagram 8.9 holds true for every $\mathcal{GL}(n, \mathcal{A})$-principal sheaf.*

If, in addition, we take into account the Frobenius condition, we obtain the analog of Corollary 8.7.5, namely:

8.8.4 Corollary. *Assume that the Frobenius condition is satisfied and the sequence (8.8.2) is exact. Then a $\mathcal{GL}(n, \mathcal{A})$-principal sheaf $\mathcal{P}$ is flat if and only if it has a flat connection. Therefore, all the conditions of Theorem 8.6.12 are equivalent and amount to the flatness of $\mathcal{P}$; in other words, Diagram 8.9 consists of equivalences everywhere.*

The more specific case of $\mathcal{GL}(1, \mathcal{A}) = \mathcal{A}^{\bullet}$ is also of interest at this point because, under suitable topological assumptions, it provides examples of principal sheaves always verifying the Frobenius condition. As a matter of fact, we prove the following analog of Lemma 8.7.1.

8.8.5 Lemma. *Let $\mathcal{A}$ be a sheaf of unital commutative and associative topological $\mathbb{C}$-algebras which are, in addition, σ-complete and locally m-convex. We further assume that the sequence of $\mathbb{C}$-vector space sheaves*

$$0 \longrightarrow d\mathcal{A} \overset{i}{\hookrightarrow} \Omega^1 \overset{d^1}{\longrightarrow} d^1\Omega^1 \longrightarrow 0 \tag{8.8.3}$$

is exact. Then, for any open $U \subseteq X$ and $\theta \in (d^1\Omega)(U)$, there is an open $V \subseteq U$ and a section $g \in \mathcal{A}^{\bullet}(V)$ such that $\widetilde{\partial}(g) = \theta|_V$, where the differential $\widetilde{\partial}$ is given by (3.2.4).

The terminology of the Lemma is explained at the end of this section.

Proof. We essentially follow Mallios [62, Vol. II, Theorem 11.9.1]: For any θ as in the statement, the exactness of (8.8.3) ensures the existence of a section $h \in \mathcal{A}(V)$, over an open $V \subseteq U$, such that

$$dh = \theta|_V. \tag{8.8.4}$$

The topological properties of $\mathcal{A}$ guarantee the existence of an ***exponential morphism***

$$\exp : \mathcal{A} \longrightarrow \mathcal{A}^{\bullet}$$

defined section-wise by

$$\exp(s) := \sum_{n=0}^{\infty} \frac{1}{n!}\, s^n,$$

for every $s \in \mathcal{A}(U)$ and every open $U \subseteq X$. Applying (the continuous morphism) d to the last expression, we check that

$$d(\exp s) = (\exp s) \cdot ds;$$

therefore, in virtue of (3.2.4),

$$\widetilde{\partial}(\exp s) = (\exp s)^{-1} \cdot d(\exp s) = ds, \tag{8.8.5}$$

for every local section as before. Now combining equalities (8.8.4) and (8.8.5), we obtain the result by setting $g := \exp h$. □

8.8.6 Theorem. *Let $\mathcal{P}$ be any $\mathcal{A}^{\bullet}$-principal sheaf endowed with a flat connection D. If $\mathcal{A}$ is a sheaf of topological algebras satisfying the assumptions of Lemma 8.8.5, then the Frobenius condition is always satisfied, thus the conclusion of Theorem 8.7.5 holds true; that is, (F. 1) – (F. 5) are all equivalent.*

Proof. Since, in the present case, $\mathcal{D} = d^1$, the Frobenius condition (8.7.2) reduces to $\operatorname{im}\partial = \ker d^1$, which is true by Lemma 8.8.5. Hence, Theorem 8.7.4 implies the result. □

8.8.7 Remarks. 1) The exponential morphism $\exp : \mathcal{A} \to \mathcal{A}^{\bullet}$ is an important morphism in the geometry of *line sheaves* (see, e.g., Mallios [64]).

2) Regarding the exactness of (8.8.2) and (8.8.3), we note that this is part of the exactness of the abstract de Rham complex

$$0 \longrightarrow \mathbb{C} \longrightarrow \mathcal{A} \xrightarrow{d} \Omega \xrightarrow{d^1} \Omega^2 \longrightarrow \cdots$$

encountered at the end of Section 2.5 (see Definition 2.5.4 and the ensuing comments). As we discussed there, in our abstract framework this exactness is not always ensured. An interesting problem is to seek (topological) algebra sheaves $\mathcal{A}$ and $\mathcal{A}$-modules satisfying it.

8.8.8 Topological algebras jargon. We give a few definitions concerning topological algebras. For full details we refer to Mallios [58]. For the general theory of topological vector spaces one may consult, e.g., Horváth [45], Schaefer [108].

A ***topological algebra*** $\mathbb{A}$ is a *topological vector space* (over $\mathbb{C}$) with an algebra multiplication $\mathbb{A} \times \mathbb{A} \ni (a, b) \mapsto a \cdot b \in \mathbb{A}$, which is *separately* continuous (i.e., continuous in each one of the variables).

A topological algebra $\mathbb{A}$ is said to be ***locally convex*** if it is a locally convex topological space. By definition, there is a fundamental system of convex neighborhoods of 0. In particular, a topological algebra is said to be ***locally m-convex*** (: *locally multiplicatively-convex*), in brief **lmc**, if the fundamental system of neighborhoods of 0 consists of multiplicative (: $U \cdot U \subseteq U$) convex sets.

Finally a topological algebra is ***σ-complete*** (: *sequentially complete*) if every Cauchy sequence in $\mathbb{A}$ converges.

Chapter 9

Chern - Weil theory

On the other hand, the latter differential geometrical method describes the way principal bundles are curved by differential forms using the concepts of a connection and the curvature. This is called the Chern - Weil theory.

S. Morita [78, p. xi].

The objective of this chapter is to give an abstract analog of the Chern-Weil homomorphism, in the context of principal sheaves and their connections. According to this, appropriate invariant symmetric multilinear maps on the sheaf of Lie algebras $\mathcal{L}$ (attached to a Lie sheaf of groups $\mathcal{G}$) determine certain cohomology classes. The latter are defined over a de Rham space X, completing the initial differential triad $(\mathcal{A}, d, \Omega)$.

We follow a variation of the geometric approach due to S. S. Chern, based on the curvature of an arbitrary connection defined on a $\mathcal{G}$-principal sheaf

$\mathcal{P}$ over X. Since the standard approach (within the context of *principal bundles*), using horizontal distributions and the covariant exterior differentiation (induced by a connection), is meaningless in our framework, we proceed by considering appropriate "local" entities over X.

The classical theory is treated in many sources. Among them we cite, for instance, Dupont [24], Greub-Halperin-Vanstone [35], Kobayashi-Nomizu [49], Naber [81]. For a local approach, within the same classical framework, the reader may consult, e.g., Nicolaescu [84] and Wells [142]. The sheaf-theoretic analog of the Chern-Weil theory in the context of *vector sheaves* is given in Mallios [62, Chapter IX].

9.1. Preliminaries

We fix throughout an algebraized space $(X, \mathcal{A})$ endowed with the differential triad $(\mathcal{A}, d, \Omega)$ and a Lie sheaf of groups $(\mathcal{G}, \rho, \mathcal{L}, \partial)$. From Section 2.5 we recall that $d = d^0$ and $\Omega \equiv \Omega^1$.

(9.1.1) We assume that the differential d of $(\mathcal{A}, d, \Omega)$ extends to operators of sufficiently higher orders, as the case may be. Furthermore, we assume the existence of a Bianchi datum $(\mathcal{G}, \mathcal{D}, \boldsymbol{d}^2)$.

At a later stage, the previous assumption about d will be strengthened by demanding X to be a de Rham space. However, (9.1.1) is sufficient for the development of the first three sections of the chapter.

From Subsection 1.3.2 we recall the notation

$$\mathcal{S}^{(k)} := \underbrace{\mathcal{S} \times_X \cdots \times_X \mathcal{S}}_{k-\text{factors}}, \tag{9.1.2}$$

applied in order to avoid confusion with the exterior power $\mathcal{S}^k = \mathcal{S} \wedge \cdots \wedge \mathcal{S}$ (k times). However, to handle more complicated formulas, we shall also use the notation

$$\prod^k \mathcal{S} := \underbrace{\mathcal{S} \times_X \cdots \times_X \mathcal{S}}_{k-\text{factors}}. \tag{9.1.2$'$}$$

9.1.1 Definition. A morphism of $\mathcal{A}$-modules $f : \mathcal{L}^{(k)} \to \mathcal{A}$ is said to be a ***k-morphism*** if it is a morphism of $\mathcal{A}$-modules with respect to each variable. Moreover, f is called ***symmetric*** if

$$f(u_{\sigma(1)}, \dots, u_{\sigma(n)}) = f(u_1, \dots, u_n),$$

for every $(u_1, \dots, u_k) \in \mathcal{L}^{(k)}$ and every permutation σ of $\{1, \dots, k\}$.

We know that a representation $\rho : \mathcal{G} \to \mathcal{A}ut(\mathcal{L})$ induces an action of $\mathcal{G}$ on the left of $\mathcal{L}$, whose result is denoted by $g.u$, for any $(g, u) \in \mathcal{G} \times_X \mathcal{L}$ (see Proposition 3.3.1). Thus, we can give the next definition:

9.1.2 Definition. A k-morphism $f : \mathcal{L}^{(k)} \to \mathcal{A}$ is said to be ***ρ-invariant*** if

$$f(g.u_1, \ldots, g.u_k) = f(u_1, \ldots, u_k),$$

for every $(g; u_1, \ldots, u_k) \in \mathcal{G} \times_X \mathcal{L}^{(k)}$.

We have already defined the algebra (see (8.1.1))

$$\Omega^{\bullet}(\mathcal{L}) := \bigoplus_{p=0}^{\infty} \Omega^p(\mathcal{L}) = \bigoplus_{p=0}^{\infty} \Omega^p \otimes_{\mathcal{A}} \mathcal{L}.$$

According to Subsection 1.3.6, it can be equivalently obtained by the sheafification of the presheaf

$$U \longmapsto \bigoplus_{p=0}^{\infty} \big(\bigwedge^p(\Omega^1(U)) \otimes_{\mathcal{A}(U)} \mathcal{L}(U)\big).$$

Given a ρ-invariant k-morphism $f : \mathcal{L}^{(k)} \to \mathcal{A}$, for each open $U \subseteq X$ we define the k-linear (with respect to $\mathcal{A}(U)$) map

$$\widehat{f_U} : \prod^{k} \Big(\bigoplus_{p=0}^{\infty} \big(\bigwedge^p(\Omega^1(U)) \otimes_{\mathcal{A}(U)} \mathcal{L}(U)\big)\Big) \longrightarrow \bigoplus_{p=0}^{\infty} \bigwedge^p(\Omega^1(U)), \tag{9.1.3}$$

by setting

$$\widehat{f_U}(\theta_1 \otimes \ell_1, \ldots, \theta_k \otimes \ell_k) := f(\ell_1, \ldots, \ell_k) \cdot \theta_1 \wedge \cdots \wedge \theta_k, \tag{9.1.4}$$

for every $\theta_i \otimes \ell_i \in \bigwedge^{p_i}(\Omega^1(U)) \otimes_{\mathcal{A}(U)} \mathcal{L}(U)$. The product on the left-hand side of (9.1.3) is the usual *cartesian product*, while the morphism f on the right-hand side of (9.1.4) is, obviously, the induced morphism of sections.

Running U in the topology of X, the presheaf morphism (9.1.3) generates the k-morphism

$$\widehat{f} := \mathbf{S}\big(\widehat{f_U}\big) : \prod^{k} \Omega^{\bullet}(\mathcal{L}) \longrightarrow \Omega^{\bullet}. \tag{9.1.5}$$

The product in the domain of $\widehat{f}$ now is the *fiber product*.

For later use, let us find the stalk-wise expression of $\widehat{f}$ for a particular type of decomposable elements, namely $\widehat{f}(w_1 \otimes u_1, \ldots, w_k \otimes u_k)$, for any $x \in X$ and $w_i \otimes u_i \in \Omega_x^{p_i} \otimes_{\mathcal{A}_x} \mathcal{L}_x \cong \Omega^{p_i}(\mathcal{L})_x$ $(i = 1, \ldots, k)$. In this case there are $U \in \mathfrak{T}_X$, $\theta_i \in \bigwedge^{p_i}(\Omega^1(U))$ and $\ell_i \in \mathcal{L}(U)$, such that $w_i = [\,\theta_i\,]_x = \widetilde{\theta}_i(x)$ and $u_i = \ell_i(x) \equiv \widetilde{\ell}_i(x)$. Hence,

$$w_i \otimes u_i = \widetilde{\theta}_i(x) \otimes \ell_i(x) \equiv \widetilde{\theta}_i(x) \otimes \widetilde{\ell}_i(x) = (\widetilde{\theta_i \otimes \ell_i})(x),$$

which leads to (see also Diagram 1.7)

$$\begin{aligned}
\widehat{f}(w_1 \otimes u_1, \ldots, w_k \otimes u_k) &= \widehat{f}\big((\widetilde{\theta_1 \otimes \ell_1})(x), \ldots, (\widetilde{\theta_k \otimes \ell_k})(x)\big) \\
&= \big(\widehat{f}_U(\theta_1 \otimes \ell_1, \ldots, \theta_k \otimes \ell_k)\big)^{\sim}(x) \\
&= \big(f(\ell_1, \ldots, \ell_k) \cdot \theta_1 \wedge \cdots \wedge \theta_k\big)^{\sim}(x) \\
&= \big(f(\ell_1, \ldots, \ell_k)\big)^{\sim}(x) \cdot (\theta_1 \wedge \cdots \wedge \theta_k)^{\sim}(x) \\
&\equiv f\,(\ell_1(x), \ldots, \ell_k(x)) \cdot \widetilde{\theta}_1(x) \wedge \cdots \wedge \widetilde{\theta}_k(x) \\
&= f(u_1, \ldots, u_k) \cdot w_1 \wedge \cdots \wedge w_k;
\end{aligned}$$

that is, we obtain

$$\widehat{f}(w_1 \otimes u_1, \ldots, w_k \otimes u_k) = f(u_1, \ldots, u_k) \cdot w_1 \wedge \cdots \wedge w_k, \tag{9.1.6}$$

for every $w_i \otimes u_i \in \Omega^{p_i}(\mathcal{L})_x$.

The action of $\mathcal{G}$ on $\Omega^1(\mathcal{L})$ and $\Omega^2(\mathcal{L})$, defined earlier, obviously extends to an action on $\Omega^p(\mathcal{L})$ and $\Omega^{\bullet}(\mathcal{L})$. Thus we prove:

9.1.3 Lemma. *Let $f : \mathcal{L}^{(k)} \to \mathcal{A}$ be a ρ-invariant k-morphism. Then $\widehat{f} : \prod^k \Omega^{\bullet}(\mathcal{L}) \to \Omega^{\bullet}$, defined by (9.1.5), is also a ρ-invariant k-morphism; that is,*

$$\widehat{f}\big(\rho(g).v_1, \ldots, \rho(g).v_k\big) = \widehat{f}(v_1, \ldots, v_k),$$

for every $(g; v_1, \ldots, v_k) \in \mathcal{G} \times_X \big(\prod^k \Omega^{\bullet}(\mathcal{L})\big)$.

Proof. Because of the $\mathcal{A}$-linearity of $\widehat{f}$, with respect to each variable, it suffices to prove the equality of the statement for elements of the form $v_i = w_i \otimes u_i \in \Omega^{p_i}(\mathcal{L})_x$ and any $\mathcal{G} \in \mathcal{G}_x$.

As in the proof of (9.1.6), there are sections $s \in \mathcal{G}(U)$, $\theta_i \in \bigwedge^{p_i}(\Omega^1(U))$ and $\ell_i \in \mathcal{L}(U)$, such that $g = s(x) \equiv \widetilde{s}(x)$, $w_i = \widetilde{\theta}_i(x)$ and $u_i = \ell_i(x) \equiv \widetilde{\ell}_i(x)$. Hence, by the analog of (8.1.8) for decomposable tensors containing forms of higher degree,

$$\rho(g).v_i = \big((1 \otimes \rho(s))(\theta_i \otimes \ell_i)\big)^{\sim}(x) = \big(\theta_i \otimes \rho(s)(\ell_i)\big)^{\sim}(x).$$

As a consequence,

$$\begin{aligned}\widehat{f}\big(\rho(g).v_1,\dots,\rho(g).v_k\big) &= \widehat{f}\big((\theta_1\otimes\rho(s)(\ell_1))^{\sim}(x),\dots,(\theta_k\otimes\rho(s)(\ell_k))^{\sim}(x)\big)\\ &= \big(\widehat{f}_U\big(\theta_1\otimes\rho(s)(\ell_1),\dots,\theta_k\otimes\rho(s)(\ell_k)\big)\big)^{\sim}(x)\\ &= f\big(\rho(s)(\ell_1),\dots,\rho(s)(\ell_k)\big)(x)\cdot(\theta_1\wedge\cdots\wedge\theta_k)^{\sim}(x).\end{aligned}$$

However, (3.3.1′) and the ρ-invariance of f yield

$$\begin{aligned}f\big(\rho(s)(\ell_1),\dots,\rho(s)(\ell_k)\big)(x) &= f\big(\rho(s)(\ell_1)(x),\dots,\rho(s)(\ell_k)(x)\big)\\ &= f\big((s.\ell_1)(x),\dots,(s.\ell_k)(x)\big)\\ &= f(s(x).\ell_1(x),\dots,s(x).\ell_k(x))\\ &= f(\ell_1(x),\dots,\ell_k(x))\\ &= f(\ell_1,\dots,\ell_k)(x).\end{aligned}$$

Therefore, combining the preceding equalities, we have that

$$\begin{aligned}\widehat{f}\big(\rho(g).v_1,\dots,\rho(g).v_k\big) &= f(\ell_1,\dots,\ell_k)(x)\cdot(\theta_1\wedge\cdots\wedge\theta_k)^{\sim}(x)\\ &\equiv f(\ell_1,\dots,\ell_k)^{\sim}(x)\cdot(\theta_1\wedge\cdots\wedge\theta_k)^{\sim}(x)\\ &= \big(\widehat{f}_U(\theta_1\otimes\ell_1,\dots,\theta_k\otimes\ell_k)\big)^{\sim}(x)\\ &= \widehat{f}(v_1,\dots,v_k).\end{aligned}$$

This completes the proof. □

From the set of all k-morphisms $f:\mathcal{L}^{(k)}\to\mathcal{A}$ we single out the ones whose corresponding $\widehat{f}$'s have an appropriate behavior with respect to the exterior product and the differential. Namely, we denote by

$$I^k(\mathcal{G}) \tag{9.1.7}$$

the set of *ρ-invariant symmetric k-morphisms* $f:\mathcal{L}^{(k)}\to\mathcal{A}$ such that the induced morphisms $\widehat{f}$ satisfy the following conditions:

(IN. 1)
$$d\big(\widehat{f}(v_1,\dots,v_k)\big)=\sum_{i=1}^{k}\widehat{f}(v_1,\dots,\boldsymbol{d}v_i,\dots,v_k),$$

for every $(v_1,\dots,v_k)\in\prod^k\Omega^2(\mathcal{L})$, and

(IN. 2)
$$\sum_{i=1}^{k}\widehat{f}(v_1,\dots,v_i\bigwedge v,\dots,v_k)=0,$$

for every $(v; v_1, \ldots, v_k) \in \Omega^1(\mathcal{L}) \times_X \prod^k \Omega^2(\mathcal{L})$.

For simplicity, in (IN. 1) we have set $d = d^{2k}$ and $\boldsymbol{d} = \boldsymbol{d}^2$. The exterior product in (IN. 2) is defined in (8.1.3). We have restricted our considerations to 2-forms since these are actually needed in what follows. Analogous formulas for forms of higher degree presuppose the existence of higher order differentials $\boldsymbol{d}^p, p \geq 2$.

Clearly, $I^k(\mathcal{G})$ is an $\mathcal{A}$-module. The notation originates from the classical case of Ad-invariant symmetric polynomials.

9.2. From $I^k(\mathcal{G})$ to closed forms

In addition to the assumptions of the previous section, we consider a fixed $\mathcal{G}$-principal sheaf $(\mathcal{P}, \mathcal{G}, X, \pi)$ endowed with a connection D. We denote by R the curvature of D and by (Ω_α) the local curvature forms defined over a (fixed) local frame $\mathcal{U} = (U_\alpha)$ of $\mathcal{P}$.

Thinking of $\Omega^2(\mathcal{L})$ as naturally imbedded in $\Omega^{\bullet}(\mathcal{L})$, for each $\alpha \in I$, we take the induced morphism of sections

$$\widehat{f}_\alpha := \overline{(\widehat{f}\,)}_{U_\alpha} : \underbrace{\Omega^2(\mathcal{L})(U_\alpha) \times \cdots \times \Omega^2(\mathcal{L})(U_\alpha)}_{k-\text{factors}} \longrightarrow \Omega^{2k}(U_\alpha)$$

and define the local forms (viz. sections)

$$\widehat{f}(\Omega_\alpha) := \widehat{f}_\alpha(\Omega_\alpha, \ldots, \Omega_\alpha) \in \Omega^{2k}(U_\alpha), \qquad \alpha \in I. \tag{9.2.1}$$

Therefore, evaluating (9.2.1) at any $x \in U_\alpha$, we see that

$$\widehat{f}(\Omega_\alpha)(x) = \widehat{f}_\alpha(\Omega_\alpha, \ldots, \Omega_\alpha)(x) = \widehat{f}(\Omega_\alpha(x), \ldots, \Omega_\alpha(x)), \tag{9.2.2}$$

where

$$\Omega_\alpha(x) \in \Omega^2(\mathcal{L})_x = \left(\Omega^2 \otimes_{\mathcal{A}} \mathcal{L}\right)_x \cong \Omega^2_x \otimes_{\mathcal{A}_x} \mathcal{L}_x.$$

9.2.1 Lemma. *The local 2k-forms given by (9.2.1) coincide on the overlappings; that is,*

$$\widehat{f}(\Omega_\alpha) = \widehat{f}(\Omega_\beta) \qquad \textit{over} \quad U_{\alpha\beta} \neq \emptyset.$$

As a result, the cochain $\big(\widehat{f}(\Omega_\alpha)\big) \in C^0(\mathcal{U}, \Omega^{2k})$ *is a 0-cocycle which determines a global 2k-form, denoted by*

$$f(D) \in \Omega^{2k}(X). \tag{9.2.3}$$

Proof. For every $x \in U_{\alpha\beta}$, the compatibility condition (8.2.4), along with the ρ-invariance of $\widehat{f}$ and (9.2.2), implies that

$$\begin{aligned}\widehat{f}(\Omega_\beta)(x) &= \widehat{f}(\Omega_\beta(x), \ldots, \Omega_\beta(x)) \\ &= \widehat{f}\big((\rho(g_{\alpha\beta}^{-1}) \cdot \Omega_\alpha)(x), \ldots, (\rho(g_{\alpha\beta}^{-1}) \cdot \Omega_\alpha)(x)\big) \\ &= \widehat{f}(\Omega_\alpha(x), \ldots, \Omega_\alpha(x)) \\ &= \widehat{f}(\Omega_\alpha)(x). \end{aligned}$$ □

Note. The notation (9.2.3) reminds us that the form (section) at hand is derived from the morphisms f and $\widehat{f}$ by means of the connection D (and its curvature).

9.2.2 Proposition. *The global form $f(D)$ of Lemma 9.2.1 is* ***closed****, i.e.,*

$$d\big(f(D)\big) = 0,$$

where d is now the induced morphism on sections

$$d \equiv \overline{(d^{2k})}_X : \Omega^{2k}(X) := (\textstyle\bigwedge^{2k}\Omega^1)(X) \longrightarrow (\textstyle\bigwedge^{2k+1}\Omega^1)(X) =: \Omega^{2k+1}(X).$$

Proof. Let x be any point of X. If, for instance, $x \in U_\alpha$, we check that

$$\begin{aligned} d\big(f(D)\big)(x) &= d\big(f(D)(x)\big) = d\big(\widehat{f}(\Omega_\alpha)(x)\big) \\ &= d\big(\widehat{f}(\Omega_\alpha(x), \ldots, \Omega_\alpha(x))\big) \\ &= \sum_{i=1}^{k} \widehat{f}\big(\Omega_\alpha(x), \ldots, \boldsymbol{d}(\Omega_\alpha(x)), \ldots, \Omega_\alpha(x)\big), \end{aligned}$$

where the summation is taken over the indices $i = 1, \ldots, k$ indicating the i-th place that $\boldsymbol{d}(\Omega_\alpha(x))$ occupies each time. Hence, the local Bianchi identities (8.3.13) and the Properties (IN. 1) – (IN. 2) transform the last equality into

$$d\big(f(D)\big)(x) = \sum_{i=1}^{k} \widehat{f}\big(\Omega_\alpha(x), \ldots, \Omega_\alpha(x) \wedge \omega_\alpha(x), \ldots, \Omega_\alpha(x)\big) = 0.$$ □

Stating the previous proposition in a different way, we have that

$$f(D) \in (\ker d)(X) \equiv (\ker d^{2k})(X) \cong \ker \overline{(d^{2k})}_X \subseteq \Omega^{2k}(X). \tag{9.2.4}$$

9.3. The effect of pull-back

This section deals with some technicalities needed in the sequel and paves the way towards our main objective, namely the construction of certain characteristic classes.

Let $\phi : Y \to X$ be a continuous map, where X is the topological space over which all the considerations of the previous sections apply. Our aim is to compare the $2k$-form $f(D)$ (see (9.2.3)) with the analogous form $\phi^*(f)(D^*) \equiv \phi^*(f)(f^*(D))$ obtained by pulling back, via ϕ, all the entities involved in the construction of $f(D)$. For the identification $D^* \equiv f^*(D)$ we refer to (6.5.1) and (6.5.2).

To this end, we first observe that the pull-back of a k-morphism $f \in I^k(\mathcal{G})$ by ϕ may be thought of as the k-morphism

(9.3.1) $$\phi^*(f) : \prod^k \phi^*(\mathcal{L}) := \underbrace{\phi^*(\mathcal{L}) \times_X \cdots \times_X \phi^*(\mathcal{L})}_{k-\text{factors}} \longrightarrow \phi^*(\mathcal{A}),$$

which, after the obvious identification

(9.3.1′) $$Y \times_X \big(\prod^k \mathcal{L}\big) \cong \prod^k (Y \times_X \mathcal{L}) = \prod^k \phi^*(\mathcal{L}),$$

is stalk-wise given by

(9.3.1″) $$\phi^*(f)\big((y, u_1), \ldots, (y, u_k)\big) = \big(y, f(u_1, \ldots, u_k)\big),$$

for every $(y, u_1, \ldots, u_k)$ with $\pi_{\mathcal{L}}(u_i) = \phi(y)$ ($\pi_{\mathcal{L}}$: the projection of $\mathcal{L}$). Thus $\phi^*(f) \in I^k(\phi^*(\mathcal{G}))$. In analogy to (9.1.5), $\phi^*(f)$ induces the k-morphism

(9.3.2) $$\widehat{\phi^*(f)} : \prod^k \phi^*(\Omega^{\bullet})(\phi^*(\mathcal{L})) \longrightarrow \phi^*(\Omega^{\bullet}).$$

By an obvious extension of Lemma 3.5.1 we have the identification (see also (8.5.9))

$$\phi^*\big(\Omega^p(\mathcal{L})\big) = \phi^*(\Omega^p \otimes_{\mathcal{A}} \mathcal{L}) \overset{\boldsymbol{\tau}_p}{\cong} \phi^*(\Omega^p) \otimes_{\phi^*(\mathcal{A})} \phi^*(\mathcal{L}) = \phi^*(\Omega^p)(\phi^*(\mathcal{L})).$$

In particular, for any $(y, w \otimes u) \in \{y\} \times \Omega^p_{\phi(y)} \otimes_{\mathcal{A}_{\phi(y)}} \mathcal{L}_{\phi(y)}$, (3.5.5′) implies

(9.3.3) $$\boldsymbol{\tau}_p(y, w \otimes u) = \big(f^*_{\Omega,y} \otimes f^*_{\mathcal{L},y}\big) = (y, w) \otimes (y, u),$$

for every $w \in \Omega^p_{\phi(y)}$ and $u \in \mathcal{L}_{\phi(y)}$; or, identifying $\boldsymbol{\tau}_p(y, w \otimes u)$ with $(y, w \otimes u)$,

(9.3.3′) $$(y, w \otimes u) \equiv (y, w) \otimes (y, u).$$

On the other hand, following an argumentation similar to that of Lemma 3.5.1, we obtain analogous results for the pull-back of the exterior powers of Ω. Namely,

$$\phi^*(\Omega^p \bigwedge_{\mathcal{A}} \Omega^q) \cong \phi^*(\Omega^p) \bigwedge_{\phi^*(\mathcal{A})} \phi^*(\Omega^q)$$

resulting from the isomorphism

$$(y, w \wedge w') \xrightarrow{\simeq} (y, w) \wedge (y, w'), \tag{9.3.4}$$

for every $y \in Y$ and $w, w' \in \Omega_{\phi(y)}$. Thus we can write

$$(y, w \wedge w') \equiv (y, w) \wedge (y, w'). \tag{9.3.4$'$}$$

Therefore, working stalk-wise, as in the proof of (9.1.6), we check that

$$\begin{aligned}
&\widehat{\phi^*(f)}\big((y, w_1) \otimes (y, u_1), \ldots, (y, w_k) \otimes (y, u_k)\big) = \\
&= \phi^*(f)\big((y, u_1), \ldots, (y, u_k)\big) \cdot (y, w_1) \wedge \ldots \wedge (y, w_k) \\
\text{(see (9.3.1$''$))} \quad &= \phi^*(f)((y, u_1, \ldots, u_k)) \cdot (y, w_1) \wedge \ldots \wedge (y, w_k) \\
\text{(see (9.3.4$'$))} \quad &= \big(y, f(u_1, \ldots, u_k)\big) \cdot (y, w_1 \wedge \ldots \wedge w_k) \\
&= \big(y, \widehat{f}(w_1 \otimes u_1, \ldots, w_k \otimes u_k)\big) \\
&= \phi^*(\widehat{f}\,)(y, w_1 \otimes u_1, \ldots, w_k \otimes u_k) \\
\text{(see (9.3.3$'$))} \quad &= \phi^*(\widehat{f}\,)\big((y, w_1) \otimes (y, u_1), \ldots, (y, w_k) \otimes (y, u_k)\big).
\end{aligned}$$

Hence, by $\mathcal{A}$-linear extension, one infers that

$$\widehat{\phi^*(f)} = \phi^*(\widehat{f}\,), \tag{9.3.5}$$

within the aforementioned identifications.

Now, given a connection D on the principal sheaf $\mathcal{P}$, we already know that $\phi^*(\mathcal{P})$ is a principal sheaf (see Example 4.1.9(c)), endowed with the connection D^* which can be identified with the morphism $\phi^*(D)$, i.e.,

$$D^*(y, p) = (\tau \circ \phi^*(D))(y, p) \equiv \phi^*(D)(y, p) = (y, D(p)), \qquad (y, p) \in Y \times_X \mathcal{P}$$

(see Proposition 6.5.1 and the ensuing comments).

By the same token, the curvature R^* of D^* identifies with the morphism $\phi^*(R)$ (see (8.5.20) – (8.5.20$''$)). The curvature forms (Ω^*_α), defined over the local frame $\mathcal{V} = \{\phi^{-1}(U_\alpha) \,|\, U_\alpha \in \mathcal{U}\}$, are given by (see (8.5.21) and (8.5.21$'$))

$$\Omega^*_\alpha(y) \equiv \phi^*_{U_\alpha}(\Omega_\alpha)(y) = \big(y, \Omega_\alpha(\phi(y))\big), \qquad y \in \phi^{-1}(U_\alpha). \tag{9.3.6}$$

As a result, using the previous identifications, we have in analogy to (9.2.1) and (9.2.2):

$$\begin{aligned}\big(\widehat{\phi^*(f)}(\Omega^*_\alpha)\big)(y) &= \widehat{\phi^*(f)}\big(\Omega^*_\alpha(y), \dots, \Omega^*_\alpha(y)\big)\\ &= \phi^*(\widehat{f}\,)\big((y, \Omega_\alpha(\phi(y))), \dots (y, \Omega_\alpha(\phi(y)))\big)\\ &= \phi^*(\widehat{f}\,)\big(y, \Omega_\alpha(\phi(y)), \dots, \Omega_\alpha(\phi(y))\big)\\ &= \Big(y, \widehat{f}\big(\Omega_\alpha(\phi(y)), \dots, \Omega_\alpha(\phi(y))\big)\Big)\\ &= \Big(y, \widehat{f}\big(\Omega_\alpha(\phi(y))\big)\Big)\\ &= \big(\phi^*_{U_\alpha}(\widehat{f}(\Omega_\alpha))\big)(y),\end{aligned}$$

for every $y \in \phi^{-1}(U_\alpha)$. Thus, within appropriate isomorphisms,

$$\widehat{\phi^*(f)}(\Omega^*_\alpha) = \widehat{\phi^*(f)}\big(\phi^*_{U_\alpha}(\Omega_\alpha)\big) = \phi^*_{U_\alpha}\big(\widehat{f}(\Omega_\alpha)\big). \tag{9.3.7}$$

Moreover, for every $y \in \phi^{-1}(U_{\alpha\beta})$, Lemma 9.2.1 and equality (9.3.7) imply

$$\begin{gathered}\big(\widehat{\phi^*(f)}(\Omega^*_\alpha)\big)(y) = \phi^*_{U_\alpha}\big(\widehat{f}(\Omega_\alpha)\big)(y) = \big(y, \widehat{f}(\Omega_\alpha)(f(y))\big) = \\ \big(y, \widehat{f}(\Omega_\beta)(f(y))\big) = \phi^*_{U_\beta}\big(\widehat{f}(\Omega_\beta)\big)(y) = \big(\widehat{\phi^*(f)}(\Omega^*_\beta)\big)(y);\end{gathered}$$

in other words,

$$\widehat{\phi^*(f)}(\Omega^*_\alpha) = \widehat{\phi^*(f)}(\Omega^*_\beta).$$

Hence, the forms (9.3.7), for all $\alpha \in I$, define a global form denoted by

$$\phi^*(f)(D^*) \equiv \phi^*(f)(\phi^*(D)) \in \phi^*(\Omega^{2k})(Y) \cong \phi^*(\Omega)^{2k}(Y).$$

This is actually the pull-back of $f(D)$, i.e.,

$$\phi^*(f)(D^*) = \phi^*(f(D)). \tag{9.3.8}$$

Indeed, for any $y \in Y$ with, say $y \in \phi^{-1}(U_\alpha)$, it follows from (9.3.6), (9.3.7), and the definition of $f(D)$:

$$\begin{gathered}\big(\phi^*(f)(D^*)\big)(y) = \big(\widehat{\phi^*(f)}(\Omega^*_\alpha)\big)(y) = \big(\phi^*_{U_\alpha}(\widehat{f}(\Omega_\alpha))\big)(y) = \\ \big(y, \widehat{f}(\Omega_\alpha)(\phi(y))\big) = \ \big(y, f(D)(\phi(y))\big) = \phi^*\big(f(D)\big)(y).\end{gathered}$$

Summarizing, we have proved the following:

9.3.1 Theorem. *Let $\mathcal{P} \equiv (\mathcal{P}, \mathcal{G}, X, \pi)$ be a principal sheaf, equipped with a connection D, with curvature R and local curvature forms (Ω_α), over a local frame $\mathcal{U}$. Also, let $f \in I^k(\mathcal{G})$ be a k-morphism and $f(D) \in \Omega^{2k}(X)$ the $2k$-form obtained from f and the 0-cocycle $(\widehat{f}(\Omega_\alpha)) \in \check{Z}^0(\mathcal{U}, \Omega^{2k})$. If $\phi : Y \to X$ is a continuous map, then the pull-back k-morphism $\phi^*(f) \in I^k(\phi^*(\mathcal{G}))$ and the pull-back connection $D^* \equiv \phi^*(D)$ on the principal sheaf $\phi^*(\mathcal{P}) \equiv (\phi^*(\mathcal{P}), \phi^*(\mathcal{G}), Y, \pi^*)$, with curvature R^* and local curvature forms $\{\Omega^*_\alpha \equiv \phi^*_{U_\alpha}(\Omega_\alpha) \,|\, \alpha \in I\}$, determine a $2k$-form*

$$\phi^*(f)(D^*) \equiv \phi^*(f)(\phi^*(D)) \in \phi^*(\Omega^{2k})(Y) \cong \phi^*(\Omega)^{2k}(Y),$$

which is obtained from the 0-cocycle $\widehat{\phi^*(f)}\big(\phi^*_{U_\alpha}(\Omega_\alpha)\big) \in \check{Z}^0(\mathcal{V}, \phi^*(\Omega^{2k}))$ and coincides with the pull-back of $f(D)$. Here $\mathcal{V} = \{\phi^{-1}(U_\alpha) \,|\, U_\alpha \in \mathcal{U}\}$.

Proposition 9.2.2 and routine calculations now yield:

9.3.2 Corollary. *The form $\phi^*(f)(\phi^*(D)) \equiv \phi^*(f)(D^*) \in \phi^*(\Omega^{2k})(Y)$ is closed with respect to the differential induced on global forms by the pull-back differential $\phi^*(d) : \phi^*(\Omega^{2k}) \to \phi^*(\Omega^{2k+1})$; that is,*

$$\phi^*(d)\big(\phi^*(f)(\phi^*(D))\big) = 0.$$

Equivalently,

$$\begin{aligned} \phi^*(f)(\phi^*(D)) \in \big(\ker \phi^*(d)\big)(Y) = \\ \big(\phi^*(\ker d)\big)(Y) = \big(Y \times_X (\ker d)\big)(Y), \end{aligned} \tag{9.3.9}$$

if, as in Proposition 9.2.2, $d \equiv \overline{(d^{2k})}_X$.

9.4. Cohomology classes from *k*-morphisms

In addition to the conditions of (9.1.1), we assume that X is a *generalized de Rham p-space*; hence, by definition, we have the exact sequence (2.5.17), repeated here for convenience, namely

$$0 \longrightarrow \ker d^0 \hookrightarrow \mathcal{A} \xrightarrow{d^0} \Omega^1 \xrightarrow{d^1} \Omega^2 \longrightarrow \cdots \longrightarrow \Omega^p \xrightarrow{d^p} d^p\Omega^p \longrightarrow 0. \tag{9.4.1}$$

Let $\omega \in \Omega^p(X)$ be a *closed* p-form, i.e., as in Proposition 9.2.2, $d\omega \equiv \overline{(d^p)}_X \omega = 0$. Then, the exactness of

$$0 \longrightarrow \ker d^p \overset{i}{\hookrightarrow} \Omega^p \xrightarrow{d^p} d^p\Omega^p \longrightarrow 0, \tag{9.4.2}$$

as part of (9.4.1), implies that

$$\omega \in (\ker d^p)(X) = (d^{p-1}\Omega^{p-1})(X). \tag{9.4.3}$$

Thus, ω may be regarded as a 0-zero cocycle, i.e., $\omega \in \check{Z}^0(\mathcal{U}, d^{p-1}\Omega^{p-1})$, for some open covering $\mathcal{U}$ of X.

We consider the order $p-1$ analog of (9.4.2), i.e.

$$0 \longrightarrow \ker d^{p-1} \overset{i}{\hookrightarrow} \Omega^{p-1} \xrightarrow{d^{p-1}} d^{p-1}\Omega^{p-1} \longrightarrow 0 \tag{9.4.4}$$

and the next diagram, consisting of commutative sub-diagrams (compare with Diagram 1.15).

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \check{C}^2(\mathcal{U}, \ker d^{p-1}) & \xrightarrow{i} & \check{C}^2(\mathcal{U}, \Omega^{p-1}) & \xrightarrow{d^{p-1}} & d^{p-1}\big(\check{C}^2(\mathcal{U}, \Omega^{p-1})\big) & \longrightarrow & 0 \\
 & & \delta \uparrow & (\mathbf{I}) & \uparrow \delta & & \uparrow \delta & & \\
0 & \longrightarrow & \check{C}^1(\mathcal{U}, \ker d^{p-1}) & \xrightarrow{i} & \check{C}^1(\mathcal{U}, \Omega^{p-1}) & \xrightarrow{d^{p-1}} & d^{p-1}\big(\check{C}^1(\mathcal{U}, \Omega^{p-1})\big) & \longrightarrow & 0 \\
 & & \delta \uparrow & (\mathbf{II}) & \uparrow \delta & (\mathbf{III}) & \uparrow \delta & & \\
0 & \longrightarrow & \check{C}^0(\mathcal{U}, \ker d^{p-1}) & \xrightarrow{i} & \check{C}^0(\mathcal{U}, \Omega^{p-1}) & \xrightarrow{d^{p-1}} & d^{p-1}\big(\check{C}^0(\mathcal{U}, \Omega^{p-1})\big) & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

DIAGRAM 9.1.

For simplicity, the morphisms between cochains in the horizontal sequences of the diagram, induced by the morphisms i and d^{p-1} of (9.4.4), are denoted by the same symbols, whereas all the coboundary operators in the vertical sequences are denoted by δ. The symbol $\check{C}$ has been used to emphasize that we are dealing with Čech cochains and, later on, with Čech cohomology.

The bijectivity of d^{p-1} in (9.4.4) and the paracompactness of X imply that $\omega \in \check{Z}^0(\mathcal{U}, d^{p-1}\Omega^{p-1}) \subseteq \check{C}^0(\mathcal{U}, d^{p-1}\Omega^{p-1})$, is *refinement-liftable* (see Subsection 1.6.2). In other words, there exists a 0-cochain $(\theta_\alpha) \in \check{C}^0(\mathcal{U}, \Omega^{p-1})$ such that $d^{p-1}((\theta_\alpha)) = (d^{p-1}\theta_\alpha) = \omega$ (see (1.6.31) and the relevant definition).

It should be noted that (θ_α) is generally defined over a refinement, say $\mathcal{V}$, of the initial covering $\mathcal{U}$, and ω is now thought of as a 0-cocycle over the same refinement. In spite of this change, for notational convenience, we retain the same symbol for the coverings, as similar changes will appear at the successive steps of the procedure we are applying.

By the commutativity of sub-diagram (III), we have that

$$d^{p-1}\big(\delta((\theta_\alpha))\big) = \delta(\omega) = 0,$$

which means that the 1-cochain $\delta((\theta_\alpha)) \in \check{C}^1(\mathcal{U}, \Omega^{p-1})$ lies in the kernel of d^{p-1}, i.e.,

$$\delta((\theta_\alpha)) \in \ker\big\{d^{p-1} : \check{C}^1(\mathcal{U}, \Omega^{p-1}) \longrightarrow d^{p-1}\big(\check{C}^1(\mathcal{U}, \Omega^{p-1})\big)\big\}.$$

Therefore, in virtue of the exactness of the middle sequence, there exists a 1-cochain

$$\eta = (\eta_{\alpha\beta}) \in \check{C}^1(\mathcal{U}, \ker d^{p-1}) \tag{9.4.5}$$

such that

$$i(\eta) = \delta((\theta_\alpha)). \tag{9.4.6}$$

On the other hand, the commutative sub-diagram (I) yields

$$\delta(i(\eta)) = i(\delta(\eta))$$

or, by (9.4.6),

$$\delta(i(\eta)) = i(\delta(\eta)) = (\delta \circ \delta)((\theta_\alpha)) = 0,$$

which implies that

$$\begin{aligned} i(\eta) \in \ker\big\{\delta : \check{C}^1(\mathcal{U}, \Omega^{p-1}) \longrightarrow \check{C}^2(\mathcal{U}, \Omega^{p-1})\big\} \\ = \check{Z}^1(\mathcal{U}, \Omega^{p-1}) \subseteq \check{C}^1(\mathcal{U}, \Omega^{p-1}). \end{aligned} \tag{9.4.7}$$

Consequently, (9.4.5) and (9.4.7), together with the exactness of (9.4.1), show that

$$\eta \in \check{Z}^1(\mathcal{U}, \ker d^{p-1}) = \check{Z}^1(\mathcal{U}, \operatorname{im} d^{p-2}) = \check{Z}^1(\mathcal{U}, d^{p-2}\Omega^{p-2}).$$

In brief, we see that, from $\omega \in (d^{p-1}\Omega^{p-1})(X)$, identified with a cocycle $\omega \in \check{Z}^0(\mathcal{U}, d^{p-1}\omega^{p-1})$, we obtain a cochain $(\theta_\alpha) \in \check{C}^0(\mathcal{U}, \Omega^{p-1})$ such that $d^{p-1}((\theta_\alpha)) = \omega$ and $\delta((\theta_\alpha)) = \eta \in \check{Z}^1(\mathcal{U}, d^{p-2}\Omega^{p-2})$.

For η we apply the same procedure using the analog of (9.4.4) for d^{p-2} and the analog of Diagram 9.1, thus we obtain a cocycle

$$\zeta \in \check{Z}^2(\mathcal{U}, \ker d^{p-2}) = \check{Z}^2(\mathcal{U}, d^{p-3}\Omega^{p-3}).$$

Repeating the same method down to $d = d^0$, we finally obtain a 1-cocycle

$$z(\omega) \in \check{Z}^p(\mathcal{U}, \ker d),$$

which determines the cohomology class

$$c(\omega) := [z(\omega)] \in \check{H}^p(X, \ker d).$$

The notations $z(\omega)$ and $c(\omega)$ are introduced in order to remind us that the cocycle and its class both originate from ω.

Regarding the previous construction, one more remark is appropriate here: The 1-cocycle $\eta = (\eta_{\alpha\beta})$ is not uniquely determined by ω, since many 0-cochains (θ_α) can be mapped (by d^{p-1}) to ω. However, η *is uniquely determined up to coboundary.*

Indeed, let $(\theta'_\alpha) \in C^0(\mathcal{U}, \Omega^{p-1})$ be another cochain with $d^{p-1}((\theta'_\alpha)) = \omega$ and let $\eta' \in Z^1(\mathcal{U}, \ker d^{p-1})$ be a cocycle with $i(\eta') = \delta((\theta'_\alpha))$. The last equality, together with (9.4.6), implies that

$$i(\eta - \eta') = \delta((\theta_\alpha)) - \delta((\theta'_\alpha)) = \delta\left((\theta_\alpha - \theta'_\alpha)\right). \tag{9.4.8}$$

Since $d^{p-1}((\theta_\alpha)) = \omega = d^{p-1}((\theta'_\alpha))$ yields $(\theta_\alpha) - (\theta'_\alpha) \in \ker d^{p-1}$, there is a $(\lambda_\alpha) \in \check{C}^0(\mathcal{U}, \ker d^{p-1})$ such that

$$i((\lambda_\alpha)) = (\theta_\alpha - \theta'_\alpha). \tag{9.4.9}$$

Hence, by (9.4.8), (9.4.9), and the commutativity of sub-diagram (II), we have that

$$(i \circ \delta)((\lambda_\alpha)) = (\delta \circ i)((\lambda_\alpha)) = \delta((\theta_\alpha - \theta'_\alpha)) = i(\eta - \eta'),$$

i.e., $\eta - \eta' = \delta((\lambda_\alpha))$, which proves the claim. The same arguments apply down to the last cocycle $z = z(\omega)$, so the class $c(\omega) = [z(\omega)]$ is *well defined.*

Our discussion constitutes the proof of the following result, stated formally for later reference.

9.4.1 Lemma. *Let X be a generalized de Rham p-space. Then a closed p-form $\omega \in \Omega^p(X)$ determines a (p-dimensional) cohomology class*

$$c(\omega) \in \check{H}^p(X, \ker d) \cong H^p(X, \ker d).$$

9.4.2 Remarks. 1) The identification of the cohomology modules in the previous statement is a consequence of the general property of cohomology (1.6.21), since (by Definition 2.5.5) X is *paracompact.*

2) If, in addition, X is a de Rham p-space, then $\ker d = \mathbb{K}$, and the class $c(\omega)$ is analogous to a class of the ordinary de Rham cohomology.

Returning to the $2k$-form $f(D) \in \Omega^{2k}(X)$, defined by Lemma 9.2.1, we see that Proposition 9.2.2 and Lemma 9.4.1 directly lead to:

9.4.3 Corollary. *If X is a generalized de Rham $2k$-space, then the closed $2k$-form $f(D) \in \Omega^{2k}(X)$ determines a $2k$-dimensional cohomology class*

$$c\big(f(D)\big) = [f(D)] \in \check{H}^{2k}(X, \ker d) \cong H^{2k}(X, \ker d). \tag{9.4.10}$$

By its construction, $f(D)$ is derived from a k-morphism $f \in I^k(\mathcal{G})$, by means of a connection D and its curvature, defined on a fixed $\mathcal{G}$-principal sheaf. Therefore, it is natural to ask whether the class (9.4.10) depends on the choice of D. The answer is negative, according to Proposition 9.4.6 below. In preparation of its proof we need the following auxiliary result:

9.4.4 Lemma. *Let D_0 and D_1 be two connections on a principal sheaf $\mathcal{P} \equiv (\mathcal{P}, \mathcal{G}, X, \pi)$. Then their convex combination*

$$D = tD_0 + (1-t)D_1; \qquad t \in [0,\ 1] \subseteq \mathbb{R},$$

is also a connection on $\mathcal{P}$.

Before the proof we explain that, for any $\lambda \in \mathbb{R}$ and any connection D, the morphism $\lambda D : \mathcal{P} \to \Omega^1(\mathcal{L})$ is defined by $(\lambda D)(p) = \lambda \cdot D(p)$, the right-hand side denoting multiplication by scalars after the natural imbedding $i_x : \mathbb{R} \cong \{x\} \times \mathbb{R} \hookrightarrow \mathcal{A}_x$, $x = \pi(p)$. Similar is the meaning of $\lambda\partial$, also needed in the proof.

Proof. We first prove that

$$t \cdot (\rho(g).w) = \rho(g).(tw), \tag{9.4.11}$$

for every $t \in \mathbb{R}$ and $(g, w) \in \mathcal{G} \times_X \Omega(\mathcal{L})$. Indeed, since $\rho(g).w = \Delta(g, w)$ (see (3.3.7)), it suffices to work with the local actions (Δ_U), generating Δ, and given by (3.3.6). Then, for any $s \in \mathcal{G}(U)$ and $\theta \in \Omega^1(U) \otimes_{\mathcal{A}(U)} \mathcal{L}(U)$,

$$(t\Delta_U)(s, \theta) = t \cdot \big((1 \otimes \rho(s))(\theta)\big) = (1 \otimes \rho(s))(t\theta) = \Delta_U(s, t\theta).$$

The proof of the statement is now a routine verification of Definition 6.6.1 by taking into account (9.4.11). □

For immediate use we also have the following, easy to prove, proposition relating the cohomology class of $f(D)$ with the class of $\phi^*(f)(f^*(D)) \equiv \phi^*(f)(D^*)$. To this end, observe that, for a continuous map $\phi : Y \to X$ and any $\mathcal{A}$-module $\mathcal{S}$, the pull-back functor ϕ^* induces the morphism of cohomology modules

$$\phi^{\#} : \check{H}^*(X, \mathcal{S}) \longrightarrow \check{H}^*(Y, \phi^*(\mathcal{S})) : [\, z\,] \mapsto \phi^{\#}([\, z\,]) := [\phi^*(z)], \tag{9.4.12}$$

for every cocycle $z \in \check{Z}^p(\mathcal{U}, \mathcal{S})$, $p \in \mathbb{Z}_0^+$, over an open covering $\mathcal{U}$ of X. More precisely, for any $(\alpha_0, \ldots, \alpha_p)$ and $y \in \phi^{-1}(U_{\alpha_0} \cap \ldots \cap U_{\alpha_p})$,

$$\phi^*(z)_{\alpha_0 \cdots \alpha_p}(y) := \big(y, z_{\alpha_0 \cdots \alpha_p}(\phi(y))\big);$$

that is, using the notation (1.4.3),

$$\phi^*(z)_{\alpha_0 \cdots \alpha_p} = \phi_U^*(z_{\alpha_0 \cdots \alpha_p}), \qquad U := U_{\alpha_0} \cap \ldots \cap U_{\alpha_p}.$$

For further details one may consult, e.g., Godement [33, p. 199].

Note. Many authors use the symbol ϕ^* in place of $\phi^{\#}$. To avoid any confusion with the pull-back functor, we adhere to the latter notation.

Combining the constructions of Section 9.3 with the previous considerations, we obtain:

9.4.5 Proposition. *Let $\mathcal{P} \equiv (\mathcal{P}, \mathcal{S}, X, \pi)$ be a principal sheaf endowed with a connection D. If X is a generalized de Rham $2k$-space and $\phi : Y \to X$ a continuous map, then*

$$\phi^{\#}\big(c(f(D))\big) = c\big(\phi^*(f)(D^*)\big) \equiv c\big(\phi^*(f)(f^*(D))\big),$$

for every k-morphism $f \in I^k(\mathcal{G})$.

Of course, one needs to prove that if X is a generalized de Rham $2k$-space, then so is Y. However, it is not hard to show that the exactness of (9.4.1) implies the exactness of the sequence obtained by pulling back (via ϕ) the modules and morphisms of the former.

We now return to the question raised after Corollary 9.4.3, namely the proof of the following result.

9.4.6 Proposition. *Let $f \in I^k(\mathcal{G})$ and a principal sheaf $\mathcal{P}$ over a generalized de Rham $2k$-space X. Then the class $c(f(D)) \in H^{2k}(X, \ker d)$ is independent of the choice of the connection D on $\mathcal{P}$.*

Proof. If D_0 and D_1 are two arbitrary connections on $\mathcal{P}$, we shall show that $c(f(D_0)) = c(f(D_1))$ by a *homotopy argument.*

More precisely, we consider the closed interval $I = [0,1] \subseteq \mathbb{R}$, the topological space $X \times I$ and the projection to the first factor

$$p_X : X \times I \longrightarrow X : (x,t) \mapsto x.$$

For each $t \in I$, we also consider the (continuous) inclusion map

$$h_t : X \longrightarrow X \times I : x \mapsto (x,t).$$

We have two immediate results: Firstly, equality

$$p_X \circ h_t = id_X, \tag{9.4.13}$$

is valid for every $t \in I$. Secondly,

(9.4.14) the maps h_0 and h_1 are *homotopic* by means of the homotopy $H := id : X \times I \longrightarrow X \times I$.

The pull-back $p_X^*(\mathcal{P})$ of $\mathcal{P} \equiv (\mathcal{P}, \mathcal{G}, X, \pi)$ is a $p_X^*(\mathcal{G})$-principal sheaf over $X \times I$, equipped with the pull-back connections $D_0^* \equiv p_X^*(D_0)$ and $D_1^* \equiv p_X^*(D_1)$ (see Section 6.5 and, in particular, the identification (6.5.2)). Hence, in virtue of Lemma 9.4.4,

$$D^* := tD_0^* + (1-t)D_1^*, \tag{9.4.15}$$

is a connection on $p_X^*(\mathcal{P})$, which can be identified with

$$D^* = tp_X^*(D_0) + (1-t)p_X^*(D_1) = p_X^*\big(tD_0 + (1-t)D_1\big). \tag{9.4.15$'$}$$

Pulling back $p_X^*(\mathcal{P})$ by h_t, we return (up to an isomorphism) to the initial principal sheaf $\mathcal{P}$, since (1.4.6) and (9.4.13) yield

$$h_t^*(p_X^*(\mathcal{P})) = (p_X \circ h_t)^*(\mathcal{P}) = id_X^*(\mathcal{P}) \cong \mathcal{P},$$

for every $t \in I$. By the same token, the connections D^*, D_0^*, D_1^* determine the corresponding pull-back connections, which can be identified, respectively, with

$$h_t^*(D^*), \quad h_t^*(D_0^*) \equiv D_0, \quad h_t^*(D_1^*) \equiv D_1.$$

Hence, setting

$$D_t := h_t^*(D^*), \tag{9.4.16}$$

and pulling-back both sides of (9.4.15′) by h_t, we have that

$$D_t = h_t^*(D^*) \equiv (p_X \circ h_t)^*\big(tD_0 + (1-t)D_1\big) \equiv tD_0 + (1-t)D_1.$$

The new connection D_t on $\mathcal{P}$ determines, in turn, the class $c(f(D_t)) \in H^{2k}(X, \ker d)$, for the given k-morphism f.

On the other hand, the pull-back connection $D^* \equiv p_X^*(D)$ on $p_X^*(\mathcal{P})$ and the k-morphism $p_X^*(f) \in I^k(p_X^*(\mathcal{G}))$ determine the class

$$c\big(p_X^*(f)(D^*)\big) \in H^{2k}\big(X \times I, \ker p_X^*(d)\big) = H^{2k}\big(X \times I, p_X^*(\ker d)\big),$$

where the equality concerning the kernels is routinely verified. Thus, by applying the cohomology functor $h_t^{\#}$, Proposition 9.4.5 and equalities (9.4.13), (9.4.16) imply that

$$h_t^{\#}\big(c\big(p_X^*(f)(D^*)\big)\big) = c\big(h_t^*\big(p_X^*(f)\big)\big(h_t^*(D^*)\big)\big) = c\big(f(D_t)\big).$$

Therefore, for $t = 0$, 1, we obtain the equalities

$$c(f(D_0)) = h_0^{\#}\big(c\big(p_X^*(f)(D^*)\big)\big), \tag{9.4.17}$$

$$c(f(D_1)) = h_1^{\#}\big(c\big(p_X^*(f)(D^*)\big)\big). \tag{9.4.18}$$

Since, by (9.4.14), h_0 and h_1 are homotopic, the general theory of cohomology (see, e.g., Spanier [116, p. 240], Mallios [62, Vol. II, p. 161]) implies that

$$h_0^{\#}\big(c\big(p_X^*(f)(D^*)\big)\big) = h_1^{\#}\big(c\big(p_X^*(t)(D^*)\big)\big),$$

which, combined with (9.4.17) and (9.4.18), leads to

$$c\big(f(D_0)\big) = c\big(f(D_1)\big).$$

This completes the proof. □

9.4.7 Remarks. 1) The vector sheaf and vector bundle analogs of Proposition 9.4.6 can be found in Mallios [62, p. 259] and Milnor-Stasheff [74, p. 298], respectively.

2) In the classical framework, the independence of $c(f(D))$ from the choice of the connection D can be shown by using the so-called *transgression formula* (see, e.g., Kobayashi-Nomizu [49, Vol. II], Naber [81]). However, this quite popular approach cannot be applied here.

The results obtained so far in the current section are summarized in the following:

9.4.8 Theorem (Chern-Weil). *Let X be a generalized de Rham $2k$-space and let $\mathcal{P}$ be a $\mathcal{G}$-principal sheaf over X, where $\mathcal{G} \equiv (\mathcal{G}, \rho, \mathcal{L}, \partial)$ is a Lie sheaf of groups. We assume that $\mathcal{P}$ admits connections and, as in (9.1.1), there is a Bianchi datum attached to $\mathcal{G}$. Then each k-morphism $f \in I^k(\mathcal{G})$ and any connection D on $\mathcal{P}$ determine a closed $2k$-form $f(D) \in \Omega^{2k}(X)$, whose corresponding cohomology class $c(f(D)) \in H^{2k}(X, \ker d)$ is independent of the choice of D.*

Owing to the independence of $c\big(f(D)\big)$ from D, we set

$$c(f, \mathcal{P}) := c\big(f(D)\big), \tag{9.4.19}$$

for any connection D on $\mathcal{P}$.

9.4.9 Definition. The cohomology class $c(f, \mathcal{P}) \in H^{2k}(X, \ker d)$ is called the ***characteristic class of $\mathcal{P}$ associated with the k-morphism*** $f \in I^k(\mathcal{G})$.

9.4.10 Proposition. *With the assumptions of Theorem 9.4.8, if $\mathcal{P}$ admits a flat connection, then $c(f, \mathcal{P}) = 0$, for every k-morphism $f \in I^k(\mathcal{G})$.*

Proof. Let D be any flat connection on $\mathcal{P}$. In virtue of Proposition 8.6.2, all the curvature forms Ω_α vanish. Therefore, by Lemma 9.2.1, $f(D) = 0$, which concludes the proof. □

9.4.11 Remark. The same result holds, of course, under any one of the (equivalent) conditions (F. 2) – (F. 5) of Theorem 8.6.12, as well as for any flat $\mathcal{G}$-principal sheaf (see Proposition 8.6.18).

Fixing a principal sheaf $\mathcal{P}$ as in Theorem 9.4.8, we shall show that the assignment of the characteristic class $c(f, \mathcal{P})$ to a k-morphism, for any k, determines a map with some nice properties. To see this, we consider the direct sum of $\mathcal{A}$-modules

$$I^*(\mathcal{G}) := \bigoplus_{k=0}^{\infty} I^k(\mathcal{G}). \tag{9.4.20}$$

It is an $\mathcal{A}$-module, in virtue of Subsection 1.3.2. As usual, $I^0(\mathcal{G}) := \mathcal{A}$.

Similarly, we define the $\mathcal{A}(X)$-modules

$$H^*(X, \ker d) = \bigoplus_{k=0}^{\infty} H^k(X, \ker d), \tag{9.4.21}$$

$$H^{**}(X, \ker d) = \bigoplus_{k=0}^{\infty} H^{2k}(X, \ker d). \tag{9.4.22}$$

Obviously,

$$H^{**}(X, \ker d) \xrightarrow{\subset} H^*(X, \ker d). \tag{9.4.23}$$

It is also clear that the modules (9.4.20) – (9.4.22) are $\mathbb{K}$-vector spaces, since $\mathbb{K}$ is canonically imbedded in $\mathcal{A}$.

By Theorem 9.4.8 we obtain the map

$$\mathfrak{W}_{\mathcal{P}} : I^*(\mathcal{G}) \longrightarrow H^*(X, \ker d) : f \mapsto \mathfrak{W}_{\mathcal{P}}(f) := c(f, \mathcal{P}). \tag{9.4.24}$$

9.4.12 Definition. Let $\mathcal{P}$ be a $\mathcal{G}$-principal sheaf over a generalized de Rham space X, as in Theorem 9.4.8. The map $\mathfrak{W}_{\mathcal{P}}$, defined by (9.4.24), is called the ***Chern - Weil map*** of $\mathcal{P}$.

9.4.13 Proposition. *The Chern - Weil map $\mathfrak{W}_{\mathcal{P}}$ is a $\mathbb{K}$-linear morphism satisfying the following functorial property*

$$\phi^{\#} \circ \mathfrak{W}_{\mathcal{P}} = \mathfrak{W}_{\phi^*(\mathcal{P})} \circ \phi^*,$$

for every continuous map $\phi : Y \to X$. Equivalently, we have the commutative diagram:

$$\begin{array}{ccc} I^*(\mathcal{G}) & \xrightarrow{\ \mathfrak{W}_{\mathcal{P}}\ } & H^*(X, \ker d) \\ \Big\downarrow{\scriptstyle \phi^*} & & \Big\downarrow{\scriptstyle \phi^{\#}} \\ I^*(\phi^*(\mathcal{G})) & \xrightarrow{\ \mathfrak{W}_{\phi^*(\mathcal{P})}\ } & H^*(Y, \ker \phi^*(d)) \end{array}$$

Diagram 9.2

The vertical map ϕ^* in Diagram 9.2 is the morphism induced by the pull-back of k-morphisms (see Section 9.3), whereas $\phi^\#$ is defined by (9.4.12). The functorial property is also known as the ***naturality*** of $\mathfrak{W}_{\mathcal{P}}$.

Proof. The first assertion results from the fact that all the maps involved in the construction of the characteristic classes, in particular the differentials d^p and the coboundary operators δ^p (see also Diagram 9.1) are linear maps with respect to $\mathbb{K}$, for all p's.

The naturality of $\mathfrak{W}_{\mathcal{P}}$ is a straightforward combination of Proposition 9.4.5 and equality (9.4.19): If $f \in I^k(\mathcal{G})$, then

$$\begin{aligned}(\phi^\# \circ \mathfrak{W}_{\mathcal{P}})(f) = \phi^\#\big(c(f,\mathcal{P})\big) = \phi^\#\big(c(f(D))\big) = \\ c\big(\phi^*(f)(f^*(D))\big) = c\big(\phi^*(f),\phi^*(\mathcal{P})\big) = \mathfrak{W}_{\phi^*(\mathcal{P})}\big(\phi^*(f)\big).\end{aligned} \qquad \square$$

We now examine the behavior of the Chern-Weil maps corresponding to two equivalent principal sheaves (see Definition 4.6.1). We always assume that the principal sheaves considered admit connections and that (9.1.1) is in force.

9.4.14 Proposition. *Let $\mathcal{P}$ and $\mathcal{P}'$ be two $\mathcal{G}$-principal sheaves over a generalized de Rham space X. If $\mathcal{P}$ and $\mathcal{P}'$ are equivalent, then*

$$\mathfrak{W}_{\mathcal{P}} = \mathfrak{W}_{\mathcal{P}'}.$$

Proof. Let $h \equiv (h, id_{\mathcal{G}}, id_{\mathcal{L}}, id_X)$ be an isomorphism defining the equivalence of $\mathcal{P}$ and $\mathcal{P}'$. Taking local frames over a common open covering $\mathcal{U} = (U_\alpha)$ of X, h is in bijective correspondence with a 0-cochain $(h_\alpha) \in C^0(\mathcal{U},\mathcal{G})$ satisfying the properties of Theorem 4.4.2.

If D' is an arbitrary connection on $\mathcal{P}'$ with corresponding curvature $R' \equiv (\Omega'_\alpha)$, then $D := D' \circ h$ is the unique connection on $\mathcal{P}$, which is h-conjugate with D' (see Definition 6.4.1). By (8.5.4′), the local curvature forms (Ω_α) of D satisfy

$$\rho(h_\alpha^{-1}).\Omega'_\alpha = \Omega_\alpha, \qquad \alpha \in I.$$

Therefore, for any $f \in I^k(\mathcal{G})$ and $x \in U_\alpha$, equality (9.2.2) and the analog of (3.3.10) for the action (8.1.7) on forms of higher degree, as well as the invariance of $\widehat{f}$ (see Lemma 9.1.3), imply that

$$\begin{aligned}\widehat{f}(\Omega_\alpha)(x) &= \widehat{f}(\Omega_\alpha(x),\dots,\Omega_\alpha(x)) \\ &= \widehat{f}\big(\rho(h_\alpha^{-1}(x))\cdot\Omega'_\alpha(x),\dots,\rho(h_\alpha^{-1}(x))\cdot\Omega'_\alpha(x)\big) \\ &= \widehat{f}(\Omega'_\alpha(x),\dots,\Omega'_\alpha(x)) \\ &= \widehat{f}(\Omega'_\alpha)(x);\end{aligned}$$

that is, $\widehat{f}(\Omega_\alpha) = \widehat{f}(\Omega'_\alpha)$, for all $\alpha \in I$. As a result, $f(D) = f(D')$, and the Chern-Weil Theorem 9.4.8 yields

$$\mathfrak{W}_{\mathcal{P}}(f) = \mathfrak{W}_{\mathcal{P}'}(f),$$

for every $f \in I^k(\mathcal{G})$ and $k \in \mathbb{Z}_0^+$. This completes the proof. □

9.5. The Chern-Weil homomorphism

We shall show that, under suitable conditions, the Chern-Weil map (9.4.24) becomes an algebra homomorphism. For the moment we remain with the assumptions of the previous sections; that is, we assume that X is a generalized de Rham space and $\mathcal{P} \equiv (\mathcal{P}, \mathcal{G}, X, \pi)$ is a principal sheaf admitting connections with curvatures, as ensured by the existence of a Bianchi datum (see (9.1.1)).

For our purpose, we supply $I^*(\mathcal{G})$ with a product in the following way: If $f \in I^k(\mathcal{G})$ and $g \in I^l(\mathcal{G})$, we define the $(k+l)$-morphism $f \odot g$ by

$$\begin{aligned}(f \odot g)(u_1, \ldots, u_k, u_{k+1}, \ldots, u_l) &:= \\ \frac{1}{(k+l)!} \sum_{\sigma \in S_{k+l}} f(u_{\sigma(1)}, \ldots, u_{\sigma(k)}) &\cdot g(u_{\sigma(k+1)}, \ldots, u_{\sigma(k+l)}),\end{aligned} \tag{9.5.1}$$

for every $(u_1, \ldots, u_{k+l}) \in \prod^{k+l} \mathcal{L}$. Here S_{k+l} denotes the group of permutations of $\{1, \ldots, k+l\}$ and the center dot on the right-hand side denotes the multiplication in $\mathcal{A}$.

It is straightforward to verify that $f \odot g \in I^{k+l}(\mathcal{G})$; thus $I^*(\mathcal{G})$, equipped with the multiplication (9.5.1), has the structure of a (graded) *commutative $\mathcal{A}$-algebra with identity*. In particular, $I^*(\mathcal{G})$ is a $\mathbb{K}$-algebra with the same properties, after the canonical imbedding $\mathbb{K} \hookrightarrow \mathcal{A}$.

Given two morphisms $f \in I^k(\mathcal{G})$ and $g \in I^l(\mathcal{G})$, using (9.2.1) and (9.2.2), we construct the forms

$$(\widehat{f \odot g})(\Omega_\alpha) \in \Omega^{2(k+l)}(U_\alpha) = (\textstyle\bigwedge^{2(k+l)} \Omega^1)(U_\alpha), \tag{9.5.2}$$

$$\begin{aligned}\widehat{f}(\Omega_\alpha) \wedge \widehat{g}(\Omega_\alpha) &\in \Omega^{2k}(U_\alpha) \textstyle\bigwedge \Omega^{2l}(U_\alpha) \\ &= (\textstyle\bigwedge^{2k} \Omega^1)(U_\alpha) \textstyle\bigwedge (\textstyle\bigwedge^{2l} \Omega^1)(U_\alpha).\end{aligned} \tag{9.5.3}$$

In order to find the relationship between these forms, we need a few preliminary remarks. For any $p \geq 2$, we know that the sheaf

$$\Omega^p = \underbrace{\Omega^1 \textstyle\bigwedge_{\mathcal{A}} \cdots \textstyle\bigwedge_{\mathcal{A}} \Omega^1}_{p-\text{factors}} = \textstyle\bigwedge^p \Omega^1$$

is generated by the presheaf

$$U \longmapsto \Omega^1(U)\bigwedge_{\mathcal{A}(U)} \cdots \bigwedge_{\mathcal{A}(U)}\Omega^1(U) = \bigwedge^p(\Omega^1(U)). \tag{9.5.4}$$

(For the sake of convenience, in what follows the index $\mathcal{A}$ will be omitted.) We denote by

$$\rho_U^p : \bigwedge^p(\Omega^1(U)) \longrightarrow (\bigwedge^p\Omega^1)(U)$$

the canonical map assigning to each "section" of the presheaf (9.5.4) the corresponding section of $\bigwedge^p\Omega$ over U (see (1.2.2)). For any decomposable element $(\theta_1 \wedge \ldots \wedge \theta_p) \in \bigwedge^p(\Omega^1(U))$ and every $x \in U$, we have that

$$\rho_U^p(\theta_1 \wedge \ldots \wedge \theta_p)(x) = \rho_{U,x}^p(\theta_1 \wedge \ldots \wedge \theta_p) = \theta_1(x) \wedge \ldots \wedge \theta_p(x), \tag{9.5.5}$$

where

$$\rho_{U,x}^p : \bigwedge^p(\Omega(U)) \longrightarrow (\bigwedge^p\Omega)_x \cong \underbrace{\Omega_x\bigwedge \ldots \bigwedge\Omega_x}_{p-\text{factors}}$$

is the canonical map into the stalk. The last term of (9.5.5) actually determines the exterior product on the stalk over x.

Analogously, thinking of $\Omega^{2k}\bigwedge\Omega^{2l}$ as the sheaf generated by the presheaf $U \mapsto \Omega^{2k}(U)\bigwedge\Omega^{2l}(U)$, we have a corresponding canonical map (for every open U)

$$\rho_U^{k,l} : \Omega^{2k}(U)\bigwedge\Omega^{2l}(U) \longrightarrow (\Omega^{2k}\bigwedge\Omega^{2l})(U) \tag{9.5.6}$$

satisfying equality

$$\rho_U^{k,l}(s \wedge t)(x) = \rho_{U,x}^{k,l}(s \wedge t) = s(x) \wedge t(x), \qquad x \in U. \tag{9.5.7}$$

Again the last exterior product is meaningful after the identification of stalks $(\Omega^{2k}\bigwedge\Omega^{2l})_x \cong \Omega_x^{2k}\bigwedge\Omega_x^{2l}$ and because s, t are sections of sheaves.

With the previous notations in mind, (9.5.2) and (9.5.3) are now related as follows.

9.5.1 Lemma. *For any $f \in I^k(\mathcal{G})$ and $g \in I^l(\mathcal{G})$, equality*

$$(\widehat{f \odot g})(\Omega_\alpha) = \rho_{U_\alpha}^{k,l}\big(\widehat{f}(\Omega_\alpha) \wedge \widehat{g}(\Omega_\alpha)\big) \tag{9.5.8}$$

is satisfied for every $\alpha \in I$.

Proof. Using (9.5.7) and(9.2.2), we see that

$$\rho^{k,l}_{U_\alpha}\big(\widehat{f}(\Omega_\alpha)\wedge\widehat{g}(\Omega_\alpha)\big)(x)=\widehat{f}(\Omega_\alpha)(x)\wedge\widehat{g}(\Omega_\alpha)(x)=$$
$$\widehat{f}(\underbrace{\Omega_\alpha(x),\dots,\Omega_\alpha(x)}_{k-\text{times}})\wedge\widehat{g}(\underbrace{\Omega_\alpha(x),\dots,\Omega_\alpha(x)}_{l-\text{times}}),$$

for every $x\in U_\alpha$. Hence, if $\Omega_\alpha=\sum_{i\in I}\theta_i\otimes u_i$,

$$\rho^{k,l}_{U_\alpha}\big(\widehat{f}(\Omega_\alpha)\wedge\widehat{g}(\Omega_\alpha)\big)(x)=$$
$$\Big(\sum_{i_1,\dots,i_k\in I} f(u_{i_1},\dots,u_{i_k})\cdot\theta_{i_1}\wedge\ldots\wedge\theta_{i_k}\Big)\wedge$$
$$\wedge\Big(\sum_{j_1,\dots,j_l\in I} g(u_{j_1},\dots,u_{j_l})\cdot\theta_{j_1}\wedge\ldots\wedge\theta_{j_l}\Big)=\tag{9.5.9}$$
$$\sum_{i_1,\dots,i_{k+l}\in I} f(u_{i_1},\dots,u_{i_k})\cdot g(u_{i_{k+1}},\dots,u_{i_{k+l}})\cdot\theta_{i_1}\wedge\ldots\wedge\theta_{i_{k+l}}.$$

On the other hand,

$$(\widehat{f\odot g})(\Omega_\alpha)(x)=(\widehat{f\odot g})(\underbrace{\Omega_\alpha(x),\dots,\Omega_\alpha(x)}_{(k+l)-\text{times}})=$$
$$\sum_{i_1,\dots,i_{k+l}\in I}(f\odot g)(u_{i_1},\dots,u_{i_{k+l}})\cdot\theta_{i_1}\wedge\ldots\wedge\theta_{i_{k+l}}=\tag{9.5.10}$$
$$\frac{1}{(k+l)!}\sum\sum_{\sigma} f(u_{\sigma_{(i_1)}},\dots,u_{\sigma_{(i_k)}})\cdot g(u_{\sigma_{(i_{k+1})}},\dots,u_{\sigma_{(i_{k+l})}})\cdot\theta_{i_1}\wedge\ldots\wedge\theta_{i_{k+l}},$$

where the first sum in the last row is taken over all $i_1,\dots,i_{k+l}\in I$, and σ is running in the group of permutations S_{k+l}.

Now, using the symmetry of both f and g, along with elementary combinatorics, we check that the coefficients of $\theta_{i_1}\wedge\ldots\wedge\theta_{i_{k+l}}$ in (9.5.9) and (9.5.10) coincide. Thus, by linear extension, we get equality (9.5.8) of the statement. □

9.5.2 Proposition. *For every $f\in I^k(\mathcal{G})$, $g\in I^l(\mathcal{G})$ and any connection D on $\mathcal{P}$, we have that*

$$(f\odot g)(D)=\rho^{k,l}_X\big(f(D)\wedge g(D)\big),\tag{9.5.11}$$

where $\rho^{k,l}_X$ is the canonical map (9.5.6) over X.

Proof. Let any $x \in X$. If $x \in U_\alpha$, for some $\alpha \in I$, then Lemmata 9.2.1, 9.5.1, and equality (9.5.7) imply that

$$(f \odot g)(D)(x) = (\widehat{f \odot g})(\Omega_\alpha)(x) = \rho^{k,l}_{U_\alpha}\big(\widehat{f}(\Omega_\alpha) \wedge \widehat{g}(\Omega_\alpha)\big)(x) =$$
$$\widehat{f}(\Omega_\alpha)(x) \wedge \widehat{g}(\Omega_\alpha)(x) = f(D)(x) \wedge g(D)(x) = \rho^{k,l}_X\big(f(D) \wedge g(D)\big)(x),$$

which concludes the proof. □

To bring to completion the aim of this section, we need to define an appropriate product on $H^*(X, \ker d)$, involving the exterior product of forms (viz. sections), since the classes we are interested in arise from such objects.

However, the cohomology classes $c(\omega) \in \check{H}^p(X, \ker d)$, obtained from the closed forms $\omega \in \Omega^p(X)$, as discussed before (and summarized in) Lemma 9.4.1, may be problematic with regard to this product. As a matter of fact, if ω_1, ω_2 are two closed forms as before, then it is not assured that $c(\omega_1 \wedge \omega_2) = c(\omega_1) \wedge c(\omega_2)$, which would be the desirable result. This is due to the fact that the class of ω is constructed by a cocycle, obtained by successive applications of the chasing diagram routine, so the shifting from the cochains of a certain degree to another may not preserve the exterior product.

To overcome this difficulty, we consider a new (equivalent) cohomology, which behaves well with respect to the exterior product, and then we transfer the latter product to $H^*(X, \ker d)$. Of course, this is done at an extra cost, as we explain in what follows.

The new cohomology is derived from the $\mathcal{A}(X)$-complex

$$0 \longrightarrow \Gamma(\mathcal{A}) \xrightarrow{\Gamma_X(d^0)} \Gamma_X(\Omega^1) \xrightarrow{\Gamma_X(d^1)} \Gamma_X(\Omega^2) \longrightarrow \cdots\ ,$$

where Γ_X is the (global) section functor. The previous complex is also written in the form

$$0 \longrightarrow \mathcal{A}(X) \xrightarrow{d^0_X} \Omega^1(X) \xrightarrow{d^1_X} \Omega^2(X) \longrightarrow \cdots \tag{9.5.12}$$

Now, in addition to the assumptions (9.1.1), we further suppose that (this is the extra cost!)

X is a de Rham space, which means that

$$0 \longrightarrow \mathcal{A} \xrightarrow{d^0} \Omega^1 \xrightarrow{d^1} \Omega^2 \longrightarrow \cdots \longrightarrow \Omega^p \xrightarrow{d^p} \cdots$$

is an *acyclic resolution* of $\ker d$.

(see Definition 2.5.5). But then we can apply the *abstract de Rham Theorem* (see the end of Subsection 1.6.3 and equality (1.6.36)), by which

$$\check{H}^p(X, \ker d) = H^p(X, \ker d) = \frac{\ker\left\{d_X^p : \Omega^p(X) \longrightarrow \Omega^{p+1}(X)\right\}}{\operatorname{im}\left\{d_X^{p-1} : \Omega^{p-1}(X) \longrightarrow \Omega^p(X)\right\}} =: \frac{\ker\left(d_X^p\right)}{\operatorname{im}\left(d_X^{p-1}\right)} \tag{9.5.13}$$

within isomorphisms of $\mathcal{A}(X)$-modules (see also Remark 9.4.2(1)). We recall that $\ker d$ is the $\mathcal{A}$-module $\ker\{d : \mathcal{A} \to \Omega^1\}$. Although it is customary to drop the superscript and subscript of d_X^p and simply write d, in order to avoid any confusion between $d : \mathcal{A} \to \Omega^1$ and $d_X^p : \Omega^p(X) \to \Omega^{p+1}(X)$, we retain the full notation of the latter differential.

If $\omega \in \Omega^p(X)$ is a closed form, we denote by

$$[\omega]_d \in \frac{\ker\left(d_X^p\right)}{\operatorname{im}\left(d_X^{p-1}\right)}$$

the cohomology class of ω defined by the sequence (9.5.12), whereas $c(\omega)$ is the class of ω in $H^p(X, \ker d)$. Hence, we have the bijective correspondence

$$H^p(X, \ker d) \ni c(\omega) \xrightarrow{h_p} [\omega]_d \in \frac{\ker\left(d_X^p\right)}{\operatorname{im}\left(d_X^{p-1}\right)}, \tag{9.5.14}$$

where h_p stands for the isomorphism (9.5.13).

Before proceeding further, let us explain how one defines the exterior product of classes of the previous kind. For two classes

$$[\omega_1]_d \in \frac{\ker\left(d_X^p\right)}{\operatorname{im}\left(d_X^{p-1}\right)} \quad \text{and} \quad [\omega_2]_d \in \frac{\ker\left(d_X^q\right)}{\operatorname{im}\left(d_X^{q-1}\right)},$$

we set (see (9.5.6) and the notations of Proposition 9.5.2)

$$[\omega_1]_d \wedge [\omega_2]_d := [\rho_X^{p,q}(\omega_1 \wedge \omega_2)]_d \in \frac{\ker\left(d_X^{p+q}\right)}{\operatorname{im}\left(d_X^{p+q-1}\right)}. \tag{9.5.15}$$

The right-hand side class is defined, because if ω_1 and ω_2 are closed forms, so is $\rho_X^{p,q}(\omega_1 \wedge \omega_2)$. Indeed, for every $x \in X$, using (9.5.7) we have:

$$\begin{gathered}\left(d_X^{p+q}(\rho_X^{p,q}(\omega_1 \wedge \omega_2))\right)(x) = d_X^{p+q}\left(\omega_1(x) \wedge \omega_2(x)\right) = \\ d_X^p(\omega_1(x)) \wedge (\omega_2(x)) + (\omega_1(x)) \wedge d_X^q(\omega_2(x)) = \\ (d_X^p\omega_1)(x) \wedge (\omega_2(x)) + (\omega_1(x)) \wedge (d_X^q\omega_2)(x) = 0.\end{gathered}$$

We prove that (9.5.15) is independent of the representatives of the classes on the left-hand side. To see this, assume that $[\omega_1]_d = [\omega_1']_d$ and $[\omega_2]_d = [\omega_2']_d$. Then there are $\theta_1 \in \Omega^{p-1}$ and $\theta_2 \in \Omega^{q-1}$ such that

$$\omega_1' = \omega_1 + d_X^{p-1}\theta_1 \quad \text{and} \quad \omega_2' = \omega_2 + d_X^{q-1}\theta_2. \tag{9.5.16}$$

We shall show that

$$\rho_X^{p,q}(\omega_1' \wedge \omega_2') = \rho_X^{p,q}(\omega_1 \wedge \omega_2) + d_X^{p+q-1}\theta, \tag{9.5.17}$$

where

$$\theta = \rho_X^{p-1,q}(\theta_1 \wedge \omega_2) + (-1)^p \rho_X^{p,q-1}(\omega_1 \wedge \theta_2) + \rho_X^{p-1,q-1}(\theta_1 \wedge \theta_2). \tag{9.5.18}$$

Thus, in virtue of (9.5.7), it suffices to show that

$$\begin{aligned} \omega_1'(x) \wedge \omega_2'(x) &= \omega_1(x) \wedge \omega_2(x) + (d_X^{p+q-1}\theta)(x) \\ &= \omega_1(x) \wedge \omega_2(x) + d_X^{p+q-1}(\theta(x)), \end{aligned} \tag{9.5.19}$$

for every $x \in X$.

Indeed, taking into account that ω_1 and ω_2 are closed, as well as that $d^{p+1} \circ d^p = 0$, (9.5.18) yields

$$\begin{gathered} d_X^{p+q-1}(\theta(x)) = \\ d_X^{p+q-1}\big(\theta_1(x) \wedge \omega_2(x) + (-1)^p\, \omega_1(x) \wedge \theta_2(x) + \theta_1(x) \wedge \theta_2(x)\big) = \\ d_X^{p-1}(\theta_1(x)) \wedge \omega_2(x) + \omega_1(x) \wedge d_X^{q-1}(\theta_2(x)) + d_X^{p-1}(\theta_1(x)) \wedge d_X^{q-1}(\theta_2(x)). \end{gathered}$$

Hence, in virtue of (9.5.16) and the preceding equality, we have

$$\begin{gathered} \omega_1(x) \wedge \omega_2(x) + d_X^{p+q-1}(\theta(x)) = \\ \big(\omega_1(x) + d_X^{p-1}(\theta_1(x))\big) \wedge \big(\omega_2(x) + d_X^{q-1}(\theta_2(x))\big) = \\ \omega_1'(x) \wedge \omega_2'(x); \end{gathered}$$

that is, we obtain (9.5.19), which, as already explained, proves (9.5.17) and the independence of (9.5.15) from the representatives of the classes.

Now, in virtue of the isomorphism (9.5.13), the exterior product of two classes $c(w_1) \in H^p(X, \ker d)$ and $c(w_2) \in H^q(X, \ker d)$ is, by definition, the class $c(w_1) \wedge c(w_2) \in H^{p+q}(X, \ker d)$ given by

$$c(w_1) \wedge c(w_2) := h_{p+q}^{-1}\big([w_1]_d \wedge [w_2]_d\big) = h_{p+q}^{-1}\big([\rho_X^{p,q}(\omega_1 \wedge \omega_2)]_d\big). \tag{9.5.20}$$

As a result, $H^*(X, \ker d)$ has the structure of a (graded) $\mathcal{A}(X)$-algebra. In particular, $H^*(X, \ker d)$ is a $\mathbb{K}$-algebra.

The previous preparation allows us to conclude with the following:

9.5.3 Theorem. *Let X be a de Rham space, $\mathcal{G}$ a Lie sheaf of groups and $(\mathcal{G}, \mathcal{D}, \boldsymbol{d}^2)$ a Bianchi datum. If $\mathcal{P}$ is a $\mathcal{G}$-principal sheaf over X admitting connections, then the Chern-Weil map $\mathfrak{W}_{\mathcal{P}} : I^*(\mathcal{G}) \to H^*(X, \ker d)$ becomes a $\mathbb{K}$-algebra morphism with respect to the products (9.5.1) and (9.5.20).*

Proof. For any $f \in I^k(\mathcal{G})$, $g \in I^l(\mathcal{G})$, and for an arbitrary connection D on $\mathcal{P}$, equalities (9.4.24) and (9.4.19) imply that

$$\mathfrak{W}_{\mathcal{P}}(f \odot g) = c\big((f \odot g)(\mathcal{P})\big) = c\big((f \odot g)(D)\big),$$

or, by (9.5.14) and (9.5.11),

$$\mathfrak{W}_{\mathcal{P}}(f \odot g) = h_{k+l}^{-1}\big([(f \odot g)(D)]_d\big) = h_{k+l}^{-1}\big(\rho_X^{k,l}[f(D) \wedge g(D)]_d\big).$$

Hence, in virtue of (9.5.20),

$$\mathfrak{W}_{\mathcal{P}}(f \odot g) = c\big(f(D)\big) \wedge c\big(g(D)\big) = \\ c(f, \mathcal{P}) \wedge c(g, \mathcal{P}) = \mathfrak{W}_{\mathcal{P}}(f) \wedge \mathfrak{W}_{\mathcal{P}}(g). \qquad \square$$

The morphism of $\mathbb{K}$-algebras $\mathfrak{W}_{\mathcal{P}}$, obtained under the conditions of the preceding theorem, is called the ***Chern-Weil homomorphism*** of $\mathcal{P}$.

Notes. 1) In the case when $\ker d = \mathbb{R}$, the previous situation is reminiscent of the classical analog within the framework of ordinary principal sheaves and Ad-invariant morphisms.

2) If we consider a principal sheaf with structure sheaf $\mathcal{GL}(n, \mathcal{A})$, then a k-morphism $f : \prod^k M_{m\times n}(\mathcal{A}) \to \mathcal{A}$ can be connected with symmetric Ad-invariant polynomials, so we obtain a Chern-Weil homomorphism involving such polynomials. Furthermore, combining Proposition 5.2.5 and Theorem 7.1.6, we obtain a Chern-Weil homomorphism in the context of vector sheaves (and polynomials). A direct proof of this is given in Mallios [62, Vol. II, p. 271].

Chapter 10

Applications and further examples

L' application pratique se trouve quand on ne la cherche pas et on peut dire que tout le programme de civilisation repose sur ce principe. ... [d' importantes recherches mathématiques] elles sont inspirées par le désir qui est le motif commun de tout travail scientifique, celui de savoir et de comprendre.

J. Hadamard [40, pp. 115–116]

This chapter contains a few applications and examples further illustrating some of the general ideas exhibited in the preceding chapters. They supplement, in a sense, the basic examples given at various stages of our exposition.

More precisely, in Section 10.1 we discuss the notion of differential triad in the context of non-commutative geometry. In Section 10.2 we examine how the connection theory of infinite-dimensional bundles fits into our scheme. This is applied to the concrete case of manifolds and bundles modelled on projective finitely generated $\mathbb{A}$-modules, where $\mathbb{A}$ is a unital commutative associative locally m-convex algebra (Section 10.3). In Section 10.4 we deal with the torsion of an $\mathcal{A}$-connection on the vector sheaf Ω^*, and we show that the local torsion forms satisfy the analogs of Cartan's structure equation and Bianchi's identity. In Section 10.5 we prove that the existence of a Riemannian metric on a vector sheaf is equivalent to the reduction of its structure sheaf to the orthogonal group sheaf.

Finally, the problems concluding the chapter raise certain research questions that have not been touched upon here. Undoubtedly, an answer to them would be an add-on to our work, enhancing the effectiveness of the present approach.

10.1. A non-commutative differential triad

Starting with a differential triad $(\mathcal{A}, d, \Omega)$ in the (original) sense of Definition 2.1.2, in Section 3.1 we constructed its matrix sheaf extension $(\mathcal{M}_n(\mathcal{A}), d, \mathcal{M}_n(\Omega))$, where $\mathcal{M}_n(\mathcal{A})$ is a non-commutative algebra sheaf. Hence, the notion of differential triad is susceptible of a generalization within the non-commutative framework. Of course, the question of how far one can go in this direction remains open.

In the present section we want to outline another example of a non-commutative differential triad derived from *non-commutative geometry.* This idea has been suggested by A. Asada ([5]) who is heartily thanked again here.

We briefly recall that a ***-algebra*** is a complex algebra $\mathbb{A}$ with an ***involution***, i.e., a map $^* : \mathbb{A} \to \mathbb{A}$ satisfying the properties:

(i) $$(a+b)^* = a^* + b^*$$

(ii) $$(\lambda a)^* = \overline{\lambda} a^*$$

(iii) $$(a \cdot b)^* = b^* \cdot a^*$$

(iv) $$(a^*)^* = a$$

for every $a, b \in \mathbb{A}$ and $\lambda \in \mathbb{C}$.

A C^****-algebra*** is a Banach *-algebra $\mathbb{A}$ whose involution satisfies, in addition, the C^**-property*:

$$\text{(v)} \qquad \| a^* \cdot a \| = \| a \|^2, \qquad a \in \mathbb{A}.$$

By the Gel'fand-Naimark theorem, a C^*-algebra is isometrically *-isomorphic with a closed subspace of the space $\mathcal{L}(H)$ of continuous (bounded) operators of a complex Hilbert space H. The space $\mathcal{L}(H)$ is a C^*-algebra whose involution associates to each operator $T \in \mathcal{L}(H)$ its adjoint T^*. To avoid any confusion, we denote by J the involution of $\mathcal{L}(H)$, thus $J(T) = T^*$.

From the very extensive literature on C^*-algebras, we cite, e.g., Bonsall-Duncan [11] and Murphy [80], where the reader is referred for details.

Let $\mathbb{A}$ be a C^*-algebra, viewed as an infinite-dimensional Hilbert space (of operators). We assume that J is *self-adjoint*, i.e. $J^* = J$. Since $J^2 = id$ (: J has square 1), we further assume that the two proper spaces

$$\{T \mid J(T) = T\} \quad \text{and} \quad \{T \mid J(T) = -T\}$$

are infinite-dimensional. In this case, the Hilbert space at hand admits a *polarization* and the space itself is said to be *polarized*. Polarized Hilbert spaces are important to loop groups and integrable systems (see [103, p. 80]).

According to A. Connes's general construction (see [21, pp. 19, 313]), the ***quantized differential*** $\boldsymbol{d}a$ of $a \in \mathbb{A}$ is defined as follows: If a is identified with an operator $T \in \mathcal{L}(H)$, then

$$\boldsymbol{d}a \equiv \boldsymbol{d}T := [J, T] = JT - TJ.$$

The previous commutator is given by

$$[J, T](S) = J(S) \circ T - T \circ J(S) = S^* \circ T - T \circ S^*, \qquad S \in \mathcal{L}(H).$$

On the other hand, we define $\boldsymbol{\Omega}$ to be the $\mathbb{A}$-module generated by the elements $[J, a] \equiv [J, T]$, for all $a \in \mathbb{A}$. As a result, we establish the map

$$\boldsymbol{d} : \mathbb{A} \ni a \longmapsto \boldsymbol{d}a = [J, a] \in \boldsymbol{\Omega},$$

which is $\mathbb{C}$-linear and satisfies the Leibniz rule. The derivation $\boldsymbol{d}$ is rich in geometric information, but this cannot be described here.

Now, let us consider a C^*-algebra sheaf $\mathcal{A}$ over a topological space $X \equiv (X, \mathfrak{T}_X)$. For each $U \in \mathfrak{T}_X$, $\mathcal{A}(U)$ is a C^*-algebra; hence, by the above constructions, we associate with it an $\mathcal{A}(U)$-module $\boldsymbol{\Omega}(U)$ and a $\mathbb{C}$-linear map $\boldsymbol{d}_U : \mathcal{A}(U) \to \boldsymbol{\Omega}(U)$ satisfying the Leibniz rule. Therefore, varying U in $\mathfrak{T}_X$, we obtain the $\mathcal{A}$-module $\Omega := \mathbf{S}\big(U \longmapsto \boldsymbol{\Omega}(U)\big)$ and the $\mathbb{C}$-linear morphism $d : \mathcal{A} \to \Omega$, with $d := \mathbf{S}\big((d_U)_{U \in \mathfrak{T}_X}\big)$, satisfying the Leibniz rule. Then $(\mathcal{A}, d, \Omega)$ is a non-commutative differential triad.

A theory of non-commutative connections on vector sheaves seems to be more complicated, since even the construction of ordinary (non sheaf-theoretic) non-commutative connections is based on particular technicalities (such as the Schatten–von Neumann ideals etc.) which are beyond the scope of this work. The interested reader is referred to Asada [4] (see also Connes [21]) for the latter terminology and related results.

Note. Non-commutative geometry is an "algebraic" approach to differential geometry within the non-commutative framework (see Connes op. cit., Madore [56] and their references). Among its ambitions are the description and understanding of the "quantum world". Without entering the discussion whether this target may or may not be achieved by the concrete methods of non-commutative geometry (cf., for instance, J. Nestruev's comments in [83, p. 141]), we would like to say that this point of view has the same philosophy as ADG; namely, to bypass the manifold structure on a space X and concentrate on the algebraic quantities over it. The reader who is familiar with non-commutative geometry, may have noticed that certain principles of it are closely related with analogous ones of ADG. However, many aspects of the former lack a convenient localization and cannot be included in our scheme.

10.2. Classical infinite-dimensional connections

As we saw in Example 6.2(a), in conjunction with Theorem 6.2.1, connections on *finite-dimensional* principal bundles can be thought of as (sheaf-theoretic) connections in the sense of Definition 6.1.1. This is based on the identification (see Example 3.3.6(a) and equality (3.3.13))

$$\Lambda^1(U, \mathbb{G}) \cong \Lambda^1(U, \mathbb{R}) \otimes_{C^\infty(U,\mathbb{R})} C^\infty(U, \mathbb{G}),$$

if $\mathbb{G}$ denotes the Lie algebra of a given (finite-dimensional) Lie group G. We recall that $\Lambda^1(U, \mathbb{G})$ is the $C^\infty(U, \mathbb{R})$-module of $\mathbb{G}$-valued differential 1-forms on an open $U \subseteq X$. Moreover, $\Omega(U) \cong \Lambda^1(U, \mathbb{R})$ if $\Omega \equiv \Omega^1_X$ is the sheaf (of germs) of $\mathbb{R}$-valued differential 1-forms on the smooth manifold X.

Although the above identification is not generally valid in the infinite-dimensional framework, we obtain a sheaf-theoretic interpretation of the connections on an infinite-dimensional principal bundle, under a slight modification of Example 6.2.(a).

More precisely, we start with a principal bundle (P, G, X, π) whose base X and structural group G have an appropriate infinite-dimensional differential structure (e.g., Banach manifold and Banach-Lie group, respectively). If $\mathbb{G}$ is the Lie algebra of G, we consider the sheaf of germs of smooth $\mathbb{G}$-valued forms defined on X

$$\Omega_X(\mathbb{G}) := \mathbf{S}\big(U \longmapsto \Lambda^1(U, \mathbb{G})\big).$$

Following Example 4.1.9(a), we obtain the principal sheaf of *smooth* sections of P

$$\mathcal{P} := \mathbf{S}\big(U \longmapsto \Gamma(U, P)\big),$$

whose structure sheaf $\mathcal{G} \equiv (\mathcal{G}, \mathcal{A}d, \mathcal{L}, \partial)$ is defined as in Example 3.3.6(a). Namely,

$$\mathcal{G} = \mathcal{C}_X^\infty(G) := \mathbf{S}\big(U \longmapsto C^\infty(U, G)\big),$$
$$\mathcal{L} = \mathcal{C}_X^\infty(\mathbb{G}) := \mathbf{S}\big(U \longmapsto C^\infty(U, \mathbb{G})\big),$$

the Maurer-Cartan differential $\partial : \mathcal{G} \to \Omega_X(\mathbb{G})$ is generated by the morphisms

$$\partial_U : C^\infty(U, G) \longrightarrow \Lambda^1(U, \mathbb{G}) : f \mapsto \partial_U(f) := f^{-1} \cdot df,$$

while the representation $\mathcal{A}d : \mathcal{G} \to \mathcal{L}$ is generated by the morphisms

$$\mathrm{Ad}_U : C^\infty(U, G) \longrightarrow \mathrm{Aut}\big(\mathcal{C}_X^\infty(\mathbb{G})|_U\big),$$

defined, in their turn, by

$$\big(\mathrm{Ad}_U(g)(f)\big)(x) := \big(\mathrm{Ad}(g(x))\big)(f(x)),$$

for every $g \in C^\infty(U, G)$, $f \in C^\infty(V, \mathbb{G})$, $x \in V$, and every open $V \subseteq U$. On the right-hand side of the preceding equality, $\mathrm{Ad} : G \to \mathrm{Aut}(\mathbb{G})$ is the ordinary adjoint representation of G.

As in the same Example 3.3.6(a), we define the local actions

$$\delta_U : C^\infty(U, G) \times \Lambda^1(U, \mathbb{G}) \longrightarrow \Lambda^1(U, \mathbb{G}),$$

with $\delta_U(g, \omega) := \mathrm{Ad}(g).\omega$, the 1-form on the right-hand side being given by

$$(\mathrm{Ad}(g).\omega)_x(v) := (\mathrm{Ad}(g))(\omega_x(v)); \qquad x \in U,\ v \in T_xX.$$

Varying U in the topology of X, we obtain an action

$$\delta : \mathcal{C}^{\infty}_X(G) \times_X \Omega_X(\mathbb{G}) \longrightarrow \Omega_X(\mathbb{G}).$$

To remind us that δ comes from the adjoint representation, we set

$$\mathcal{A}d(a).w := \delta(a,w), \qquad (a,w) \in \mathcal{G} \times_X \Omega_X(\mathbb{G}).$$

Then ∂ satisfies the fundamental property

$$\partial(a \cdot b) = \mathcal{A}d(b^{-1}).\partial(a) + \partial(b); \qquad (a,b) \in \mathcal{G} \times_X \mathcal{G}.$$

Now let us turn to the connections of the principal sheaf $\mathcal{P}$ constructed from the bundle P. First recall that (see (4.1.9)) the natural sections $s_\alpha \in \mathcal{P}(U_\alpha)$ of $\mathcal{P}$ are given by $s_\alpha = \widetilde{\sigma}_\alpha$, if $\sigma_\alpha \in \Gamma(U_\alpha, P)$ are the (smooth) natural sections of P. Similarly, by (4.3.7), the transition sections of $\mathcal{P}$ are the sections $\gamma_{\alpha\beta} = \widetilde{g}_{\alpha\beta} \in \mathcal{G}(U_\alpha)$, if $g_{\alpha\beta} \in C^\infty(U_\alpha, G)$ are the transition functions of G.

A connection on $\mathcal{P}$ will be a morphism $D : \mathcal{P} \to \Omega_X(\mathbb{G})$ such that

$$D(p \cdot g) = \mathcal{A}d(g^{-1}).D(p) + \partial(g), \qquad (p,g) \in \mathcal{P} \times_X \mathcal{G}.$$

As in the general theory of Section 6.1, D is equivalent to the existence of a 0-cochain $(\omega_\alpha) \in C^0(\mathcal{U}, \Omega_X(\mathbb{G}))$ satisfying the compatibility condition

$$\omega_\beta = \mathcal{A}d(\gamma^{-1}_{\alpha\beta}).\omega_\alpha + \partial(\gamma_{\alpha\beta}), \tag{10.2.1}$$

over $U_{\alpha\beta} \neq \emptyset$, for all $\alpha, \beta \in I$. In fact, $\omega_\alpha = D(s_\alpha)$. The first term on the right-hand side of (10.2.1) is the section defined by

$$(\mathcal{A}d(\gamma^{-1}_{\alpha\beta}).\omega_\alpha)(x) = \mathcal{A}d(\gamma^{-1}_{\alpha\beta}(x)).\omega_\alpha(x).$$

By the construction of $\Omega_X(\mathbb{G})$, we may write $\omega_\alpha = \widetilde{\theta}_\alpha$ and $\omega_\beta = \widetilde{\theta}_\beta$ for $\theta_\alpha \in \Lambda^1(U_\alpha, \mathbb{G})$ and $\theta_\beta \in \Lambda^1(U_\beta, \mathbb{G})$. Hence, (10.2.1) takes the form

$$\widetilde{\theta}_\beta = \mathcal{A}d(\widetilde{g}^{-1}_{\alpha\beta}).\widetilde{\theta}_\alpha + \partial(\widetilde{g}^{-1}_{\alpha\beta}) = \big(\mathrm{Ad}(g^{-1}_{\alpha\beta}).\theta_\alpha + \partial_{U_{\alpha\beta}}(g^{-1}_{\alpha\beta})\big)^{\widetilde{}}.$$

Since the presheaf of $\mathbb{G}$-valued forms generating $\Omega_X(\mathbb{G})$ is complete, the canonical map

$$\Lambda^1(U, \mathbb{G}) \longrightarrow \Omega_X(\mathbb{G})(U) : \theta \mapsto \widetilde{\theta}$$

is a bijection, thus the last equality implies

$$\theta_\beta = \mathrm{Ad}(g^{-1}_{\alpha\beta}).\theta_\alpha + \partial_{U_{\alpha\beta}}(g^{-1}_{\alpha\beta}) = \mathrm{Ad}(g^{-1}_{\alpha\beta}).\theta_\alpha + g^{-1}_{\alpha\beta}.dg_{\alpha\beta};$$

that is, the 1-forms $\theta_\alpha \in \Lambda^1(U_\alpha, \mathbb{G})$, $\alpha \in I$, determine a connection on P with connection forms θ_α. The converse is proved by reversing the previous arguments.

Hence, analogously to Theorem 6.2.1, one infers:

> *The connections on an infinite-dimensional principal bundle P are in bijective correspondence with the connections $D : \mathcal{P} \to \Omega_X(\mathbb{G})$ on the sheaf $\mathcal{P}$ of germs of smooth sections of P.*

However, we do not know a reasonable way to express $\Omega_X(\mathbb{G})$ in terms of Ω and $\mathcal{L}$, as in the finite-dimensional case, already described in the beginning of this section.

In the next section we apply the previous discussion to the frame bundles of a particular category of infinite-dimensional vector bundles.

10.3. On the geometry of $\mathbb{A}$-bundles

We outline an application whose full details can be found in Vassiliou-Papatriantafillou [134]. We are mainly interested in studying the connections on the frame bundle $F(E)$ of a vector bundle whose fiber type is a projective finitely generated module over an appropriate topological algebra $\mathbb{A}$. As we shall see, the topological structure of $\mathbb{A}$ is crucial for our considerations.

Before approaching our objective, let us say a few words about the bundles in the title, in order to clarify when and how these bundles are connected with our sheaf-theoretic techniques.

In many areas of pure mathematics and theoretical physics, one frequently encounters vector spaces having the additional structure of a module over a topological algebra $\mathbb{A}$. A number of problems related with, e.g., operator theory, theoretical physics, differential topology, to name but a few, have been successfully treated by taking into account this additional structure (see Kaplansky [47], Selesnick [110], Miščenko [75], respectively).

Of particular interest, in the same direction, are manifolds and vector bundles, whose models are *projective finitely generated* $\mathbb{A}$*-modules* (the relevant definitions will be given below). For the sake of brevity, such manifolds and bundles henceforth are called $\mathbb{A}$-***manifolds*** and $\mathbb{A}$-***bundles***, respectively. We refer to Kobayashi [48], Shurygin [111], Prastaro [102] for some applications of this aspect in differential geometry, PDEs, and mechanics.

The structure and classification of *topological* $\mathbb{A}$-bundles have been studied by A. Mallios (see [59] and the references therein), whereas *differentiable*

$\mathbb{A}$-bundles have been considered by M. Papatriantafillou, within an appropriate differential framework (see [90], [95], [96]). In both aspects, $\mathbb{A}$ is a *locally m-convex algebra*, a case encompassing all the aforementioned examples and applications.

As already proven in [96], differentiable $\mathbb{A}$-bundles admit linear connections, which are equivalent to covariant $\mathbb{A}$-derivations (see [95]), in contrast to the general infinite-dimensional case (cf., e.g., Flaschel-Klingenberg [29], Vilms [138]). But, if we want to view an infinite-dimensional vector bundle as a bundle associated with its principal bundle of frames, and to reduce linear connections on the former to connections on the latter, a serious obstacle arises: If the fiber of the bundle is a *non-Banachable* infinite dimensional vector space P, then the general linear group $\mathrm{GL}(P)$ of P is not necessarily a Lie group, not even a topological one as is, for instance, the case of a Fréchet space. Thus the frame bundle of a vector bundle of this type may not be smooth or topological, let alone have connections.

However, in the context of an $\mathbb{A}$-bundle E of fiber type P, the structural group of the frame bundle $F(E)$ is the group $\mathrm{GL}_{\mathbb{A}}(P)$ of $\mathbb{A}$-linear automorphisms of P, which *is always a topological group*. In addition, if $\mathbb{A}$ is a ***Q-algebra***, i.e., the set $\mathbb{A}^{\bullet}$ of invertible elements of $\mathbb{A}$ is *open* in $\mathbb{A}$, then $\mathrm{GL}_{\mathbb{A}}(P)$ is proved to be a Lie group and $F(E)$ becomes a smooth principal bundle. Moreover, each linear connection ∇ on P induces a 0-cochain of local forms (ω_α) satisfying the compatibility condition (6.2.1) and vice-versa. The same family of forms globalizes to a principal connection form ω on $F(E)$, thus the linear connections on E are in bijective correspondence with the connections on the frame bundle $F(E)$.

Unfortunately, in the most important and frequently met examples of $\mathbb{A}$-bundles, $\mathbb{A}$ is either the algebra $\mathcal{C}(X)$ of continuous functions on a topological space X, or the algebra $\mathcal{C}^\infty(X)$ of smooth functions on a smooth manifold X. In both cases, $\mathbb{A}$ is *not* a Q-algebra, unless X is compact (as a matter of fact, the previous functional algebras are Q if and only if X is compact; see Mallios [58, Scholium 1.1, p. 221] and [62, Vol. II, (11.39), p. 371]). As a result, in the general (non-compact) case, $F(E)$ is treated only as a *topological bundle*, in which case (ω_α) cannot be globalized to a connection form on $F(E)$ as before.

Yet, the sheaf of germs $\mathcal{F}(E)$ of certain continuous sections of $F(E)$ (which play the rôle of "differentiable" sections) admits a connection D in the sense of Section 10.2, fully determined by (and determining) the local forms (ω_α). Therefore, the linear connections on E are in bijective correspondence with the connections D on $\mathcal{F}(E)$. Of course, if $\mathbb{A}$ is a Q-algebra, the connections D coincide, within a bijection, with the ordinary connections on the smooth bundle of frames $F(E)$.

In what follows we fix a *unital commutative associative lmc (: locally m-convex)* algebra $\mathbb{A}$ (see also the brief comments in 8.8.8), and we denote by $\mathcal{PFG}(\mathbb{A})$ the category of ***projective finitely generated*** $\mathbb{A}$***-modules***. We recall that, if $M \in \mathcal{PFG}(\mathbb{A})$, then (by definition)

$$M \oplus M_1 = \mathbb{A}^m \tag{10.3.1}$$

for some $M_1 \in \mathcal{PFG}(\mathbb{A})$ and $m \in \mathbb{N}$. Obviously, $\mathbb{A} \in \mathcal{PFG}(\mathbb{A})$.

Since the objects of $\mathcal{PFG}(\mathbb{A})$ will be used as models of the manifolds and bundles considered below, we need to define a *topology* and a method of *differentiation* on them.

We topologize an M, satisfying (10.3.1), by taking the relative topology τ_M on M, induced by the product topology of $\mathbb{A}^m$. It turns out that τ_M does not depend on M_1 or m. Moreover, it is the unique topology making M a topological $\mathbb{A}$-module and any $\mathbb{A}$-multilinear map on M continuous. We call τ_M the ***canonical topology of*** M. For details we refer to Papatriantafillou [89] in conjunction with Mallios [57].

The description of the differentiation method is a little more complicated. It is due to Vu Xuan Chi [139], originally defined on modules over a *topological ring*. To this end, for any $M, N \in \mathcal{PFG}(\mathbb{A})$, we denote by $\mathrm{L}_{\mathbb{A}}(M, N)$ the projective finitely generated $\mathbb{A}$-module of $\mathbb{A}$-linear maps from M into N. As usual, $\mathcal{N}(x)$ stands for the filter of open neighborhoods of $x \in M$ and 0_M is the zero element of M.

With these notations, for $x \in M$ and $W \in \mathcal{N}(x)$, we say that a map $f : W \to N$ is $\mathbb{A}$***-differentiable*** at x if there is a $Df(x) \in \mathrm{L}_{\mathbb{A}}(M, N)$ such that the *remainder* ϕ *of* f at x, given by

$$\phi(h) := f(x+h) - f(x) - Df(x)(h); \qquad h \in M \quad \text{with} \quad x + h \in U,$$

satisfies the following *infinitesimality* condition:

$$\forall\, V \in \mathcal{N}(0_N)\ \exists\, U \in \mathcal{N}(0_M) : \quad \forall\, B \in \mathcal{N}(0_{\mathbb{A}})\ \exists\, A \in \mathcal{N}(0_{\mathbb{A}}) :$$
$$a \in A\ \Rightarrow\ \phi(aU) \subset aBV.$$

This differentiation, applied to our case, where the structure of the modules is richer than the one considered by Vu Xuan Chi op. cit., has a number of fundamental properties, missing both in the context of the latter work and that of locally convex spaces. In particular,

i) $\mathbb{A}$-differentiability implies continuity.

ii) The composition and the evaluation maps are $\mathbb{A}$-differentiable.

iii) The chain rule holds true for every order of differentiation.

The proofs can be found in Papatriantafillou [90], where an analogous differentiation is introduced on *-algebras.

An infinitely $\mathbb{A}$-differentiable map will be called $\mathbb{A}$-***smooth***.

Following the standard pattern, we obtain the category $\mathcal{M}an(\mathbb{A})$ of $\mathbb{A}$-***manifolds***, modelled on the objects of $\mathcal{PFG}(\mathbb{A})$, and $\mathbb{A}$-***smooth morphisms***.

We construct the tangent space T_xX at a point x of $X \in \mathcal{M}an(\mathbb{A})$ by considering classes of equivalent $\mathbb{A}$-curves. An $\mathbb{A}$-***curve*** is an $\mathbb{A}$-smooth map $\alpha : V \to \mathbb{A}$, where $V \in \mathcal{N}(0_{\mathbb{A}})$. If X is modelled on M and (U, ϕ) is a chart at $x \in X$, then the bijection

$$\bar{\phi} : T_xX \xrightarrow{\simeq} M : [(\alpha, x)] \mapsto D(\phi \circ \alpha)(0)$$

provides T_xX with the structure of a (projective finitely generated) $\mathbb{A}$-module. Furthermore, the tangent bundle TX of X is an $\mathbb{A}$-manifold. If $f : X \to Y$ is $\mathbb{A}$-smooth, the differential $df : TX \to TY$ is also $\mathbb{A}$-smooth and its restrictions to the tangent spaces are $\mathbb{A}$-linear maps.

Vector $\mathbb{A}$-bundles are defined similarly: Let $X, E \in \mathcal{M}an(\mathbb{A})$, $\pi : E \to X$ be $\mathbb{A}$-smooth and $P \in \mathcal{PFG}(\mathbb{A})$. We assume that

$$E_x := \pi^{-1}(x) \in \mathcal{PFG}(\mathbb{A}); \qquad x \in X,$$

and that there exist an open covering $\mathcal{U} := \{U_\alpha\}_{\alpha \in I}$ of X and (*trivializing*) $\mathbb{A}$-diffeomorphisms

$$\tau_\alpha : \pi^{-1}(U_\alpha) \xrightarrow{\simeq} U_\alpha \times P\,; \qquad \alpha \in I,$$

such that $\mathrm{pr}_1 \circ \tau_\alpha = \pi$, with the restrictions $\tau_{\alpha x} : E_x \to \{x\} \times P$ of τ_α to the fibers E_x, $x \in U_\alpha$, being $\mathbb{A}$-module isomorphisms.

A triplet $E \equiv (E, \pi, X)$ satisfying the previous properties is called an $\mathbb{A}$-***bundle***. An obvious example is the tangent bundle of an $\mathbb{A}$-manifold.

It is worth noticing that the definition of infinite dimensional vector bundles (from Banach ones and beyond) includes as an axiom the differentiability of the transition functions (cf. condition (VB. 3) in Lang [54]).

However, in our context, although the underlying vector spaces of the models are infinite-dimensional locally convex spaces, this condition is now a consequence of the properties of the projective finitely generated $\mathbb{A}$-modules. In fact, we have the next result, whose proof is included here just to give an idea of the mechanism used in this framework.

10.3.1 Lemma. *Let M, N, $P \in \mathcal{PFG}(\mathbb{A})$, an open $U \subseteq M$ and an $\mathbb{A}$-smooth map $f : U \times N \to P$ such that the partial maps $f_x : N \to P$, with $f_x(y) := f(x,y)$, $y \in N$, are $\mathbb{A}$-linear for all $x \in U$. Then the map*

$$F : U \longrightarrow \mathrm{L}_{\mathbb{A}}(N,P) : x \mapsto f_x$$

is $\mathbb{A}$-smooth.

Proof. Let $N_1, P_1 \in \mathcal{PFG}(\mathbb{A})$ and $n, p \in \mathbb{N}$, with $N \oplus N_1 = \mathbb{A}^n$ and $P \oplus P_1 = \mathbb{A}^p$. The $\mathbb{A}$-smooth extension of f

$$\overline{f} : U \times (N \oplus N_1) \to P \oplus P_1 : (x, y, y_1) \mapsto (f(x,y), 0),$$

induces the map $\overline{F} : U \to \mathrm{L}_{\mathbb{A}}(\mathbb{A}^n, \mathbb{A}^p) : x \mapsto \overline{f}_x$. It is elementary to show that $\mathrm{L}_{\mathbb{A}}(N,P)$ is a direct factor of $\mathrm{L}_{\mathbb{A}}(\mathbb{A}^n, \mathbb{A}^p)$ and $F = \mathrm{pr} \circ \overline{F}$, where $\mathrm{pr} : \mathrm{L}_{\mathbb{A}}(\mathbb{A}^n, \mathbb{A}^p) \to \mathrm{L}_{\mathbb{A}}(N,P)$ denotes the respective projection. Therefore, F is $\mathbb{A}$-smooth if and only if $\overline{F}$ is $\mathbb{A}$-smooth, the other component of $\overline{F}$ vanishing. Since $\mathrm{L}_{\mathbb{A}}(\mathbb{A}^n, \mathbb{A}^p)$ is $\mathbb{A}$-isomorphic with the $\mathbb{A}$-module $M_{n\times p}(\mathbb{A})$ of $n \times p$ matrices with entries in $\mathbb{A}$, it suffices to prove that

$$U \longrightarrow M_{n\times p}(\mathbb{A}) : x \mapsto \big(a_{ij}(x)\big) := \big(\mathrm{pr}_j \circ \overline{f}_x(e_i)\big)$$

is $\mathbb{A}$-smooth. This is a consequence of the $\mathbb{A}$-smoothness of the maps

$$\overline{f}_i : U \to \mathbb{A}^p : x \mapsto \overline{f}(x, e_i) = (a_{i1}(x), \ldots, a_{ip}(x)),$$

for all indices $i = 1, \ldots, n$. □

Consequently, we prove at once the following:

10.3.2 Proposition. *Let $E \equiv (E, \pi, X)$ be an $\mathbb{A}$-bundle of fiber type $P \in \mathcal{PFG}(\mathbb{A})$, with a trivializing covering $\{(U_\alpha, \tau_\alpha) \,|\, \alpha \in I\}$. Then the transition functions*

$$g_{\alpha\beta} : U_{\alpha\beta} \longrightarrow \mathrm{L}_{\mathbb{A}}(P) := \mathrm{L}_{\mathbb{A}}(P,P) : x \mapsto \tau_{\alpha x} \circ \tau_{\beta x}^{-1} \tag{10.3.2}$$

are $\mathbb{A}$-smooth.

By their definition, the transition functions (10.3.2) take their values in the group $\mathrm{GL}_{\mathbb{A}}(P)$ of invertible elements of the algebra $\mathrm{L}_{\mathbb{A}}(P)$. The latter, being an object of $\mathcal{PFG}(\mathbb{A})$, admits the canonical topology, and the algebra multiplication

$$\mathrm{L}_{\mathbb{A}}(P) \times \mathrm{L}_{\mathbb{A}}(P) \ni (f, g) \longmapsto g \circ f \in \mathrm{L}_{\mathbb{A}}(P) \tag{10.3.3}$$

is continuous (as an $\mathbb{A}$-bilinear map). On the other hand, $\mathrm{L}_{\mathbb{A}}(P)$, being also a unital *lmc*-algebra, has a *continuous inversion* (see Mallios [58, Chap. II, Lemma 3.1]), thus $\mathrm{GL}_{\mathbb{A}}(M)$, topologized with the relative topology induced by the canonical topology of $\mathrm{L}_{\mathbb{A}}(P)$, is a *topological group*. Therefore, the cocycle $(g_{\alpha\beta}) \in Z^1(\mathcal{U}, \mathrm{GL}_{\mathbb{A}}(P))$ determines a *topological principal bundle* $F(E) \equiv (F(E), \mathrm{GL}_{\mathbb{A}}(P), X, p)$, called the ***bundle of frames*** or ***frame bundle*** of E.

In summary, we obtain:

10.3.3 Theorem. *If E is an $\mathbb{A}$-bundle of fiber type $P \in \mathcal{PFG}(\mathbb{A})$, then the bundle of frames $F(E)$ is a topological principal bundle with structural group the general $\mathbb{A}$-linear group $\mathrm{GL}_{\mathbb{A}}(P)$.*

Now assume that $\mathbb{A}$ is, in particular, a ***Q-algebra*** (see p. 376). This property is inherited by the algebra $\mathrm{L}_{\mathbb{A}}(P)$, for every $P \in \mathcal{PFG}(\mathbb{A})$ (see Swan [120, Corollary 1.2] and Mallios [57, Theorem 1.1]). Thus the restriction of (10.3.2) to $\mathrm{GL}_{\mathbb{A}}(P)$ yields a corresponding $\mathbb{A}$-smooth "multiplication"

$$\mathrm{GL}_{\mathbb{A}}(P) \times \mathrm{GL}_{\mathbb{A}}(P) \longrightarrow \mathrm{GL}_{\mathbb{A}}(P) : (f, g) \mapsto g \circ f.$$

To view $\mathrm{GL}_{\mathbb{A}}(P)$ as a Lie group, we should prove the differentiability of the inversion. Unfortunately, this cannot be deduced from the differentiability of the multiplication, since such a deduction is based on the inverse mapping theorem, which is generally not valid in this context. However, the differentiability in question is shown straightforwardly in the next:

10.3.4 Lemma. *Let $\mathbb{A}$ be a commutative associative lmc Q-algebra with unit and $P \in \mathcal{PFG}(\mathbb{A})$. Then the inversion map* $\mathrm{inv} : f \mapsto f^{-1}$ *is $\mathbb{A}$-smooth in* $\mathrm{GL}_{\mathbb{A}}(P)$.

Proof. Let $f \in \mathrm{GL}_{\mathbb{A}}(P)$ and set $D\mathrm{inv}(f)(h) := -f^{-1} \circ h \circ f^{-1}$, for every $h \in \mathrm{L}_{\mathbb{A}}(P)$. Following the standard pattern (see, e.g., Cartan [19, Théorème 2.4.4, p. 34] we check that the remainder of inv at f

$$\phi(h) := \mathrm{inv}(f + h) - \mathrm{inv}(f) - D\mathrm{inv}(f)(h)$$

can be transformed to $\phi(h) = f^{-1} \circ \psi(h) \circ f^{-1}$, where $\psi(h) = h \circ (f+h)^{-1} \circ h$.

We prove that ψ is infinitesimal: Let $\mathbf{0}$ be the zero element of $\mathrm{L}_{\mathbb{A}}(P)$ and $V \in \mathcal{N}(\mathbf{0})$. The continuity of the composition at $(\mathbf{0}, f^{-1}, \mathbf{0})$ implies the existence of $U_1 \in \mathcal{N}(\mathbf{0})$, $V_1 \in \mathcal{N}(f^{-1})$ with $U_1 \circ V_1 \circ U_1 \subseteq V$. Since $V_1 \in \mathcal{N}(f^{-1})$ and inv is continuous, there exists $V_2 \in \mathcal{N}(f)$ with $V_2^{-1} \subseteq V_1$. The continuity of the $\mathbb{A}$-module operations also determine $A_1 \in \mathcal{N}(0_{\mathbb{A}})$ and $U_2 \in \mathcal{N}(\mathbf{0})$, with $A_1 U_2 \subseteq V_2 - f \in \mathcal{N}(\mathbf{0})$. We set $U := U_1 \cap U_2$, and, for $B \in \mathcal{N}(0_{\mathbb{A}})$, $A := A_1 \cap B$. Then, for any $a \in A$ and $h \in U$, we have that

$$\begin{aligned} \psi(ah) &= ah \circ (f+ah)^{-1} \circ ah = a^2 h \circ (f+ah)^{-1} \circ h \\ &\in aAU_1 \circ (f + A_1 U_2)^{-1} \circ U_1 \subseteq aBU_1 \circ V_2^{-1} \circ U_1 \\ &\subseteq aBU_1 \circ V_1 \circ U_1 \subseteq aBV, \end{aligned}$$

which proves the assertion. Since an $\mathbb{A}$-linear combination of infinitesimal maps is infinitesimal, ϕ is also infinitesimal. This completes the proof. □

Therefore, we are led to the following:

10.3.5 Theorem. *Let $\mathbb{A}$ be a unital commutative associative lmc Q-algebra and let $P \in \mathcal{PFG}(\mathbb{A})$. Then $\mathrm{GL}_{\mathbb{A}}(P)$ is a Lie group. Therefore, for every $\mathbb{A}$-bundle E of fiber type P, the corresponding bundle of frames $F(E)$ is an $\mathbb{A}$-smooth principal bundle.*

Given an $\mathbb{A}$-bundle $E \equiv (E, \pi, X)$, the set of $\mathbb{A}$-*smooth sections of* E will be denoted by $\Gamma(X, E)$. An $\mathbb{A}$-***connection*** on E is defined to be an $\mathbb{A}$-bilinear map

$$\nabla^E : \Gamma(X, TX) \times \Gamma(X, E) \longrightarrow \Gamma(X, E) : (\xi, s) \mapsto \nabla^E_\xi s,$$

satisfying the following properties:

$$\nabla^E_{f\xi} s = f \cdot \nabla^E_\xi s,$$
$$\nabla^E_\xi (fs) = f \cdot \nabla^E_\xi s + (df \circ \xi) \cdot s,$$

for every $\xi \in \Gamma(X, TX)$, $s \in \Gamma(X, E)$, and every $\mathbb{A}$-smooth map $f : X \to \mathbb{A}$.

Here a connection is essentially identified with a covariant derivation. This well known fact in finite dimensional bundles (owing to the existence of bases in the models), is not necessarily true in the infinite dimensional case, even for Banach bundles (cf. Flaschel-Klingenberg [29], Vilms [138]). In our context, although bases do not exist, we recover this property of finite dimensional bundles by showing that ∇^E amounts to a family of ***(generalized) Christoffel symbols***,

– firstly, by imbedding the given bundle in one with bases; and

– secondly, by extending ∇^E to a suitable map,

under the condition that the base manifold admits $\mathbb{A}$-bump functions (see below). The technical details are given in Vassiliou-Papatriantafillou [134, Theorem 4.2]. Here we merely recall that the Christoffel symbols now have the form

$$\Gamma_\alpha : U_\alpha \to L^2_{\mathbb{A}}(M \times P, P) \cong \mathrm{L}_{\mathbb{A}}(M, \mathrm{L}_{\mathbb{A}}(P)); \qquad \alpha \in I,$$

and satisfy the compatibility conditions

$$(10.3.4) \qquad \Gamma_\beta(x)(h,k) = \big(g_{\beta\alpha}(x) \circ \Gamma_\alpha(x) - D(g_{\beta\alpha} \circ \phi_\alpha^{-1})(\phi_\alpha(x))\big)\big((\bar{\phi}_\alpha \circ \bar{\phi}_\beta^{-1})(h), g_{\alpha\beta}(x)(k)\big),$$

for every $h \in M$, $k \in P$, $x \in U_{\alpha\beta}$ and α, $\beta \in I$. The previous expressions are defined with respect to a smooth atlas $\{(U_\alpha, \phi_\alpha) \,|\, \alpha \in I\}$ of X and a trivializing covering $\{(U_\alpha, \tau_\alpha) \,|\, \alpha \in I\}$ of E. The isomorphism $\bar{\phi}_\alpha$ has been explained in the discussion concerning the tangent space of an $\mathbb{A}$-manifold (see p. 378).

The equivalence of ∇^E with the family (Γ_α), as already commented, relies on the existence of appropriate bump functions. In fact, analogously to the standard case, we say that a manifold $X \in \mathcal{M}an(\mathbb{A})$ *admits* $\mathbb{A}$-***bump functions***, if

> for every open $U \subseteq X$ and $x \in U$, there is an open $V \subseteq X$, with $x \in V \subseteq \overline{V} \subseteq U$, and an $\mathbb{A}$-smooth map $f : X \to \mathbb{A}$ such that $f|_{\overline{V}} = 1$, $f|_{X \setminus U} = 0$.

The existence of $\mathbb{A}$-bump functions on $\mathbb{A}$-manifolds is ensured if $\mathbb{A}$ coincides with the algebra $\mathcal{C}(X)$ of continuous functions on a *completely regular Hausdorff* topological space X, or with the algebra $\mathcal{C}^\infty(X)$ of smooth functions on a *compact* manifold X (see Papatriantafillou [93] and [92], respectively). The general case still remains open.

Similarly, an $\mathbb{A}$-connection is equivalent to a splitting of the short exact sequence of $\mathbb{A}$-bundles (see Papatriantafillou [95])

$$0 \xrightarrow{\ i\ } VE \longrightarrow TE \xrightarrow{\ T\pi!\ } \pi^*(TM) \longrightarrow 0,$$

where VE is the vertical subbundle of TE, and $T\pi! := (\tau_E, T\pi)$ is the morphism induced by the universal property of the pull-back (cf. also Vilms

[138] regarding the case of connections on Banach bundles). Again this result is valid for finite-dimensional bundles but not necessarily for infinite-dimensional ones.

Concerning the connections of the bundle of frames, we have the next:

10.3.6 Theorem. *Let $\mathbb{A}$ be a unital commutative associative lmc Q-algebra and let $E \equiv (E, \pi, X)$ be an $\mathbb{A}$-bundle of fiber type $P \in \mathcal{PFG}(\mathbb{A})$. Then the $\mathbb{A}$-connections on E correspond bijectively to the connections on the bundle of frames $F(E) \equiv (F(E), \mathrm{GL}_{\mathbb{A}}(P), X, p)$.*

Proof. Let $\Lambda^1(U_\alpha, \mathrm{L}_{\mathbb{A}}(P))$ denote the $\mathbb{A}$-module of $\mathrm{L}_{\mathbb{A}}(P)$-valued $\mathbb{A}$-smooth 1-forms on U_α, $\alpha \in I$.

Given a connection ∇^E on E, its Christoffel symbols (Γ_α) induce the 1-forms $\omega_\alpha \in \Lambda^1(U_\alpha, \mathrm{L}_{\mathbb{A}}(P))$ defined by

$$(10.3.5) \qquad \big(\omega_{\alpha,x}(v)\big)(h) := \Gamma_\alpha(x)(\bar{\phi}_\alpha(v), h),$$

for every $x \in U_\alpha$, $v \in T_xX$ and $h \in P$.

We check that (10.3.4) yields the compatibility condition (: gauge transformation)

$$(10.3.6) \qquad \omega_\beta = \mathrm{Ad}(g_{\alpha\beta}^{-1}).\omega_\alpha + g_{\alpha\beta}^{-1} \cdot dg_{\alpha\beta}.$$

We recall that the last term on the right-hand side of (10.3.6) is the total differential mentioned in Example 3.3.6(a). Therefore, as in the classical case (see, e.g., Kobayashi-Nomizu [49], Pham Mau Quan [101], Sulanke-Wintgen [118]), we obtain a global connection form $\omega \in \Lambda^1(F(E), \mathrm{L}_{\mathbb{A}}(P))$ by setting

$$(10.3.7) \qquad \omega|_{\pi^{-1}(U_\alpha)} := \mathrm{Ad}(g_\alpha^{-1}).\pi^*\omega_\alpha + g_\alpha^{-1} \cdot dg_\alpha,$$

where $g_\alpha : \pi^{-1}(U_\alpha) \to \mathrm{GL}_{\mathbb{A}}(P)$ is the $\mathbb{A}$-smooth map defined by the equality $p = s_\alpha(\pi(p)) \cdot g_\alpha(p)$, for every $p \in \pi^{-1}(U_\alpha)$, if (s_α) are the natural sections of $F(E)$ over the open covering $\mathcal{U} = \{U_\alpha \mid \alpha \in I\}$. The local connection forms of ω are precisely the given ω_α's.

Conversely, starting with a connection form ω and its local connection forms (ω_α) on the bundle of frames, we see that equalities (10.3.5), for all $\alpha \in I$, determine a family of Christoffel symbols satisfying (10.3.4), as a result of (10.3.6). Hence, we obtain an $\mathbb{A}$-connection on E. The bijectivity of the statement is routinely checked. □

The natural question now arising is what happens if $\mathbb{A}$ is not a Q-algebra. In this case, $\mathrm{GL}_{\mathbb{A}}(P)$ may not be a Lie group with corresponding Lie algebra $\mathrm{L}_{\mathbb{A}}(P)$. As before, an $\mathbb{A}$-connection ∇^E on E determines the local forms (10.3.5) satisfying (10.3.6), but (10.3.7) has no meaning, so the family (ω_α) does not globalize to a connection form ω on $F(E)$. However, using the ideas of Section 10.2, each ∇^E will be associated with an abstract connection D on an appropriate sheaf.

To this end, we say that a map $g : U \to \mathrm{GL}_{\mathbb{A}}(P)$ ($U \subseteq X$ open) is ***invertibly*** $\mathbb{A}$***-smooth***, if g and g^{-1} (with $g^{-1}(x) := g(x)^{-1}$, $x \in U$) are $\mathbb{A}$-smooth when they are considered as taking values in $\mathrm{L}_{\mathbb{A}}(P)$. Similarly, a local section of $F(E)$ will be called invertibly $\mathbb{A}$-smooth, if its principal parts are invertibly $\mathbb{A}$-smooth. By the principal part of a section of $F(E)$, relative to a trivialization $\Phi_\alpha : F(E)|_{U_\alpha} \to \mathrm{GL}_{\mathbb{A}}(P)$, we mean the map $\mathrm{pr}_2 \circ \Phi_\alpha \circ s : U \cap U_\alpha \to \mathrm{GL}_{\mathbb{A}}(P)$.

If $\Gamma(U_\alpha, F(E))$, $\alpha \in I$, are the sets of invertibly $\mathbb{A}$-smooth sections of $F(E)$ over the trivializing open covering $\mathcal{U}$ of both E and $F(E)$, we obtain the presheaf $U_\alpha \longmapsto \Gamma(U_\alpha, F(E))$ generating a sheaf $\mathcal{F}(E)$. Similarly, $\mathcal{GL}_{\mathbb{A}}(P)$ denotes the sheaf of germs of invertibly $\mathbb{A}$-smooth $\mathrm{GL}_{\mathbb{A}}(P)$-valued maps on X. Taking into account the structure of the principal bundle of frames $F(E)$ and following the general construction of Example 4.1.9(a), we see that the quadruple

$$\mathcal{F}(E) \equiv (\mathcal{F}(E), \mathcal{GL}_{\mathbb{A}}(P), X, \tilde{\pi})$$

is a principal sheaf. The structure sheaf $\mathcal{GL}_{\mathbb{A}}(P)$ is a Lie sheaf of groups of the form

$$(\mathcal{GL}_{\mathbb{A}}(P), \mathcal{A}d, \mathcal{L}_{\mathbb{A}}(P), \partial),$$

where $\mathcal{L}_{\mathbb{A}}(P)$ is the sheaf of germs of $\mathbb{A}$-smooth $\mathrm{L}_{\mathbb{A}}(P)$-valued maps on X, $\mathcal{A}d : \mathcal{GL}_{\mathbb{A}}(P) \to \mathcal{A}ut\big(\mathcal{L}_{\mathbb{A}}(P)\big)$ the sheafification of the usual adjoint representation of $\mathrm{GL}_{\mathbb{A}}(P)$ on $\mathrm{L}_{\mathbb{A}}(P)$, and $\partial : \mathcal{GL}_{\mathbb{A}}(P) \to \Omega^1_X(\mathrm{L}_{\mathbb{A}}(P))$ is the Maurer-Cartan differential obtained by the sheafification of the ordinary total differential (with respect to the invertibly $\mathbb{A}$-smoothness). Here, $\Omega^1_X(\mathrm{L}_{\mathbb{A}}(P))$ is the sheaf of germs of $\mathrm{L}_{\mathbb{A}}(P)$-valued $\mathbb{A}$-smooth 1-forms on X, thus we have the canonical identifications

$$\big(\Omega^1_X(\mathrm{L}_{\mathbb{A}}(P))\big)(U) \cong \Lambda^1(U, \mathrm{L}_{\mathbb{A}}(P)); \qquad U \in \mathfrak{T}_X. \tag{10.3.8}$$

Moreover, under (10.3.8), the local connection forms (ω_α) determine a 0-cochain $(\widetilde{\omega}_\alpha) \in C^0\big(\mathcal{U}, \Omega^1_X(\mathrm{L}_{\mathbb{A}}(P))\big)$, which define a unique connection $D : \mathcal{F}(E) \to \Omega^1_X(\mathrm{L}_{\mathbb{A}}(P))$ with local connection forms $(\widetilde{\omega}_\alpha)$.

Reversing our arguments, we check that a connection D on $\mathcal{F}(E)$ determines a unique $\mathbb{A}$-connection ∇^E on E.

To conclude, we summarize the main results of the present section in the following statement.

10.3.7 Theorem. *Let* $\mathbb{A}$ *be an arbitrary unital commutative associative lmc algebra. Also, let* E *be an* $\mathbb{A}$*-bundle,* $F(E)$ *its bundle of frames, and* $\mathcal{F}(E)$ *the principal sheaf of germs of invertibly* $\mathbb{A}$*-smooth sections of* $F(E)$*. Then there exists a bijective correspondence between the* $\mathbb{A}$*-connections* ∇^E *on* E *and the connections* D *on* $\mathcal{F}(E)$*. In particular, if* $\mathbb{A}$ *is a* Q*-algebra, then both* ∇^E *and* D *correspond bijectively to a global connection (form)* ω *on the bundle of frames* $F(E)$*.*

The previous discussion shows that the (abstract) connections D on a *topological object*, namely $\mathcal{F}(E)$, describe, through appropriate isomorphisms, the connections of E in all of their equivalent forms, as well as, in the case of a Q-algebra, the corresponding connections of the (smooth) bundle of frames $F(E)$. Therefore, the example of $\mathbb{A}$-bundles illustrates, once again, the efficiency of the sheaf-theoretic approach in enlarging certain aspects of the ordinary differential geometry to a non-smooth context.

10.4. The torsion of a linear connection on Ω^*

If X is a smooth manifold, a linear connection on the tangent bundle TX induces the corresponding torsion and its torsion form. This is a standard fact found in most of the books dealing with connections. The purpose of the present section is to obtain the analogous notions in our abstract framework.

To explain the title and the general setting of the section, let us recall Example 2.1.4(a): Given a smooth manifold X, we define the sheaf of germs of its differential 1-forms $\Omega := \Omega^1_X$, which is a $\mathcal{C}^\infty_X$-module. In particular, if X is finite-dimensional, Ω is a vector sheaf. The dual module Ω^* is identified with the sheaf of germs of smooth sections of the tangent bundle TX, in other words the sheaf of germs of smooth vector fields of X. Therefore, the abstract analog of the torsion will be obtained by considering $\mathcal{A}$-connections on Ω^*.

Here, starting with a differential triad $(\mathcal{A}, d, \Omega)$ whose Ω is assumed to be a vector sheaf, we define the torsion as an appropriate morphism on the sheaf of frames of Ω^*.

Before moving on to the main subject, we prove a few auxiliary results concerning vector sheaves in general.

10.4.1 Lemma. *Let $\mathcal{E} \equiv (\mathcal{E}, \pi_{\mathcal{E}}, X)$ be a vector sheaf of rank n. Then the $\mathcal{A}$-module $\mathcal{E}^* := \mathcal{H}om_{\mathcal{A}}(\mathcal{E}, \mathcal{A})$ is also a vector sheaf of rank n.*

Proof. Let $\mathcal{U} \equiv ((U_\alpha), (\psi_\alpha))$ be a local frame of $\mathcal{E}$. We already know (see Section 5.1) that each local coordinate $\psi_\alpha : \mathcal{E}|_{U_\alpha} \xrightarrow{\simeq} \mathcal{A}^n|_{U_\alpha}$ determines the basis $e^\alpha = (e_i^\alpha)$, $1 \leq i \leq n$, of $\mathcal{E}(U_\alpha)$, given by $e_i^\alpha = \psi_\alpha^{-1}(\epsilon_i|_{U_\alpha})$, if (ϵ_i) is the natural basis of $\mathcal{A}^n(X) \cong \mathcal{A}(X)^n$.

Without loss of generality, we may assume that (U_α) is a basis for the topology of X (see also the comments preceding (4.1.5)). Thus, according to the conclusion of Subsection 1.2.2, $\mathcal{E}$ is identified with the sheaf generated by the presheaf of its sections over $\mathcal{U}$. The same principle applies to every other sheaf considered in the sequel.

With the previous remarks in mind, we have that $\mathcal{E}^*$ is generated by the presheaf of $\mathcal{A}(U_\alpha)$-modules $U_\alpha \mapsto \mathrm{Hom}_{\mathcal{A}|_{U_\alpha}}(\mathcal{E}|_{U_\alpha}, \mathcal{A}|_{U_\alpha})$. Therefore, for every $V \in \mathcal{U}$ with $V \subseteq U_\alpha$, we can consider the $\mathcal{A}(V)$-isomorphism

(10.4.1)
$$\begin{aligned} \psi^*_{\alpha,V} : \mathrm{Hom}_{\mathcal{A}|_V}(\mathcal{E}|_V, \mathcal{A}|_V) &\xrightarrow{\simeq} \mathcal{A}(V)^n : \\ f &\longmapsto \big(f(e_1^\alpha|_V), \dots, f(e_n^\alpha|_V)\big), \end{aligned}$$

where the map f in the target is, obviously, the induced morphism of sections.

The inverse of (10.4.1) is obtained as follows: If $(\alpha_1, \dots, \alpha_n) \in \mathcal{A}(V)^n$, the morphism $(\psi^*_{\alpha,V})^{-1}(\alpha_1, \dots, \alpha_n) \in \mathrm{Hom}_{\mathcal{A}|_V}(\mathcal{E}|_V, \mathcal{A}|_V)$ is determined by

(10.4.1′)
$$\big((\psi^*_{\alpha,V})^{-1}(\alpha_1, \dots, \alpha_n)\big)(u) := \sum_{i=1}^n u_i \alpha_i(x),$$

for every $u = \sum_{i=1}^n u_i e_i^\alpha(x) \in \mathcal{E}_x$ and $x \in V$.

Varying V in U_α, we obtain a presheaf isomorphism generating an $\mathcal{A}|_{U_\alpha}$-isomorphism

(10.4.2)
$$\psi^*_\alpha : \mathcal{E}^*|_{U_\alpha} \xrightarrow{\simeq} \mathcal{A}^n|_{U_\alpha}.$$

The local frame $\big((U_\alpha), (\psi^*_\alpha)\big)$ determines the desired vector sheaf structure on $\mathcal{E}^*$. □

In virtue of the previous lemma we call $\mathcal{E}^*$ the ***dual vector sheaf*** of $\mathcal{E}$.

Each coordinate ψ^*_α of $\mathcal{E}^*$ induces the basis $e^*_\alpha = ({}^*e^\alpha_i)$, $1 \leq i \leq n$, of $\mathcal{E}^*(U_\alpha)$, with

$$({}10.4.3) \qquad {}^*e^\alpha_i := (\psi^*_\alpha)^{-1}(\epsilon_i|_{U_\alpha}).$$

As expected, e^*_α coincides –up to isomorphism– with the ***dual basis*** of e_α. Indeed, after the identification

$$\mathcal{E}^*(U_\alpha) = \mathcal{H}om_{\mathcal{A}}(\mathcal{E},\mathcal{A})(U_\alpha) \cong \mathrm{Hom}_{\mathcal{A}|_{U_\alpha}}(\mathcal{E}|_{U_\alpha},\mathcal{A}|_{U_\alpha}),$$

each ${}^*e^\alpha_i$ is viewed as an element of $\mathrm{Hom}_{\mathcal{A}|_{U_\alpha}}(\mathcal{E}|_{U_\alpha},\mathcal{A}|_{U_\alpha})$. Thus, the induced morphism of sections (see also (1.1.3)) ${}^*e^\alpha_i = \overline{({}^*e^\alpha_i)} : \mathcal{E}(U_\alpha) \to \mathcal{A}(U_\alpha)$ can be evaluated at the sections of the basis $e^\alpha = (e^\alpha_1,\ldots,e^\alpha_n)$. Therefore, for every $x \in U_\alpha$, (10.4.3) and (10.4.1′), along with the completeness of the presheaves involved and (1.2.17), yield:

$$\begin{aligned} {}^*e^\alpha_i(e^\alpha_j)(x) &= \Big((\psi^*_\alpha)^{-1}(\epsilon_i|_{U_\alpha})\Big)\big(e^\alpha_j(x)\big) \\ &\equiv \Big((\psi^*_{\alpha,U_\alpha})^{-1}(\epsilon_i|_{U_\alpha})\Big)\big(e^\alpha_j(x)\big) \\ &= \delta_{ij}(x), \end{aligned}$$

from which we obtain

$$(10.4.4) \qquad {}^*e^\alpha_i(e^\alpha_j) \equiv \delta_{ij}|_{U_\alpha}; \qquad i,j = 1,\ldots,n.$$

The second auxiliary result is:

10.4.2 Lemma. *Let $\mathcal{E}$ be a vector sheaf. Then*

$$(10.4.5) \qquad (\mathcal{E}{\textstyle\bigwedge_{\mathcal{A}}}\mathcal{E})(U_\alpha) \cong \mathcal{E}(U_\alpha){\textstyle\bigwedge_{\mathcal{A}(U_\alpha)}}\mathcal{E}(U_\alpha),$$

for every (U_α,ψ_α) in the local frame of $\mathcal{E}$.

Proof. For convenience, we omit the index of the exterior product. Then, as in the proof of Lemma 10.4.1, we assume that $\mathcal{U}$ is a basis for the topology of X. Since $\psi_\alpha : \mathcal{E}|_{U_\alpha} \xrightarrow{\simeq} \mathcal{A}^n|_{U_\alpha}$ is an $\mathcal{A}|_{U_\alpha}$-isomorphism, it follows that

$$(10.4.6) \qquad \begin{aligned} (\mathcal{E}{\textstyle\bigwedge}\mathcal{E})(U_\alpha) &= (\mathcal{E}{\textstyle\bigwedge}\mathcal{E})|_{U_\alpha}(U_\alpha) \\ &= (\mathcal{E}|_{U_\alpha}{\textstyle\bigwedge}\mathcal{E}|_{U_\alpha})(U_\alpha) \\ &\cong (\mathcal{A}^n|_{U_\alpha}{\textstyle\bigwedge}\mathcal{A}^n|_{U_\alpha})(U_\alpha) \\ &= (\mathcal{A}^n{\textstyle\bigwedge}\mathcal{A}^n)(U_\alpha). \end{aligned}$$

On the other hand, by (5.1.2),

$$\mathcal{A}^n(U_\alpha)\bigwedge\mathcal{A}^n(U_\alpha) \cong \mathcal{A}(U_\alpha)^n\bigwedge\mathcal{A}(U_\alpha)^n \cong \mathcal{A}(U_\alpha)^{\binom{n}{2}}.$$

Thus the sheaf $\mathcal{A}^n\bigwedge\mathcal{A}^n$, generated by the presheaf $U_\alpha \longmapsto \mathcal{A}^n(U_\alpha)\bigwedge\mathcal{A}^n(U_\alpha)$, can be identified with the sheaf $\mathcal{A}^{\binom{n}{2}}$, generated by $U_\alpha \longmapsto \mathcal{A}(U_\alpha)^{\binom{n}{2}}$; hence, by the completeness of the latter,

$$(\mathcal{A}^n\bigwedge\mathcal{A}^n)(U_\alpha) \cong \mathcal{A}(U_\alpha)^{\binom{n}{2}},$$

from which, along with (10.4.6), we get

$$(\mathcal{E}\bigwedge\mathcal{E})(U_\alpha) \cong \mathcal{A}(U_\alpha)^{\binom{n}{2}}. \tag{10.4.7}$$

By the same token,

$$\begin{aligned} \mathcal{E}(U_\alpha)\bigwedge\mathcal{E}(U_\alpha) &\cong \mathcal{A}^n(U_\alpha)\bigwedge\mathcal{A}^n(U_\alpha) \\ &\cong \mathcal{A}(U_\alpha)^n\bigwedge\mathcal{A}(U_\alpha)^n \\ &\cong \mathcal{A}(U_\alpha)^{\binom{n}{2}}. \end{aligned} \tag{10.4.8}$$

The isomorphisms (10.4.7) and (10.4.8) prove the lemma. □

To define the abstract torsion, we start with a differential triad $(\mathcal{A}, d, \Omega)$, where Ω is a *vector sheaf of rank n.*

As in Section 8.1, in order to avoid confusion with the exterior powers Ω^n and $\Omega(U)^n$, $U \subseteq X$ open, we set

$$\Omega^{(n)} = \underbrace{\Omega \times_X \cdots \times_X \Omega}_{n-\text{factors}} \quad \text{and} \quad \Omega(U_\alpha)^{(n)} = \underbrace{\Omega(U_\alpha) \times \cdots \times \Omega(U_\alpha)}_{n-\text{factors}},$$

where the second product is now the usual cartesian product.

Since Ω^* is a vector sheaf of rank n, we obtain the corresponding principal sheaf of frames $\mathcal{P}(\Omega^*) \equiv (\mathcal{P}(\Omega^*), \mathcal{GL}(n, \mathcal{A}), X, \tilde{\pi})$. We denote by (ψ_α), (ψ^*_α) and (Ψ^*_α) the coordinates of Ω, Ω^* and $\mathcal{P}(\Omega^*)$, respectively, over a common open covering $\mathcal{U} = (U_\alpha)$, which is a *basis for the topology* of X. The coordinates (ψ_α) and (ψ^*_α) induce the bases

$$\theta_\alpha := (\theta^\alpha_1, \ldots, \theta^\alpha_n) \quad \text{and} \quad \theta^*_\alpha := ({}^*\theta^\alpha_1, \ldots, {}^*\theta^\alpha_n)$$

of $\Omega(U_\alpha)$ and $\Omega^*(U_\alpha)$, respectively.

For each $U_\alpha \in \mathcal{U}$, we define the map

$$F_{U_\alpha} : \mathrm{Iso}_{\mathcal{A}|_{U_\alpha}}\big(\mathcal{A}^n|_{U_\alpha}, \Omega^*|_{U_\alpha}\big) \longrightarrow \Omega(U_\alpha)^{(n)} \tag{10.4.9}$$

by setting

$$F_{U_\alpha}(f) := (f_{ij})^{-1} \cdot (\theta_\alpha)^T, \tag{10.4.10}$$

where the matrix $(f_{ij}) \in \mathrm{GL}(n, \mathcal{A}(U_\alpha))$ is determined by

$$f(\epsilon_i|_{U_\alpha}) = \sum_{j=1}^{n} f_{ji} \cdot {}^*\theta_i^\alpha, \tag{10.4.11}$$

and $(\theta_\alpha)^T$ denotes the transpose of $\theta_\alpha = (\theta_1^\alpha, \dots, \theta_n^\alpha)$. It is clear that (F_{U_α}), with U_α running in $\mathcal{U}$, is a presheaf morphism.

10.4.3 Definition. The ***canonical morphism of*** $\mathcal{P}(\Omega^*)$ is the morphism (of sheaves of sets)

$$F : \mathcal{P}(\Omega^*) \longrightarrow \Omega^{(n)} \tag{10.4.12}$$

generated by the presheaf morphism (F_{U_α}).

Let us compute the images $F(\sigma_\alpha^*) \in \Omega^{(n)}(U_\alpha)$ of the natural sections σ_α^* of $\mathcal{P}(\Omega^*)$, since they can be used to determine F in a direct way.

By (5.2.6′), $\sigma_\alpha^* = \widetilde{(\psi_\alpha^*)^{-1}}$, where $\psi_\alpha^* \in \mathrm{Iso}_{\mathcal{A}|_{U_\alpha}}(\Omega^*|_{U_\alpha}, \mathcal{A}^n|_{U_\alpha})$ is the coordinate of Ω^* over U_α. Hence Diagram 1.7 yields

$$\begin{aligned} F(\sigma_\alpha^*) &= F\big(\widetilde{(\psi_\alpha^*)^{-1}}\big) = \big(F_{U_\alpha}\big((\psi_\alpha^*)^{-1}\big)\big)^\sim \\ \text{(see (10.4.10))} \qquad &= \big((\psi_{ij}^*) \cdot (\theta_\alpha)^T\big)^\sim. \end{aligned}$$

where (ψ_{ij}^*) is the matrix of ψ_α^*, determined by reversing the analog of (10.4.11). But the definition of θ_α^* implies that $(\psi_{ij}^*) = \mathrm{I}$ (: the identity matrix), from which it follows that

$$F(\sigma_\alpha^*) = \big((\theta_\alpha)^T\big)^\sim \equiv \widetilde{\theta_\alpha} = \big(\theta_1^\alpha, \dots, \theta_n^\alpha\big)^\sim, \qquad \alpha \in I. \tag{10.4.13}$$

After the natural identification

$$\Omega(U_\alpha)^{(n)}(s_1, \dots, s_n) \xrightarrow{\ \simeq\ } (s_1, \dots, s_n)^\sim \in \Omega^{(n)}(U_\alpha), \tag{10.4.14}$$

we can write $\mathsf{F}(\sigma_\alpha^*) \equiv \theta_\alpha$ and call the previous sections the ***canonical local forms*** of $\mathcal{P}(\Omega^*)$, under an obvious abuse of terminology, of course.

We shall show that F is tensorial, with respect to appropriate actions. To this end, observe that each group $\mathrm{GL}(n, \mathcal{A}(U_\alpha))$ acts on the left of $\Omega(U_\alpha)^{(n)}$ in the following way:

$$\begin{gathered} \mathrm{GL}(n, \mathcal{A}(U_\alpha)) \times \Omega(U_\alpha)^{(n)} \longrightarrow \Omega(U_\alpha)^{(n)} : \\ \big((g_{ij}), (\omega_1, \ldots, \omega_n)\big) \longmapsto (g_{ij})^{-1} \cdot (\omega_1, \ldots, \omega_n)^T. \end{gathered} \tag{10.4.15}$$

The previous local actions determine a presheaf morphism generating an action of $\mathcal{GL}(n, \mathcal{A})$ on the left of $\Omega^{(n)})$.

10.4.4 Proposition. *The canonical morphism* F *is tensorial with respect to the action of* $\mathcal{GL}(n, \mathcal{A})$ *on the right of* $\mathcal{P}(\Omega^*)$*, and the action of* $\mathcal{GL}(n, \mathcal{A})$ *on the left of* $\Omega^{(n)}$*, generated by the local actions (10.4.15).*

Proof. It suffices to work locally. Indeed, for any $h \in \mathrm{Iso}_{\mathcal{A}|_{U_\alpha}}(\mathcal{A}^n|_{U_\alpha}, \Omega^*|_{U_\alpha})$ and $g \in \mathrm{GL}(n, \mathcal{A}(U_\alpha))$, we have that

$$\mathsf{F}_{U_\alpha}(h \cdot g) := \mathsf{F}_{U_\alpha}(h \circ g) = \big((h \circ g)_{ij}\big)^{-1} \cdot (\theta_\alpha)^T = (g_{ij})^{-1} \cdot \mathsf{F}_{U_\alpha}(h),$$

which yields the result. □

The tensoriality of F and (1.4.13) essentially determine F. As a matter of fact, if $p \in \mathcal{P}(\Omega^*)$ with $\tilde{\pi}(x) \in U_\alpha$, there is a unique $g_\alpha \in \mathcal{GL}(n, \mathcal{A})_x$ such that $p = \sigma_\alpha^* \cdot g_\alpha$. Hence,

$$\mathsf{F}(p) = g_\alpha^{-1} \cdot \mathsf{F}(\sigma_\alpha^*) = g_\alpha^{-1} \cdot \big((\theta_\alpha)^T\big)^{\sim}(x). \tag{10.4.16}$$

But, if $x \in U_{\alpha\beta}$, we also have

$$\mathsf{F}(p) = g_\beta^{-1} \cdot \big((\theta_\beta)^T\big)^{\sim}(x). \tag{10.4.16$'$}$$

However, the two expressions of F coincide. This is proved by taking into account the following facts:

i) $g_\beta = g_{\alpha\beta}^*(x)^{-1} \cdot g_\alpha$, where $(g_{\alpha\beta}^*)$ is the cocycle of $\mathcal{P}(\Omega^*)$.

ii) By Corollary 5.2.3, $g_{\alpha\beta}^*$ is identified with the transition matrix, say $\big({}^*\psi_{ij}^{\alpha\beta}\big)$, of the coordinate transformation $\psi_{\alpha\beta}^*$ of Ω^*.

iii) If we denote by $\big(\psi_{ij}^{\alpha\beta}\big)$ the transition matrix of the coordinate transformation $\psi_{\alpha\beta}$ of Ω, the definition of (θ_α), $\alpha \in I$, and elementary calculations show that

$$(\theta_\beta)^T = \big(\psi_{ij}^{\alpha\beta}\big)^T \cdot (\theta_\alpha)^T.$$

iv) Since the bases (θ^*_α) and (θ_α) are dual, for all $\alpha \in I$ (see(10.4.4)), it follows that

$$({}^*\psi^{\alpha\beta}_{ij}) = \big((\psi^{\alpha\beta}_{ij})^T\big)^{-1}.$$

Therefore,

$$\begin{aligned} g_\beta^{-1} \cdot \big((\theta_\beta)^T\big)^\sim(x) &= g_\alpha^{-1} \cdot \Big(({}^*\psi^{\alpha\beta}_{ij})\Big)^\sim(x) \cdot \Big((\psi^{\alpha\beta}_{ij})^T \cdot (\theta_\alpha)^T\Big)^\sim(x) \\ &= g_\alpha^{-1} \cdot \Big(({}^*\psi^{\alpha\beta}_{ij}) \cdot (\psi^{\alpha\beta}_{ij})^T \cdot (\theta_\alpha)^T\Big)^\sim(x) \\ &= g_\alpha^{-1} \cdot \big((\theta_\alpha)^T\big)^\sim(x). \end{aligned}$$

The previous arguments actually prove:

10.4.5 Corollary. *The canonical morphism F is completely determined by the family (of local canonical forms) (θ_α) via (10.4.16).*

Proof. Let $\mathsf{F}' : \mathcal{P}(\Omega^*) \to \Omega^{(n)}$ be the map with $\mathsf{F}'(p) := g_\alpha^{-1} \cdot \big((\theta_\alpha)^T\big)^\sim(x)$, for any p as in (10.4.16). F' is well defined. It is continuous, since over each $U_\alpha \in \mathcal{U}$,

$$\mathsf{F}'(p) = \boldsymbol{k}\big(p, \sigma^*_\alpha(\pi(p))\big)^{-1} \cdot \theta_\alpha(\pi(p))$$

(see Proposition 4.1.4); thus F' is a continuous morphism. By its construction, F' is tensorial.

Moreover, (10.4.13) and the definition of F' imply that

$$\mathsf{F}'(\sigma^*_\alpha) = \big((\theta_\alpha)^T\big)^\sim = \mathsf{F}(\sigma^*_\alpha).$$

Hence, the preceding equality and the tensoriality of both F and F' imply that $\mathsf{F} = \mathsf{F}'$. □

Now, we further assume that

> Ω^* admits an $\mathcal{A}$-connection ∇. We recall that ∇ is completely determined by the connection matrices $\omega^\alpha := (\omega^\alpha_{ij}) \in M_n(\Omega(U_\alpha))$, $\alpha \in I$ (see (7.1.4) and Theorem 7.1.4). The corresponding curvature R is also completely determined by the curvature matrices $R^\alpha = (R^\alpha_{ij}) \in M_n(\Omega^2(U_\alpha))$ (see (8.5.23) and its subsequent discussion).

To define the torsion, we need an appropriate exterior product: After the identification $M_n(\Omega(U_\alpha)) \cong \Omega(U_\alpha)^{(n^2)}$, analogously to (8.1.19) we have the exterior products

$$\wedge_{U_\alpha} : \Omega(U_\alpha)^{(n^2)} \times \Omega(U_\alpha)^{(n)} \longrightarrow \big(\Omega(U_\alpha)^2\big)^{(n)} = \big(\Omega(U_\alpha) \textstyle\bigwedge_{\mathcal{A}(U_\alpha)} \Omega(U_\alpha)\big)^{(n)},$$

for all $U_\alpha \in \mathcal{U}$, which generate an exterior product

$$\wedge : \Omega^{(n^2)} \times \Omega^{(n)} \longrightarrow (\Omega^2)^{(n)}. \tag{10.4.17}$$

Using the later and the previous assumptions about Ω^*, we define the $\mathbb{K}$-morphism, called ***Cartan (first) structure operator***,

$$\mathsf{W} : \Omega^{(n)} \longrightarrow (\Omega^2)^{(n)},$$

given by the ***Cartan (first) structure equation***

$$\mathsf{W}(a) := d^1 a + \omega^\alpha(x) \wedge a, \qquad a \in (\Omega^{(n)})_x \cong (\Omega_x)^{(n)}. \tag{10.4.18}$$

We clarify that if $a = (a_1, \ldots, a_n)$, then $d^1 a = (d^1 a_1, \ldots, d^1 a_n)$.

10.4.6 Definition. The ***torsion*** of an $\mathcal{A}$-connection ∇ on Ω^* (equivalently, of a connection D on $\mathcal{P}(\Omega^*)$) is the morphism of (sheaves of sets) $\mathsf{T} := \mathsf{W} \circ \mathsf{F}$, also shown in Diagram 10.1. Accordingly, by abuse of terminology, we call ***local torsion forms*** of ∇ the local sections

$$\Theta_\alpha := \mathsf{T}(\sigma_\alpha^*) \in (\Omega^2)^{(n)}(U_\alpha), \qquad \alpha \in I.$$

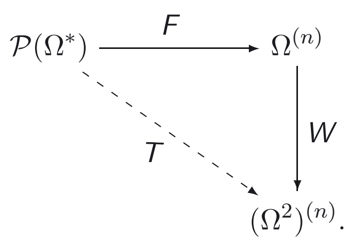

DIAGRAM 10.1

10.4.7 Proposition. *The local torsion forms satisfy the* ***local Cartan (first) structure equations***

$$\Theta_\alpha = \big(d^1\theta_\alpha + \omega^\alpha \wedge \theta_\alpha\big)^\sim; \qquad \alpha \in I, \tag{10.4.19}$$

within appropriate identifications.

In the previous statement, by the notation introduced in ($\lozenge$), p. 104, the superscript "$^\sim$" represents the map

$$\big(\Omega(U_\alpha)^2\big)^{(n)} \ni t \to \widetilde{t} \in (\Omega^2)^{(n)}(U_\alpha).$$

Recall that $U_\alpha \longmapsto \big(\Omega(U_\alpha)^2\big)^{(n)}$ generates $(\Omega^2)^{(n)}$ (in this respect see also the Subsection 1.3.6). Moreover, for convenience, we write $\wedge$ instead of $\wedge_{U_\alpha}$.

Proof. In virtue of (10.4.13) and (10.4.18), we have that

$$\begin{aligned}\Theta_\alpha(x) &= (\mathsf{W}\circ\mathsf{F})(\sigma^*_\alpha)(x) = \mathsf{W}(\widetilde{\theta_\alpha})(x) = \mathsf{W}(\widetilde{\theta_\alpha(x)})\\ &= \mathsf{W}(\theta_\alpha(x)) = d^1(\theta_\alpha(x)) + \omega^\alpha(x)\wedge\theta_\alpha(x),\end{aligned}$$

after $(\Omega^{(n)})_x \cong (\Omega_x)^{(n)}$. But $\omega^\alpha(x)\wedge\theta_\alpha(x) = (\omega^\alpha\wedge_{U_\alpha}\theta_\alpha)^\sim(x)$, since $\omega^\alpha\wedge_{U_\alpha}\theta_\alpha \in \big(\Omega(U_\alpha)^2\big)^{(n)}$. On the other hand, by Lemma 10.4.2, $d^1\theta_\alpha \in \big(\Omega^2(U_\alpha)\big)^{(n)} \cong \big(\Omega(U_\alpha)^2\big)^{(n)}$, i.e., $d^1\theta_\alpha$ may be thought of as element of $\big(\Omega(U_\alpha)^2\big)^{(n)}$, thus $(d^1\theta_\alpha)^\sim \in (\Omega^2)^{(n)}(U_\alpha)$. Consequently,

$$\begin{aligned}\Theta_\alpha(x) &\equiv d^1(\theta_\alpha(x)) + \omega^\alpha(x)\wedge\theta_\alpha(x)\\ &= (d^1\theta_\alpha)^\sim(x) + (\omega^\alpha\wedge_{U_\alpha}\theta_\alpha)^\sim(x)\\ &= \big(d^1\theta_\alpha + \omega^\alpha\wedge\theta_\alpha\big)^\sim(x),\end{aligned}$$

which yields the result. □

As we saw before, $\Theta_\alpha \in (\Omega^2)^{(n)}(U_\alpha)$, $d^1\theta_\alpha \in \big(\Omega^2(U_\alpha)\big)^{(n)}$ and $\omega^\alpha\wedge_{U_\alpha}\theta_\alpha \in \big(\Omega(U_\alpha)^2\big)^{(n)}$. However, if we take into account the second order analog of (10.4.14) and Lemma 10.4.2, we obtain the isomorphisms

$$(\Omega^2)^{(n)}(U_\alpha) \cong \big(\Omega^2(U_\alpha)\big)^{(n)} \cong \big(\Omega(U_\alpha)^2\big)^{(n)}.$$

Therefore, we prove the following:

10.4.8 Corollary. *Within an isomorphism, (10.4.9) also takes the form:*

(10.4.19′) $$\Theta_\alpha = d^1\theta_\alpha + \omega^\alpha\wedge\theta_\alpha, \qquad \alpha\in I.$$

To differentiate Θ_α, we need a second order differential (same symbol!) d^2 on the sections of $(\Omega^2)^{(n)}$. Thus, if we are given a Bianchi datum $(\mathcal{A}, d, \Omega, d^1, \Omega^2, d^2, \Omega^3)$ (see the relevant comments concerning the terminology in the beginning of Section 8.3), the differential d^2 induces the differentials

$$\begin{aligned}d^2 : \big(\Omega^2(U_\alpha)\big)^{(n)} &\longrightarrow \big(\Omega^3(U_\alpha)\big)^{(n)} :\\ (\zeta_1,\dots,\zeta_n) = \zeta &\longmapsto d^2(\zeta) := (d^2\zeta_1,\dots,d^2\zeta_n),\end{aligned}$$

for every $\alpha\in I$. By the identification $\big(\Omega^p(U_\alpha)\big)^{(n)} \cong \big(\Omega(U_\alpha)^p\big)^{(n)}$ (see Lemma 10.4.2), we can also interpret the previous operators as morphisms of the form

$$d^2 : \big(\Omega(U_\alpha)^2\big)^{(n)} \longrightarrow \big(\Omega(U_\alpha)^3\big)^{(n)}, \qquad \alpha\in I.$$

Moreover, we need the second order analogs of the local exterior products generating (10.4.17). Namely, following the same pattern, we obtain the morphisms (same unadorned symbol for both!)

$$\begin{aligned}\wedge : \big(\Omega(U_\alpha)^2\big)^{(n^2)} \times \Omega(U_\alpha)^{(n)} &\longrightarrow \big(\Omega(U_\alpha)^3\big)^{(n)},\\ \wedge :\ \Omega(U_\alpha)^{(n^2)} \times \big(\Omega(U_\alpha)^2\big)^{(n)} &\longrightarrow \big(\Omega(U_\alpha)^3\big)^{n}.\end{aligned}$$

10.4.9 Theorem. *Let Ω be a vector sheaf of rank n, equipped with an $\mathcal{A}$-connection and a Bianchi datum $(\mathcal{A}, d, \Omega, d^1, \Omega^2, d^2, \Omega^3)$. Then, within isomorphism, the following* ***local Bianchi identities***

$$d^2\Theta_\alpha = \big(R^\alpha \wedge \theta_\alpha - \omega^\alpha \wedge \Theta_\alpha)\big)^{\sim} \equiv R^\alpha \wedge \theta_\alpha - \omega^\alpha \wedge \Theta_\alpha$$

are satisfied, for all $\alpha \in I$.

Proof. Considering the isomorphisms preceding Corollary 10.4.8, we take

$$\begin{aligned}d^1\theta_\alpha &\in \big(\Omega(U_\alpha)^2\big)^{(n)},\\ \omega^\alpha \wedge \theta_\alpha &\in \big(\Omega(U_\alpha)^2\big)^{(n)} \cong \big(\Omega^2(U_\alpha)\big)^{(n)},\\ d^2\Theta_\alpha &\in (\Omega^2)^{(n)}(U_\alpha) \cong \big(\Omega^2(U_\alpha)\big)^{(n)} \cong \big(\Omega(U_\alpha)^2\big)^{(n)},\end{aligned}$$

and we differentiate (10.4.19′) by using the appropriate differential and exterior product each time. Hence, elementary calculations and the properties of d^2 listed before (8.3.5), show that

$$d^2\Theta_\alpha = d^2(d^1\theta_\alpha + \omega^\alpha \wedge \theta_\alpha) = d^2(\omega^\alpha \wedge \theta_\alpha) = (d^1\omega^\alpha) \wedge \theta_\alpha - \omega^\alpha \wedge d^1\theta_\alpha.$$

Applying Cartan's second structure equation of ∇ (see (8.5.24)) and, once more, equality (10.4.19′), we obtain

$$\begin{gathered}(d^1\omega^\alpha) \wedge \theta_\alpha - \omega^\alpha \wedge d^1\theta_\alpha =\\ (R^\alpha - \omega^\alpha \wedge \omega^\alpha) \wedge \theta_\alpha - \omega^\alpha \wedge (\Theta_\alpha - \omega^\alpha \wedge \theta_\alpha) =\\ R^\alpha \wedge \theta_\alpha - \omega^\alpha \wedge \Theta_\alpha,\end{gathered}$$

which, substituted in the preceding series of equalities, leads to the identities of the statement. □

Note. In Mallios [62, Vol. II, p. 236], the local torsion forms (Θ_α) (called local torsions therein) are *defined* by (10.4.19′), starting with the local bases

(θ_α) of Ω, relative to $\mathcal{U}$. In our approach we have shown that the forms (θ_α) define the torsion as a *global* $(\Omega^2)^{(n)}$-valued morphism $\mathcal{T}$ on $\mathcal{P}(\Omega^*)$, from which we obtain the local torsion forms.

In the remainder of this section we outline the construction of the torsion morphism derived from the classical torsion form of the principal bundle of linear frames of a smooth manifold.

So we begin with an n-dimensional smooth manifold X and denote by $TX \equiv (TX, \tau, X)$ its tangent bundle. The (principal) ***bundle of (linear) frames*** of X is denoted by $L(X) \equiv (L(X), \mathrm{GL}(n, \mathbb{R}), X, \pi)$. Its total space consists of all the bases (: ***linear frames***) of the tangent spaces T_xX, for all $x \in X$. Equivalently, a basis of T_xX is identified with a linear isomorphism $\mathbb{R}^n \to T_xX$. If (U_α, ϕ_α) is a chart of X, with coordinates $(x_i^\alpha)_{i=1,\dots,n}$, then $L(X)$ is trivial over U_α by means of the $\mathrm{GL}(n, \mathbb{R})$-equivariant diffeomorphism

$$\Phi_\alpha : \pi^{-1}(U_\alpha) \longrightarrow U_\alpha \times \mathrm{GL}(n, \mathbb{R}) : u \mapsto (x, (u_{ij})),$$

if $u : \mathbb{R}^n \xrightarrow{\simeq} T_xX$. The matrix figuring in the image of Φ_α is determined by

$$u(e_i) = \sum_{j=1}^{n} u_{ji} \frac{\partial}{\partial x_j^\alpha}\Big|_x ,$$

where $\big(\frac{\partial}{\partial x_j^\alpha}\big|_x\big)$ is the natural basis of T_xX with respect to (U_α, ϕ_α).

For any natural section (alias *moving frame*) $s_\alpha \in \Gamma(U_\alpha, L(X))$ and any $x \in U_\alpha$, $s_\alpha(x)$ is the linear isomorphism $\mathbb{R}^n \to T_xX$ satisfying equality

$$(s_\alpha(x))(v) = \sum_{i=1}^{n} v_i \frac{\partial}{\partial x_i^\alpha}\Big|_x , \tag{10.4.20}$$

for every $v = \sum_{i=1}^{n} v_i e_i \in \mathbb{R}^n$.

In this context, the ***canonical form of*** $L(X)$ is defined to be the 1-form $\theta \in \Lambda^1(L(X), \mathbb{R}^n)$ defined by (see, e.g., Kobayashi-Nomizu [49, p. 118], Bleecker [10, p. 78])

$$\theta_u(\xi) := u^{-1}(T_u\pi(\xi)),$$

for every $\xi \in T_uL(X)$ and $u \in L(X)$ interpreted as a linear isomorphism $u : \mathbb{R}^n \to T_xX$, with $T_u\pi = \pi_{*,u} = d_u\pi$ denoting the differential of π at u. The form θ is tensorial, or $\mathrm{GL}(n, \mathbb{R})$-equivariant, in the sense that

$$(R_g^*\theta)_u(\xi) = g^{-1}.\theta_u(\xi) := g^{-1} \circ \theta_u(\xi); \qquad \xi \in T_uL(X),\ u \in L(X),$$

if R_g denotes the right translation of $\mathrm{GL}(n,\mathbb{R})$ by g, and g identifies with a linear automorphism of $\mathbb{R}^n$.

Taking the pull-back of θ by the local sections (s_α) of $L(X)$, we obtain the ***canonical local forms of*** $L(X)$

$$\underline{\theta}_\alpha = s_\alpha^*\theta \in \Lambda^1(U_\alpha, \mathbb{R}^n), \qquad \alpha \in I.$$

A simple calculation shows that

$$\underline{\theta}_{\alpha,x} = s_\alpha(x)^{-1}\,; \qquad x \in U_\alpha,$$

thus (10.4.20) yields the equalities

$$\underline{\theta}_{\alpha,x}\Big(\frac{\partial}{\partial x_i^\alpha}\Big|_x\Big) = e_i\,; \qquad x \in U_\alpha,\ 1 \leq i \leq n.$$

Hence, the smooth map $\underline{\theta}_\alpha(\frac{\partial}{\partial x_i^\alpha}) \in C^\infty(U_\alpha, \mathbb{R}^n)$ reduces to the constant e_i.

Moreover, writing $\underline{\theta}_\alpha$ as an n-tuple of $\mathbb{R}$-valued 1-forms, i.e.,

$$\Lambda^1(U,\mathbb{R}^n) \ni \underline{\theta}_\alpha \equiv (\underline{\theta}_1^\alpha, \ldots, \underline{\theta}_n^\alpha) \in \big(\Lambda^1(U,\mathbb{R})\big)^n,$$

we find that

$$\underline{\theta}_\alpha = (dx_1^\alpha, \ldots, dx_n^\alpha), \qquad \alpha \in I. \tag{10.4.21}$$

In other words, relative to the chart (U_α, ϕ_α), each local torsion form $\underline{\theta}_\alpha$ is precisely the ordinary basis of $\Lambda^1(U_\alpha, \mathbb{R})$, which is the dual of the basis $(\frac{\partial}{\partial x_i^\alpha})$ of TU_α.

To translate the previous classical apparatus into our abstract setting, we consider the following sheaves (see also Example 2.1.4(a)):

- $\mathcal{A} \equiv \mathcal{C}_X^\infty := \mathbf{S}\big(U_\alpha \longmapsto C^\infty(U_\alpha, \mathbb{R})\big)$, i.e., the sheaf of germs of real-valued smooth functions on X;
- $\Omega \equiv \Omega_X^1 := \mathbf{S}\big(U_\alpha \longmapsto \Lambda^1(U_\alpha, \mathbb{R})\big)$, i.e., the sheaf of germs of differential 1-forms on X;
- $\mathfrak{L} := \mathbf{S}\big(U_\alpha \longmapsto \Gamma(U_\alpha, L(X))\big)$, i.e., the sheaf of germs of smooth sections of $L(X)$;
- $\mathcal{GL}(n,\mathbb{R}) := \mathbf{S}\big(U_\alpha \longmapsto C^\infty(U_\alpha, \mathrm{GL}(n,\mathbb{R}))\big)$, i.e., the sheaf of germs of $\mathrm{GL}(n,\mathbb{R})$-valued functions of X.

Note that all the presheaves involved above are defined with respect to the open covering $\mathcal{U} = (U_\alpha)$, determined by the smooth structure of X, thus $\mathcal{U}$ is a basis for the topology of X.

We observe that any section $s \in \Gamma(U_\alpha, L(X))$ corresponds bijectively to a unique invertible matrix $(s_{ij}) \in \mathrm{GL}(n, C^\infty(U_\alpha, \mathbb{R}))$, determined by

$$s(x)(e_i) = \sum_{j=1}^{n} s_{ji}(x) \frac{\partial}{\partial x_j^\alpha}\Big|_x .$$

This fact implies the identifications

$$\mathfrak{L}(U_\alpha) \cong \mathrm{GL}(n, C^\infty(U_\alpha, \mathbb{R})) \cong$$
$$\mathrm{Iso}_{\mathcal{A}|_{U_\alpha}}(\mathcal{A}^n|_{U_\alpha}, \Omega^*|_{U_\alpha}) \cong \mathcal{P}(\Omega^*)(U_\alpha),$$

for every $U_\alpha \in \mathcal{U}$; hence,

$$\mathfrak{L} \cong \mathcal{P}(\Omega^*).$$

Similarly,

$$C^\infty(U_\alpha, \mathrm{GL}(n, \mathbb{R})) \cong C^\infty(U_\alpha, \mathbb{R})^{n^2} \cong$$
$$\mathrm{GL}(n, C^\infty(U_\alpha, \mathbb{R})) \cong \mathrm{GL}(n, \mathcal{A}(U_\alpha)),$$

for every $U_\alpha \in \mathcal{U}$. Thus,

$$\mathcal{GL}(n, \mathbb{R}) \cong \mathcal{GL}(n, \mathcal{A}).$$

Now, given the canonical form θ of $L(X)$, the corresponding canonical local forms $(\underline{\theta}_\alpha)$ determine a unique basis (θ_α) of $\Omega(U_\alpha)$, the elements θ_i^α of each θ_α being defined by the isomorphism (see also 10.4.21)

$$\Lambda^1(U_\alpha, \mathbb{R}) \xrightarrow{\ \simeq\ } \Omega_X^1(U_\alpha) \equiv \Omega(U_\alpha) : dx_i^\alpha \mapsto \widetilde{dx_i^\alpha} =: \theta_i^\alpha .$$

As a result, we state the following conclusion:

10.4.10 Theorem. *Let X be an n-dimensional smooth manifold. Then the canonical form $\theta \in \Lambda^1(L(X), \mathbb{R}^n)$, on the bundle of linear frames $L(X)$ of X, determines the canonical morphism $\mathsf{F} : \mathfrak{L} \cong \mathcal{P}(\Omega^*) \to \Omega^{(n)}$ and the torsion $\mathsf{W} : \mathfrak{L} \cong \mathcal{P}(\Omega^*) \to (\Omega^2)^{(n)}$ on the sheaf $\mathfrak{L}$ of germs of smooth sections of $L(X)$.*

10.5. Riemannian metrics

We intend to show that, under suitable conditions on a given algebraized space $(X, \mathcal{A})$, the existence of a Riemannian metric on a vector sheaf $(\mathcal{E}, \pi, X)$ of rank n amounts to the reduction of $\mathcal{GL}(n, \mathcal{A})$ to the orthogonal group sheaf $\mathcal{O}(n)$.

Before the main results, we give a short account of Riemannian metrics, and we prove some auxiliary results serving the purpose of this section.

If $(\mathcal{E}, \pi, X)$ is an arbitrary $\mathcal{A}$-module, our aim is to define an inner product on $\mathcal{E}$. Classically, an inner product (on a real vector space) is a positive-definite real-valued function on the given space. In our context, where the coefficients live in the algebra sheaf $\mathcal{A}$ (instead of the classical field of reals), we need beforehand an ordering in $\mathcal{A}$, so that in the latter we may consider positive and negative elements.

Motivated by certain algebraic structures, such as ordered fields (see, e.g., Lang [53, p. 390]), and taking into account the structure of $\mathcal{A}$, we say that $(X, \mathcal{A})$ is a ***preordered*** algebraized space if there is a subsheaf $\mathcal{S}$ of $\mathcal{A}$ (called ***preorder***) with the following properties:

$$\mathcal{S} + \mathcal{S} \subseteq \mathcal{S}, \tag{10.5.1}$$

$$\lambda \mathcal{S} \subseteq \mathcal{S}, \tag{10.5.2}$$

$$\mathcal{S} \cdot \mathcal{S} \subseteq \mathcal{S}, \tag{10.5.3}$$

for every $\lambda \in \mathbb{R}_+ \hookrightarrow \mathcal{A}$. All the algebraic operations listed above are defined stalk-wise and $\mathbb{R}_+$ is the short-hand notation for the constant sheaf $\mathbb{R}_+ \times X$. Such a preordered algebraized space is conveniently denoted by $(X, \mathcal{A}, \mathcal{S})$.

A preordered algebraized space $(X, \mathcal{A}, \mathcal{S})$ is called ***(partially) ordered*** if $\mathcal{S}$ satisfies the additional condition

$$\mathcal{S} \cap (-\mathcal{S}) = 0, \tag{10.5.4}$$

where 0 is the constant sheaf 0, also identified with the (image of the) zero section of $\mathcal{S}$ over X.

For the sake of completeness, we add that a sheaf $\mathcal{S}$ satisfying the properties (10.5.1) – (10.5.3) is also known as the (pointed) ***multiplicative convex cone*** of $\mathcal{A}$, whereas (10.5.1) – (10.5.4) characterize $\mathcal{S}$ as the (pointed) ***multiplicative convex salient cone*** of $\mathcal{A}$.

If $(X, \mathcal{A}, \mathcal{S})$ is an ordered algebraized space, then the ***positive*** elements of $\mathcal{A}$ are, by definition, the elements of $\mathcal{S}$ itself. It is customary to set

$\mathcal{A}^+ := \mathcal{S}$. Analogously, the ***negative*** elements of $\mathcal{A}$ are the elements of $\mathcal{A}^- := -\mathcal{S}$. Accordingly, sections of $\mathcal{A}$ taking values in $\mathcal{A}^+$ or $\mathcal{A}^-$ are called ***positive*** or ***negative sections*** respectively.

Now, turning to $\mathcal{A}$-modules (in particular vector sheaves), we first define the notion of inner product.

10.5.1 Definition. Let $(X, \mathcal{A}, \mathcal{S})$ be an ordered algebraized space and let $\mathcal{E} \equiv (\mathcal{E}, \pi, X)$ be an $\mathcal{A}$-module. A (sheaf) morphism $\rho : \mathcal{E} \times_X \mathcal{E} \longrightarrow \mathcal{A}$ is called an $\mathcal{A}$-valued ***inner product*** if it is

i) symmetric and $\mathcal{A}$-*bilinear*,

ii) positive-definite; that is,

$$\begin{cases} \rho(u,u) \in \mathcal{S}_x = (\mathcal{A}^+)_x \cong (\mathcal{A}_x)^+, \\ \rho(u,u) = 0_x \quad \Longleftrightarrow \quad u = 0_x, \end{cases} \tag{10.5.5}$$

for every $u \in \mathcal{E}_x$ and every $x \in X$. In the preceding equivalence, the first 0_x is clearly the zero element of $\mathcal{A}_x$, whereas the second one is the zero of $\mathcal{E}_x$.

An inner product ρ on $\mathcal{E}$ induces a canonical $\mathcal{A}$-morphism

$$\widetilde{\rho} : \mathcal{E} \longrightarrow \mathcal{E}^* := \mathcal{H}om_{\mathcal{A}}(\mathcal{E}, \mathcal{A}) \tag{10.5.6}$$

in the following way: Let U be a fixed open subset of X and let $s \in \mathcal{E}(U)$ be a fixed section. For each open $V \subseteq U$, we define the $\mathcal{A}(V)$-linear map

$$\widetilde{\rho}_U(s)_V : \mathcal{E}(V) \longrightarrow \mathcal{A}(V) : \; t \mapsto \rho(s|_V, t). \tag{10.5.7}$$

Varying V in U, we obtain a presheaf morphism generating a morphism of sheaves

$$\widetilde{\rho}_U(s) \in \mathrm{Hom}_{\mathcal{A}|_U}(\mathcal{E}|_U, \mathcal{A}|_U).$$

Therefore, varying s in $\mathcal{E}(U)$, we obtain a new presheaf morphism.

$$\widetilde{\rho}_U : \mathcal{E}(U) \longrightarrow \mathrm{Hom}_{\mathcal{A}|_U}(\mathcal{E}|_U, \mathcal{A}|_U),$$

where each $\widetilde{\rho}_U$ is $\mathcal{A}(U)$-linear. Running U in the topology of X, $(\widetilde{\rho}_U)$ generates the desired $\mathcal{A}$-morphism $\widetilde{\rho}$.

Based on equality (10.5.5), it is not hard to show that $\widetilde{\rho}$ is an *injective* $\mathcal{A}$-morphism but not necessarily a surjective one. As in the case of infinite-dimensional vector spaces, the surjectivity of $\widetilde{\rho}$ characterizes a particular class of inner products. The abstract homologue of this classical case is described in the following:

10.5.2 Definition. An inner product ρ is said to be ***strongly non-degenerate*** if the canonical morphism $\widetilde{\rho}$ is surjective; hence, ρ is an $\mathcal{A}$-isomorphism. Using a terminology suggested by ordinary differential geometry, we also say that such a ρ is a ***Riemannian $\mathcal{A}$-metric*** and the pair $(\mathcal{E}, \rho)$ a ***Riemannian $\mathcal{A}$-module***. In the same spirit, an inner product, whose canonical morphism $\widetilde{\rho}$ is only injective, is called ***weakly non-degenerate***.

If $\mathcal{E} = \mathcal{A}^n$ (i.e., $\mathcal{E}$ is a *free module*), a Riemannian $\mathcal{A}$-metric on $\mathcal{E}$ is completely known by its matrix $(\rho_{ij}) \in \mathrm{GL}(n, \mathcal{A}(X)) \cong \mathcal{GL}(n, \mathcal{A})(X)$, where $\rho_{ij} := \rho(\epsilon_i, \epsilon_j)$. Here ρ denotes the induced morphism of sections, and $\epsilon_i \in \mathcal{A}^n(X) \cong \mathcal{A}(X)^n$, $1 \leq i \leq n$, are the natural (global) sections of $\mathcal{A}^n$.

Similarly, if $\mathcal{E}$ is a *Riemannian vector sheaf* with corresponding coordinates $\psi_\alpha : \mathcal{E}|_{U_\alpha} \to \mathcal{A}^n|_{U_\alpha}$ over $\mathcal{U} = (U_\alpha)$, then ρ is completely determined by the family of local metrics $\rho_\alpha : \mathcal{E}|_{U_\alpha} \times_{U_\alpha} \mathcal{E}|_{U_\alpha} \to \mathcal{A}|_{U_\alpha}$, restrictions of ρ to the indicated domains. Each ρ_α is equivalently defined by the (local) matrix $(\rho^\alpha_{ij}) \in \mathrm{GL}(n, \mathcal{A}(U_\alpha)) \cong \mathcal{GL}(n, \mathcal{A})(U_\alpha)$, whose entries are given by

$$\rho^\alpha_{ij} := \rho\big(e^\alpha_i, e^\alpha_j\big) = \rho\big(\psi^{-1}_\alpha(\epsilon_i|_{U_\alpha}), \psi^{-1}_\alpha(\epsilon_j|_{U_\alpha})\big); \qquad 1 \leq i, j \leq n,$$

(see (5.1.3)). A direct consequence of the definitions is that the invertible matrices (ρ_{ij}) and (ρ^α_{ij}) are *symmetric*, i.e.,

$$(\rho_{ij}) = (\rho_{ij})^T \quad \text{and} \quad (\rho^\alpha_{ij}) = (\rho^\alpha_{ij})^T,$$

where the superscript T denotes the transpose of the matrix involved.

If we start with an ordered algebraized space $(X, \mathcal{A}, \mathcal{S})$ where, in addition, $\mathcal{A}$ *is assumed to be a Riemannian $\mathcal{A}$-module* with metric ρ, then the free $\mathcal{A}$-module $\mathcal{A}^n$ is provided with a natural Riemannian metric, called ***extension*** of ρ,

$$\rho^n : \mathcal{A}^n \times_X \mathcal{A}^n \longrightarrow \mathcal{A} : (a, b) \mapsto \rho^n(a, b) := \sum_{i=1}^{n} \rho(a_i, b_i), \tag{10.5.8}$$

for every $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$ in the given domain.

On the other hand, assuming again that $(\mathcal{A}, \rho)$ is a Riemannian module, a vector sheaf $\mathcal{E}$ (of rank n) can be equipped with a Riemannian metric in various ways. One routine is, roughly speaking, the following: We choose a local frame $(U_\alpha, (\psi_\alpha))$ and restrict the metric ρ^n , defined by (10.5.8), to every $\mathcal{A}^n|_{U_\alpha}$. Transferring the latter metrics by the coordinates, we obtain a family of local Riemannian metrics, say $\rho_\alpha : \mathcal{E}|_{U_\alpha} \times_{U_\alpha} \mathcal{E}|_{U_\alpha} \to \mathcal{A}|_{U_\alpha}$,

$\alpha \in I$. We get a (global) Riemannian metric $\rho_{\mathcal{E}}$ on $\mathcal{E}$ by gluing together the ρ_α's, a procedure applicable if, for instance, $\mathcal{A}$ admits a partition of unity subordinate to $\mathcal{U}$.

The idea just described hides certain subtleties. Firstly, to define a partition of unity subordinate to $\mathcal{U}$, the covering need to be *locally finite*. For this purpose, X is assumed to be a *(Hausdorff) paracompact* space. Then, for any local frame $(\mathcal{V}, (\chi_\alpha))$ of $\mathcal{E}$, there is a locally finite refinement $\mathcal{U}$ of $\mathcal{V}$, which yields (by appropriate restrictions) a local frame $(\mathcal{U}, (\psi_\alpha))$ of $\mathcal{E}$ with the desired property.

Secondly, the morphism $\rho_{\mathcal{E}}$, obtained by the gluing process via a partition of unity, is, in general, only symmetric and bilinear. We do get a (positive-definite) inner product if we further assume that the partition of unity, say (f_α), is ***strictly positive***; that is,

$$f_\alpha(u) \in \mathcal{A}_x^+ \cap \mathcal{A}_x^\bullet, \quad \text{for every} \quad u \in \mathcal{A}_x,\, x \in X,\, \alpha \in I.$$

Notice that

$$\mathcal{A}_x^+ := (\mathcal{A}^+)_x \cong (\mathcal{A}_x)^+ \quad \text{and} \quad \mathcal{A}_x^\bullet := (\mathcal{A}^\bullet)_x \cong (\mathcal{A}_x)^\bullet.$$

We do not give further details (which can otherwise be found in [62, Vol. I, pp. 316–330]; see in particular Theorem 8.3, p. 328 ibid.) because, as explained earlier, our intention is to obtain Riemannian metrics from a reduction of the structure sheaf. However, there are still a couple of important notions needed later on.

10.5.3 Definition. An ordered algebraized space $(X, \mathcal{A}, \mathcal{S})$ *has square roots* if the following conditions are satisfied:

i) The strictly positive elements of $\mathcal{A}$ are *invertible*, i.e., $\mathcal{A}^+ - 0 \subseteq \mathcal{A}^\bullet$;

ii) For every $a \in \mathcal{A}^+$, there is a unique $b \in \mathcal{A}^+$ such that

$$(a, b) \in \mathcal{A}^+ \times_X \mathcal{A}^+ \quad \text{and} \quad b^2 = a.$$

In this case, b is called the ***square root*** of a, also denoted by $\sqrt{a}$.

The square root of a section is defined in the obvious way. We note that the existence of square roots is not guaranteed for sheaves of arbitrary algebras. The sheaves of germs of $\mathbb{R}$-valued continuous (resp. smooth maps) on a Hausdorff paracompact topological space (resp. smooth manifold) X are typical examples of algebraized spaces with square roots.

10.5.4 Definition. If $(\mathcal{A}, \rho)$ is a Riemannian $\mathcal{A}$-module also admitting square roots, then the ***$\mathcal{A}$-valued norm on $\mathcal{A}^n$*** is the morphism

$$\|\cdot\| : \mathcal{A}^n \longrightarrow \mathcal{A}^+ \subset \mathcal{A} : a \mapsto \|a\| := \sqrt{\rho^n(a, a)}.$$

With the foregoing machinery at our disposal (: existence of square roots, norms etc.), we can perform the analog of the ***Gram-Schmidt orthogonalization*** process to any basis of sections of $\mathcal{A}^n$. As a matter of fact, given a basis $(\sigma_i) \in \mathcal{A}^n(X) \cong \mathcal{A}(X)^n$, $1 \leq i \leq n$, we construct an ***orthonormal basis*** (s_i), $1 \leq i \leq n$; in other words,

$$\| s_i \| = \sqrt{\rho^n(s_i, s_i)} = 1|_X, \quad \rho^n(s_i, s_j) = \delta_{ij}.$$

The steps of the process follow the standard inductive pattern of the classical case, under the necessary modifications. The same procedure works for any *free* $\mathcal{A}$-module.

Let us fix a Riemannian $\mathcal{A}$-module $(\mathcal{A}, \rho)$ with square roots and an orthonormal basis $(\epsilon_i) \in \mathcal{A}^n(X)$. An ***isometry*** of $\mathcal{A}^n$ (with respect to ρ^n) is an $\mathcal{A}$-morphism $f : \mathcal{A}^n \to \mathcal{A}^n$ satisfying

$$\rho^n(f(u), f(v)) = \rho^n(u, v), \qquad (u, v) \in \mathcal{A}^n \times_X \mathcal{A}^n.$$

It corresponds bijectively to a matrix

$$(f_{ij}) \in \mathrm{GL}(n, \mathcal{A}(X)) \quad \text{such that} \quad (f_{ij}) \cdot (f_{ij})^T = \mathrm{I},$$

where I is the identity matrix and $f_{ij} = \rho^n(\epsilon_i, \epsilon_j)$. Thus

$$(f_{ij}) \in \mathrm{O}(n, \mathcal{A}(X)),$$

the latter symbol denoting the ***orthogonal group*** with coefficients in $\mathcal{A}(X)$. The ***orthogonal group sheaf*** $\mathcal{O}(n, \mathcal{A})$ is the sheaf of groups generated by the complete presheaf of (local) orthogonal groups

$$U \longmapsto \mathrm{O}(n, \mathcal{A}(U)).$$

The preceding preliminary (though lengthy) discussion brings us to the main target of the present section. Before stating its results, let us gather all the properties of $\mathcal{A}$ needed henceforth. So, we assume that:

(10.5.9) $(X, \mathcal{A}, \mathcal{S})$ is an ordered algebraized space such that $(\mathcal{A}, \rho)$ is a Riemannian $\mathcal{A}$-module (with a Riemannian metric ρ), also admitting square roots. If ρ^n is the Riemannian metric on $\mathcal{A}^n$, defined by (10.5.8), we fix an orthonormal basis $(\epsilon_i)_{1 \leq i \leq n}$ of $\mathcal{A}(X)^n$, relative to ρ^n, by applying the Gram-Schmidt orthogonalization process.

10.5.5 Proposition. *With the assumptions (10.5.9), we consider a vector sheaf $\mathcal{E} = (\mathcal{E}, \pi, X)$ of rank n and a local frame $(\mathcal{U}, (\psi_\alpha))$ of it. If the coordinate transformations $\psi_{\alpha\beta} := \psi_\alpha \circ \psi_\beta^{-1} : \mathcal{A}^n|_{U_{\alpha\beta}} \to \mathcal{A}^n|_{U_{\alpha\beta}}$ are isometries (with respect to ρ^n), then $\mathcal{E}$ is provided with a Riemannian $\mathcal{A}$-metric.*

Proof. For each $\alpha \in I$, we define the morphism $\rho_{\mathcal{E}}^\alpha : \mathcal{E}|_{U_\alpha} \times_{U_\alpha} \mathcal{E}|_{U_\alpha}$ by setting $\rho_{\mathcal{E}}^\alpha := \rho^n \circ (\psi_\alpha \times \psi_\alpha)$, as in the diagram

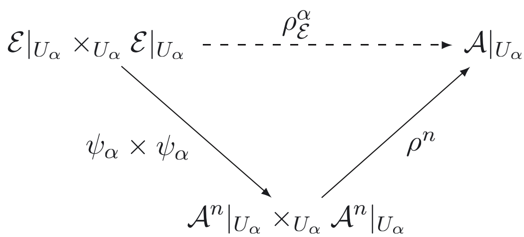

Diagram 10.2

where, for the sake of simplicity, we have omitted the symbol of restriction from ρ^n.

We shall show that $\rho_{\mathcal{E}}^\alpha = \rho_{\mathcal{E}}^\beta$ over $\mathcal{E}|_{U_{\alpha\beta}} \times_{U_{\alpha\beta}} \mathcal{E}|_{U_{\alpha\beta}}$. Indeed, for every (u, v) in the last sheaf, we have that

$$\begin{aligned} \rho_{\mathcal{E}}^\beta(u, v) &= \rho^n\big(\psi_\beta(u), \psi_\beta(v)\big) \\ &= \rho^n\big(\psi_{\alpha\beta}(\psi_\beta(u), \psi_{\alpha\beta}(\psi_\beta(v))\big) \\ &= \rho^n\big(\psi_\alpha(u), \psi_\alpha(v)\big) \\ &= \rho_{\mathcal{E}}^\alpha(u, v). \end{aligned}$$

Therefore, gluing the morphisms $(\rho_{\mathcal{E}}^\alpha)_{\alpha \in I}$ together, we obtain a morphism $\rho : \mathcal{E} \times_X \mathcal{E}$. It is a Riemannian $\mathcal{A}$-metric since all the previous partial morphisms have the same property, in virtue of our assumptions. □

Proposition 10.5.5 implies that the transition matrix $\big(\psi_{ij}^{\alpha\beta}\big)$ corresponding to $\psi_{\alpha\beta}$ is an orthogonal matrix, i.e.,

$$\big(\psi_{ij}^{\alpha\beta}\big) \in \mathrm{O}(n, \mathcal{A}(U_{\alpha\beta}));$$

thus, identifying the coordinate transformations with the transition matrices (recall Proposition 5.1.4 and its preceding discussion), we see that the cocycle of $\mathcal{E}$ now has the form

$$(\psi_{\alpha\beta}) \in Z^1(\mathcal{U}, \mathcal{O}(n, \mathcal{A})). \tag{10.5.10}$$

Consequently, the previous proposition can be restated as follows:

10.5.5 Proposition (restated). *If the cocycle of $\mathcal{E}$ has coefficients in $\mathcal{O}(n, \mathcal{A})$, then $\mathcal{E}$ admits a Riemannian $\mathcal{A}$-metric $\rho_{\mathcal{E}}$.*

10.5.6 Corollary. *Under the assumptions of Propositions 10.5.5, the coordinates $\psi_\alpha : \mathcal{E}|_{U_\alpha} \to \mathcal{A}^n|_{U_\alpha}$, $\alpha \in I$, are isometries with respect to the metrics $\rho_{\mathcal{E}}$ and ρ^n.*

Proof. Reverting to the beginning of the proof of Proposition 10.5.5, we see that the isometry property of ψ_α is merely the definition of $\rho^\alpha_{\mathcal{E}}$. □

10.5.7 Corollary. *Let $\mathcal{E}$ be a vector sheaf of rank n, equipped with a Riemannian $\mathcal{A}$-metric $\rho_{\mathcal{E}}$. If the coordinates (ψ_α) are isometries (with respect to $\rho_{\mathcal{E}}$ and ρ^n), then $(\psi_{\alpha\beta}) \in Z^1(\mathcal{U}, \mathcal{O}(n, \mathcal{A}))$.*

Proof. For every $(u, v) \in \mathcal{E}|_{U_{\alpha\beta}} \times_{U_{\alpha\beta}} \mathcal{E}|_{U_{\alpha\beta}}$, the assumption implies that

$$\rho_{\mathcal{E}}(u, v) = \rho^n\big(\psi_\alpha(u), \psi_\alpha(v)\big) = \rho^n\big(\psi_\beta(u), \psi_\beta(v)\big),$$

for all $\alpha, \beta \in I$. Therefore, for every $(a, b) \in \mathcal{A}^n|_{U_{\alpha\beta}} \times_{U_{\alpha\beta}} \mathcal{A}^n|_{U_{\alpha\beta}}$,

$$\begin{aligned} \rho^n\big(\psi_{\alpha\beta}(a), \psi_{\alpha\beta}(b)\big) &= \rho^n\big(\psi_\alpha(\psi_\beta^{-1}(a)), \psi_\alpha(\psi_\beta^{-1}(b))\big) \\ &= \rho_{\mathcal{E}}\big(\psi_\beta^{-1}(a), \psi_\beta^{-1}(b)\big) \\ &= \rho^n\big(\psi_\beta(\psi_\beta^{-1}(a)), \psi_\beta(\psi_\beta^{-1}(b))\big) \\ &= \rho^n(a, b), \end{aligned}$$

which means that $\psi_{\alpha\beta} \in \mathrm{O}(n, \mathcal{A}(U_{\alpha\beta})) \cong \mathcal{O}(n, \mathcal{A}(U_{\alpha\beta}))$. □

In virtue of Corollaries 10.5.6 and 10.5.7 we also have:

10.5.8 Corollary. *Let $\mathcal{E}$ be a vector sheaf of rank n and $(\mathcal{U}, (\psi_\alpha))$ a given local frame of it. Then the following conditions are equivalent:*

i) $(\psi_{\alpha\beta}) \in Z^1(\mathcal{U}, \mathcal{O}(n, \mathcal{A}))$.

ii) There exists a Riemannian $\mathcal{A}$-metric $\rho_{\mathcal{E}}$ on $\mathcal{E}$ such that the coordinates (ψ_α) are isometries (with respect to $\rho_{\mathcal{E}}$ and ρ^n).

With the assumptions (10.5.9) always in force, the converse of Proposition 10.5.5 is stated in the following way.

10.5.9 Proposition. *Let $\mathcal{E}$ be a vector sheaf of rank n admitting a Riemannian $\mathcal{A}$-metric $\rho_{\mathcal{E}}$. Then there always exists a local frame $(\mathcal{U}, (\psi_\alpha))$ of $\mathcal{E}$ such that the coordinates (ψ_α) are isometries and the respective cocycle has coefficients in $\mathcal{O}(n, \mathcal{A})$.*

Proof. Let $(\mathcal{U}, (\chi_\alpha))$ be a local frame of $\mathcal{E}$. We consider the canonical orthonormal basis (ϵ_i) of $\mathcal{A}^n(X) \cong \mathcal{A}(X)^n$ and the basis $(\bar{e}_i^\alpha)$ of $\mathcal{E}(U_\alpha)$ with $\bar{e}_i^\alpha := \chi_\alpha^{-1}(\epsilon_i|_{U_\alpha})$, $1 \leq i \leq n$. Applying the Gram-Schmidt orthogonalization process to $(\bar{e}_i^\alpha)$, we obtain an orthonormal basis (e_i^α) of $\mathcal{E}(U_\alpha)$. Then the maps $\psi_\alpha : \mathcal{E}|_{U_\alpha} \to \mathcal{A}^n|_{U_\alpha}$, defined by $\psi_\alpha(e_i^\alpha) = \epsilon_i|_{U_\alpha}$, for all $1 \leq i \leq n$ and $\alpha \in I$, induce coordinates which are isometries with respect to the metrics involved. By Corollary 10.5.8 we get a cocycle of the form (10.5.10). □

Using a suitable terminology, a ***Riemannian local frame*** is a frame satisfying the properties of Proposition 10.5.9.

We now formulate the existence of Riemannian metrics in terms of principal sheaves and also the reduction of their structure sheaf (see Section 4.7).

We start with a Riemannian vector sheaf $(\mathcal{E}, \rho_\mathcal{E})$ of rank n. Mimicking the general technique of the construction of the principal sheaf of frames $\mathcal{P}(\mathcal{E})$ (see Section 5.2), we define the ***sheaf of orthonormal frames*** $\mathcal{P}_\mathrm{o}(\mathcal{E})$ of $\mathcal{E}$. As a matter of fact, we choose a Riemannian local frame $(\mathcal{U}, (\psi_\alpha))$ of $\mathcal{E}$ so that $\mathcal{U}$ be the basis for the topology of X. Also, we denote by

$$\mathrm{Isom}_\mathcal{A}(\mathcal{A}^n, \mathcal{E}) \subset \mathrm{Iso}_\mathcal{A}(\mathcal{A}^n, \mathcal{E})$$

the $\mathcal{A}(X)$-module of isometries (with respect to ρ^n and $\rho_\mathcal{E}$). Localizing the previous module over $\mathcal{U}$, we obtain the presheaf of modules

$$U_\alpha \longmapsto \mathrm{Isom}_{\mathcal{A}|_{U_\alpha}}(\mathcal{A}^n|_{U_\alpha}, \mathcal{E}|_{U_\alpha}).$$

The sheafification of the latter is, by definition, the sheaf $\mathcal{P}_\mathrm{o}(\mathcal{E})$.

As in the case of the ordinary sheaf of frames, for each $\alpha \in I$, we define the coordinate

$$\Phi_\alpha^\mathrm{o} : \mathcal{P}_\mathrm{o}(\mathcal{E})|_{U_\alpha} \xrightarrow{\simeq} \mathcal{O}(n, \mathcal{A})|_{U_\alpha},$$

generated by the $\mathcal{A}(V)$-isomorphisms, when V is varying in U_α,

$$\Phi_{\alpha,V}^\mathrm{o} : \mathrm{Isom}_{\mathcal{A}|_V}(\mathcal{A}^n|_V, \mathcal{E}|_V) \longrightarrow \mathrm{O}(n, \mathcal{A}(V)) : f \to \psi_\alpha \circ f,$$

the isometry $\psi_\alpha \circ f$ being identified with its orthogonal matrix. Therefore, it is routinely checked that

$$\mathcal{P}_\mathrm{o}(\mathcal{E}) \equiv (\mathcal{P}_\mathrm{o}(\mathcal{E}), \mathcal{O}(n, \mathcal{A}), X, \pi_\mathrm{o})$$

is a principal sheaf, *subsheaf* of $\mathcal{P}(\mathcal{E})$.

Let us denote by $g^{\mathrm{o}}_{\alpha\beta} \in \mathcal{O}(n,\mathcal{A})(U_{\alpha\beta})$ the transition sections of $\mathcal{P}_{\mathrm{o}}(\mathcal{E})$, and by $g_{\alpha\beta} \in \mathcal{GL}(n,\mathcal{A})(U_{\alpha\beta})$ the transition sections of $\mathcal{P}(\mathcal{E})$. Then, identifying $\mathcal{G}$ with the sheaf of germs of its sections, Diagram 1.7 and Corollary 5.2.3 imply that

$$\begin{aligned} g^{\mathrm{o}}_{\alpha\beta} &= \big(\Phi^{\mathrm{o}}_{\alpha} \circ (\Phi^{\mathrm{o}}_{\beta})^{-1}\big)(\mathbf{1}|_{U_{\alpha\beta}}) \\ &= \big(\big(\Phi^{\mathrm{o}}_{\alpha,U_{\alpha\beta}} \circ (\Phi^{\mathrm{o}}_{\beta,U_{\alpha\beta}})^{-1}\big)(\mathbf{1}|_{U_{\alpha\beta}})\big)^{\sim} \\ &= (\psi_{\alpha} \circ \psi_{\beta}^{-1})^{\sim} = \widetilde{\psi_{\alpha\beta}} = g_{\alpha\beta}. \end{aligned}$$

Thus, in virtue of Proposition 4.7.2 and Corollary 4.7.5,

> *the principal sheaf of frames $\mathcal{P}(\mathcal{E})$ of a Riemannian vector sheaf $\mathcal{E}$ reduces to the principal (sub)sheaf of orthonormal frames $\mathcal{P}_{\mathrm{o}}(\mathcal{E})$; equivalently, $\mathcal{GL}(n,\mathcal{A})$ reduces to $\mathcal{O}(n,\mathcal{A})$.*

Conversely, assume that $\mathcal{GL}(n,\mathcal{A})$ reduces to $\mathcal{O}(n,\mathcal{A})$. Then there exists a $\mathcal{O}(n,\mathcal{A})$-principal subsheaf $\mathcal{F}$ of $\mathcal{P}(\mathcal{E})$. If $(g_{\alpha\beta})$ denotes the cocycle of $\mathcal{F}$, we have that $(g_{\alpha\beta}) \in Z^1(\mathcal{U},\mathcal{O}(n,\mathcal{A}))$. Since $(g_{\alpha\beta})$ coincides with the cocycle of $\mathcal{P}(\mathcal{E})$, and the latter coincides –up to isomorphism– with the cocycle $(\psi_{\alpha\beta})$ of $\mathcal{E}$, we have that $(\psi_{\alpha\beta}) = (g_{\alpha\beta}) \in Z^1(\mathcal{U},\mathcal{O}(n,\mathcal{A}))$; hence, in virtue of Corollary 10.5.8, $\mathcal{E}$ has a Riemannian $\mathcal{A}$-metric. However, we can select as local frame of $\mathcal{E}$ a Riemannian one, whose open sets form the basis for the topology of X. In this case, we construct, as earlier, the principal sheaf of orthonormal frames $\mathcal{P}_{\mathrm{o}}(\mathcal{E})$. Since its cocycle also coincides with $(\psi_{\alpha\beta}) = (g_{\alpha\beta})$, we conclude that $\mathcal{P}_{\mathrm{o}}(\mathcal{E}) = \mathcal{F}$, within an isomorphism. In other words,

> *the reduction of $\mathcal{GL}(n,\mathcal{A})$ to $\mathcal{O}(n,\mathcal{A})$ implies that $\mathcal{P}(\mathcal{E})$ reduces to $\mathcal{P}_{\mathrm{o}}(\mathcal{E})$, and $\mathcal{E}$ is a Riemannian vector sheaf.*

Putting together the previous italicized conclusions, we state the following main result, closing the present section.

10.5.10 Theorem. *Let $\mathcal{E}$ be a vector sheaf of rank n. With the assumptions (10.5.9) on $\mathcal{A}$, the following conditions are equivalent:*

i) The sheaf $\mathcal{E}$ admits a Riemannian $\mathcal{A}$-metric.

ii) The sheaf of frames $(\mathcal{P}(\mathcal{E}),\mathcal{GL}(n,\mathcal{A}),X,\pi)$ reduces to the sheaf of orthonormal frames $(\mathcal{P}_{\mathrm{o}}(\mathcal{E}),\mathcal{O}(n,\mathcal{A}),X,\pi_{\mathrm{o}})$.

iii) The general linear group sheaf $\mathcal{GL}(n,\mathcal{A})$ reduces to the orthogonal group sheaf $\mathcal{O}(n,\mathcal{A})$.

10.6. Problems for further investigation

The following list of selected problems presents a research interest, and complements some of the ideas expounded in the main part of this work.

1. Define the notion of Grassmann sheaf and classify vector sheaves by means of it.

2. Construct universal sheaves and connections.

3. Define the abstract notions of parallel translation and holonomy, and relate them with connections.

4. Relate flat connections, and relative notions of flatness, with representations of the fundamental group of the base space of a principal sheaf on its structure sheaf.

5. Find conditions under which it is possible to determine connections with prescribed curvature.

6. Develop a general theory of $\mathcal{G}$-structures, where $\mathcal{G}$ is Lie sheaf of groups.

7. Develop a sheaf-theoretic approach to symplectic geometry and geometric quantization.

8. Develop a sheaf-theoretic approach to gauge theory.

Certain problems stated above might be hard to answer and represent large undertakings. Nevertheless, their investigation would be a valuable contribution towards the same direction of research. The interested readers are kindly invited to explore the new territory opened by ADG.

Σὰν βγεῖς στὸν πηγαιμὸ γιὰ τὴν Ἰθάκη,
νὰ εὔχεσαι νἆναι μακρὺς ὁ δρόμος,
γεμάτος περιπέτειες, γεμάτος γνώσεις.

As you set out for Ithaka
hope the voyage is a long one,
full of adventure, full of discovery.

(From Constantine Cavafy's (1863–1933) *Ithaka**)

*C. P. Kavafy, Collected Poems. Translated by Edmund Keeley and Philip Sherrard. Princeton University Press, Princeton N. J., 1992

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List of symbols

The list contains the main symbols, with a fixed meaning, together with a brief description and the page of their first appearance.

Chapter 1

$\mathfrak{T}_X$	topology of X	2
$\mathcal{S} \equiv (\mathcal{S}, \pi, X)$	sheaf over X	2
$\pi : \mathcal{S} \to X$	projection of $\mathcal{S}$	2
$\mathcal{S}_x := \pi^{-1}(x)$	stalk of $\mathcal{S}$ at x	2
$\mathcal{S}\|_U := \pi^{-1}(U)$	restriction of $\mathcal{S}$ to U	2
$\phi : \mathcal{S} \to \mathcal{T}$	sheaf morphism	3
$\phi_x : \mathcal{S}_x \to \mathcal{T}_x$	restriction of a sheaf morphism to the stalk at x	3
$\mathrm{Hom}(\mathcal{S}, \mathcal{T})$	set of sheaf morphisms of $\mathcal{S}$ into $\mathcal{T}$	3
$\mathcal{S}h_X$	category of sheaves over X	3
$\mathcal{S} \times_X \mathcal{T}$	fiber product of sheaves over X	4
$\mathcal{G} \equiv (\mathcal{G}, \pi, X)$	sheaf of groups	4
$\gamma : \mathcal{G} \times_X \mathcal{G} \to \mathcal{G}$	multiplication of $\mathcal{G}$	4
$\alpha : \mathcal{G} \to \mathcal{G}$	inversion of $\mathcal{G}$	4
$\mathcal{A} \equiv (\mathcal{A}, \pi, X)$	sheaf of algebras	4
$\mathcal{E} \equiv (\mathcal{E}, \pi, X)$	$\mathcal{A}$-module (also vector sheaf, p. 164)	4

Chapter 2

Chapter 3

	$f^*(\mathcal{S} \otimes_{\mathcal{A}} \mathcal{T}) \cong f^*(\mathcal{S}) \otimes_{f^*(\mathcal{A})} f^*(\mathcal{T})$	116
∂^*	Maurer-Cartan differential of $f^*(\mathcal{G})$	118
ρ^*	representation of $f^*(\mathcal{G})$ on $f^*(\mathcal{L})$	123
Δ^*	action of $f^*(\mathcal{G})$ on $f^*(\Omega)\,(f^*(\mathcal{L}))$	123

Chapter 4

$(\mathcal{P}, \mathcal{G}, X, \pi)$	principal sheaf	132
$\mathcal{U} = \{U_\alpha \subseteq X \mid \alpha \in I\}$	(open) covering of X	132
$\phi_\alpha : \mathcal{P}\vert_{U_\alpha} \xrightarrow{\simeq} \mathcal{G}\vert_{U_\alpha}$	coordinate of a principal sheaf $\mathcal{P}$	132
$\mathcal{U} \equiv (\mathcal{U}, (\phi_\alpha))$	local frame with its coordinates	133
$\boldsymbol{k} : \mathcal{P} \times_X \mathcal{P} \to \mathcal{G}$	morphism connecting elements in the same stalks of a principal sheaf	134
(s_α)	natural sections of $\mathcal{P}$	135
(s^*_α)	natural sections of $f^*(\mathcal{P})$	139
$(f, \phi, \overline{\phi}, id_X)$	morphism of principal sheaves	140
(f, ϕ, id_X)	particular case of the above	140
$f^*_{\mathcal{S}}$	equivalent to $\mathrm{pr}_2\vert_{f^*(\mathcal{S})}$	140
$(g_{\alpha\beta}) \in Z^1(\mathcal{U}, \mathcal{G})$	1-cocycle with values in $\mathcal{G}$	144
$\boldsymbol{P}_{\mathcal{G}}(X)$	set of equivalence classes of $\mathcal{G}$-isomorphic principal sheaves over X	156

Chapter 5

$\mathcal{E} \equiv (\mathcal{E}, \pi, X)$	vector sheaf (also $\mathcal{A}$-module, p. 4)	164
$\psi_\alpha : \mathcal{E}\vert_{U_\alpha} \xrightarrow{\simeq} \mathcal{A}^n\vert_{U_\alpha}$	coordinate of a vector sheaf $\mathcal{E}$	164
$e^\alpha_i \in \mathcal{E}(U_\alpha)$, $i \in I$	natural sections of a vector sheaf	165
$\psi_{\alpha\beta} = \psi_\alpha \circ \psi^{-1}_\beta$	coordinate transformation of a vector sheaf	166
$(g^{\alpha\beta}_{ij})$	transition matrix of $\mathcal{E}$	167

Chapter 6

Chapter 9

Chapter 10

Subject index